1.06 *Radicals and Fractional Exponents*

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Before beginning a study of **radicals** (or **roots**), it is appropriate to begin with some necessary terminology. The expression $\sqrt{X}$ or $\sqrt[2]{X}$ means the "square root of X" or "what squared would equal X?" The quantity inside the radical sign (or in this case **X)** is called the **radicand,** and the **2** (in this case) is the **index of the radical.** The expression $\sqrt[3]{X}$ is called the "cube root of X," and it asks the question, "What cubed would equal X?" In general, $\sqrt[n]{X}$ means the **"nth root of X,"** where the **radicand** is **X,** and the **index of the radical** is **"n".**

Remember also that the operations of **square root, cube root, fourth root, etc.** are actually **inverse operations** for the operations of **squaring, cubing, raising to the fourth power, etc.** When taking a **square root,** it is essential to be familiar with the **perfect squares: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144,** and **169.** Also, remember that the **even powers** (X^2, X^4, X^6, X^8, X^{10}, **etc.)** are perfect squares. When taking a **cube root,** it is essential to be familiar with (i.e., **memorize them!!**) the **perfect cubes: 1, 8, 27, 64,** and **125.** Powers that are **divisible by three** (X^3, X^6, X^9, X^{12}, X^{15}, **etc.)** are perfect cubes. When taking **fourth roots, fifth roots, etc,** remember the **perfect fourth powers:** $2^4 = 16$, $3^4 = 81$, and powers that are **divisible by four** (X^4, X^8, X^{12}, **etc.)**, and the **perfect fifth powers:** $2^5 = 32$, and powers that are **divisible by five** (X^5, X^{10}, X^{15}, **etc.)**.

Simplify each of the following:

1. $\sqrt{25X^6}$ 2. $\sqrt{49X^{12}}$ 3. $\sqrt{16X^{16}}$ 4. $\sqrt{25X^{100}}$

5. $\sqrt[3]{125X^6}$ 6. $\sqrt[3]{8X^{12}}$ 7. $\sqrt[3]{27X^{27}}$ 8. $\sqrt[3]{64X^{51}}$

9. $\sqrt[4]{16X^{16}}$ 10. $\sqrt[4]{81X^{12}}$ 11. $\sqrt[5]{32X^{20}}$ 12. $\sqrt[5]{32X^{60}}$

If the root to be taken is not a "perfect power," then sometimes it can be simplified by using the **product property of radicals.**

Product Property of Equations

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$
$$\sqrt[3]{a \cdot b} = \sqrt[3]{a} \cdot \sqrt[3]{b}$$
$$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

Because the product property of radicals is a property of **real numbers**, if the **index** of the radical is **even**, then the **radicands** must be positive.

To simplify a square root by this property, it may be helpful to think of the "radical two-step": 1. Sort; 2. Sqrt. In the first step, you must **"sort"** the radical, placing the **"perfect squares"** in the first radical and the other "leftover" factors in the second radicals. In the second step, you take the square root of the perfect square, and just bring down the "leftover" radical. For higher roots, the process is analagous.

13. $\sqrt{125X^6} = \sqrt{25X^6} \cdot \sqrt{5}$

$= \_\_\_\_\_\_\_\_\_\_$

14. $\sqrt{48X^{13}} = \sqrt{16X^{12}} \cdot \sqrt{3X}$

$= \_\_\_\_\_\_\_\_\_\_$

15. $\sqrt{72X^9} =$

16. $\sqrt{50X^7} =$

17. $\sqrt{75X^8Y^9} =$

18. $\sqrt{40X^{11}Y^6} =$

19. $\sqrt{98X^7Y^{13}} =$

20. $\sqrt{300X^{15}Y^{25}} =$

21. $\sqrt[3]{54X^6Y^{10}} = \sqrt[3]{27X^6Y^9} \cdot \sqrt[3]{2Y}$

$= \_\_\_\_\_\_\_\_\_\_$

22. $\sqrt[3]{16X^7Y^{12}} =$

23. $\sqrt[3]{72X^5Y^8} =$

24. $\sqrt[3]{80X^4Y^{14}} =$

25. $\sqrt[4]{32X^8Y^6} =$

26. $\sqrt[4]{48X^5Y^{16}} =$

27. $\sqrt[4]{162X^9Y^{10}} =$

28. $\sqrt[4]{405X^7Y^{14}} =$

29. $\sqrt[5]{96X^{12}Y^9} =$

30. $\sqrt[5]{64X^{25}Y^{13}} =$

After simplification of radicals, the next step is **operations with radicals**--that is, addition, subtraction, multiplication, and division of radicals. Addition and subtraction of radicals is just like combining like terms. Even as 3X + 4X = 7X, so it is true that $3\sqrt{2} + 4\sqrt{2} = 7\sqrt{2}$ and $3\sqrt[3]{2} + 4\sqrt[3]{2} = 7\sqrt[3]{2}$. The expression $3\sqrt{2} + 4\sqrt{3}$ cannot be combined because $\sqrt{2}$ and $\sqrt{3}$ are "unlike radicals." Similarly, $(3\sqrt[3]{2} + 4\sqrt[3]{3})$ and $(3\sqrt[3]{5}\ \ 4\sqrt{5})$ cannot be combined since $\sqrt[3]{2}$ and $\sqrt[3]{3}$ are not like radicals and $\sqrt[3]{5}$ and $\sqrt{5}$ are unlike radicals.

What about $3\sqrt{2} + 4\sqrt{8}$? At first glance, it appears that $\sqrt{2}$ and $\sqrt{8}$ are unlike radicals. However, since $\sqrt{8}$ simplifies to $2\sqrt{2}$, the expression <u>can</u> be combined!

Can **6 - 4√2** be simplified to **2√2**? This is a very common error! Even as **6 - 4X** cannot be combined, neither can **6 - 4√2**. It is possible to factor the common factor of **2** from **6 - 4√2** and write **2(3 - 2√2)**. There will be more on factoring later.

Simplify each of the following radical expressions if possible.

31. $24 - 4\sqrt{18}$

32. $60 - 10\sqrt{32}$

33. $3\sqrt{2} + 4\sqrt{8}$

34. $3\sqrt{75} + 4\sqrt{48}$

35. $\sqrt[3]{16} + \sqrt[3]{54}$

36. $2\sqrt[3]{81} - 3\sqrt[3]{375}$

37. $7\sqrt[3]{40} + 3\sqrt[3]{320}$

38. $5\sqrt[3]{108} - 4\sqrt[3]{32}$

39. $7\sqrt[4]{32} - 3\sqrt[4]{162}$

40. $5\sqrt[5]{5} + 4\sqrt[5]{160}$

41. $7X^2\sqrt{24XY^6} + 8Y^3\sqrt{54X^5}$ 42. $5XY\sqrt{20X^7Y^5} - 4\sqrt{45X^9Y^7}$

43. $5X^2Y\sqrt[3]{54X^7Y^5} - 4XY^2\sqrt[3]{16X^{10}Y^2}$ 44. $7X\sqrt[3]{16XY^3} + 8Y\sqrt[3]{54X^4}$

45. $3X^2Y\sqrt{20XY^4} - 2X\sqrt{45X^3Y^6}$ 46. $3X^2Y\sqrt{20XY^4} + 2X\sqrt{45X^3Y^6}$

When multiplying radicals, use the **product property of radicals** in reverse: $\sqrt[n]{a}\cdot\sqrt[n]{b} = \sqrt[n]{ab}$, where "a" and "b" are non-negative quantities, and "n" represents the **index** or the **order of the radical.**

47. $\sqrt{5}\cdot\sqrt{7}$ 48. $\sqrt{3}\cdot\sqrt{11}$ 49. $\sqrt{6}\cdot\sqrt{10}$ 50. $\sqrt{15}\cdot\sqrt{6}$

51. $\sqrt[3]{5} \cdot \sqrt[3]{7}$ 52. $\sqrt[3]{3} \cdot \sqrt[3]{11}$ 53. $\sqrt[3]{12} \cdot \sqrt[3]{6}$ 54. $\sqrt[3]{50} \cdot \sqrt[3]{5}$

In the next examples (see #55), notice that if you just multiply the numbers together (like 35 times 77), you would get a very large number (like 2695) that will be difficult to break down and simplify. So, instead of multiplying it out, **why not factor the numbers first?** In the process, for square root problems (as this one is), be looking for "**pairs**" of numbers; for cube root problems, be looking for "**triplets**" or "**three of a kind**;" fourth roots, look for "**four of a kind**;" etc. Remember, "if you ain't got no pair, then you ain't got no square!"

55. $\sqrt{35} \cdot \sqrt{77}$ 56. $\sqrt{55} \cdot \sqrt{33}$ 57. $\sqrt{46} \cdot \sqrt{69}$

$= \sqrt{7 \cdot 5 \cdot 7 \cdot 11}$

$= \sqrt{7^2} \cdot \sqrt{55}$

= ____________

58. $\sqrt{85} \cdot \sqrt{34}$ 59. $\sqrt{92} \cdot \sqrt{69}$ 60. $\sqrt{155} \cdot \sqrt{124}$

61. $\sqrt[3]{35} \cdot \sqrt[3]{50}$

$= \sqrt[3]{5 \cdot 7 \cdot 5^2 \cdot 2}$

$= \sqrt[3]{5^3} \cdot \sqrt[3]{7 \cdot 2}$

= ____________

62. $\sqrt[3]{98} \cdot \sqrt[3]{35}$

63. $\sqrt[3]{75} \cdot \sqrt[3]{15}$

64. $\sqrt[3]{105} \cdot \sqrt[3]{45}$

65. $\sqrt[3]{105} \cdot \sqrt[3]{50}$

66. $\sqrt[3]{242} \cdot \sqrt[3]{55}$

67. $4\sqrt{3} \cdot 6\sqrt{15}$

$= 24\sqrt{45}$

= ____________

= ____________

68. $2\sqrt{6} \cdot 9\sqrt{10}$

69. $6\sqrt{35} \cdot 5\sqrt{42}$

70. $8\sqrt[3]{65} \cdot 2\sqrt[3]{50}$

71. $15\sqrt[3]{98} \cdot 4\sqrt[3]{35}$

72. $4\sqrt[3]{21} \cdot 4\sqrt[3]{45}$

Frequently, problems involve the **distributive property**, and the familiar process known as "**F OI L**" is used to find the products of radicals.

73. $8\sqrt{10}\,(2\sqrt{6} - 3\sqrt{2})$

74. $2\sqrt{6}\,(4\sqrt{3} + 5\sqrt{2})$

75. $4\sqrt{10}\,(8\sqrt{15} + 9\sqrt{30})$

76. $3\sqrt{20}\,(5\sqrt{2} - 8\sqrt{15})$

77. $(4 + 5\sqrt{6})\,(8 + 2\sqrt{6})$

78. $(4 - 5\sqrt{6})\,(3 + 2\sqrt{6})$

79. $(5\sqrt{3} + 2\sqrt{6})\,(8\sqrt{3} - 5\sqrt{6})$

80. $(4\sqrt{5} - 5\sqrt{15})\,(3\sqrt{5} + 2\sqrt{15})$

81. $(6\sqrt{3} - 2\sqrt{15})^2$

82. $(4\sqrt{6} + 5\sqrt{2})^2$

83. $(6\sqrt{3} - 2\sqrt{15})(6\sqrt{3} + 2\sqrt{15})$

84. $(4\sqrt{15} - 5\sqrt{6})(4\sqrt{15} + 5\sqrt{6})$

85. $(6 + 2\sqrt[3]{15})^2$

86. $(4\sqrt[3]{6} - 5)^2$

87. $(4 - \sqrt[3]{2})(16 + 4\sqrt[3]{2} + \sqrt[3]{4})$

88. $(5 + \sqrt[3]{5})(25 - 5\sqrt[3]{5} + \sqrt[3]{25})$

89. $(3\sqrt{2} - 2\sqrt{3})^3$

90. $(4\sqrt{6} + 5\sqrt{3})^3$

In 91 - 96, simplify the radicals and reduce the fractions if possible.

91. $\frac{\sqrt{72} + \sqrt{27}}{12}$

92. $\frac{\sqrt{48} - \sqrt{80}}{20}$

93. $\dfrac{6\sqrt{300} - 5\sqrt{8}}{10}$

94. $\dfrac{6\sqrt{24} + 4\sqrt{50}}{12}$

95. $\dfrac{(6\sqrt{3} - 2\sqrt{15})^2}{24}$

96. $\dfrac{(3\sqrt{6} + 5\sqrt{2})^2}{24}$

RATIONALIZING DENOMINATORS

There is a tradition in mathematics of eliminating the radicals from the denominators (or numerators) of fractions. This process is called **rationalizing the denominator (or numerator) of the fraction.** This may be done to simplify the radical expression or to make calculation of the expression easier, especially in days when calculators were not available. For example, knowing the value of $\sqrt{2}$ to be approximately 1.414, to calculate $\dfrac{20}{\sqrt{2}}$ without a calculator would require **long division** of 20 divided by 1.414. It is much easier to multiply numerator and denominator by $\sqrt{2}$,

$$\frac{20}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{20\sqrt{2}}{2} = 10\sqrt{2}\ .$$

It is much easier to calculate **10(1.414)**, than to divide $\dfrac{20}{1.414}$.

When rationalizing a **monomial** square root denominator, multiply numerator and denominator by "something" that makes the denominator result in a perfect square. For monomial cube root denominators, multiply numerator and denominator by "something" that makes the denominator a perfect cube, etc.

In each of the following exercises, rationalize the denominators:

1. $\frac{6}{\sqrt{2}}$ 2. $\frac{20}{\sqrt{5}}$ 3. $\frac{20}{\sqrt{6}}$ 4. $\frac{6}{\sqrt{10}}$

Note: In the next exercises, it is usually a good idea to simplify the radical first, then rationalize the denominator.

5. $\frac{6}{\sqrt{18}}$ 6. $\frac{12}{\sqrt{20}}$ 7. $\frac{12}{\sqrt{45X}}$

8. $\frac{8}{\sqrt{80X}}$ 9. $\frac{15}{\sqrt{72X^3}}$ 10. $\frac{10}{\sqrt{75X^3}}$

Consider the problem $\frac{6}{\sqrt[3]{2}}$.

A common error is to multiply numerator and denominator by $\sqrt[3]{2}$.

$$\frac{6}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{2}}{\sqrt[3]{2}} = \frac{6\sqrt[3]{2}}{\sqrt[3]{4}}$$

This does <u>not</u> help, because it does <u>not</u> eliminate the radical from the denominator! The denominator should end up a **perfect cube** (like "8"!). To do this, you should multiply numerator and denominator by $\sqrt[3]{4}$ as follows:

11. $\frac{6}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{4}}{\sqrt[3]{4}} = \frac{6\sqrt[3]{4}}{\sqrt[3]{8}}$

$= \_\_\_\_\_\_$

$= \_\_\_\_\_\_$

12. $\frac{6}{\sqrt[3]{4}} \cdot \frac{\sqrt[3]{2}}{\sqrt[3]{2}} = \_\_\_\_\_\_$

$= \_\_\_\_\_\_$

$= \_\_\_\_\_\_$

13. $\frac{6}{\sqrt[3]{3}}$

14. $\frac{6}{\sqrt[3]{9}}$

15. $\frac{10}{\sqrt[3]{25}}$

16. $\frac{10}{\sqrt[3]{5}}$

17. $\frac{35}{\sqrt[3]{49}}$

18. $\frac{35}{\sqrt[3]{7}}$

Consider the problem 19. $\frac{6}{\sqrt[3]{9X^2Y}}$

In this denominator, you need to build the 9 up to 27, X^2 up to X^3, and Y up to Y^3. This means you need to multiply by $\sqrt[3]{3XY^2}$, since 3, X, and Y^2, since these are the "missing factors" needed to form the perfect cube $27X^3Y^3$. In #20, the perfect cube is $27X^3Y^6$.

19. $\frac{6}{\sqrt[3]{9X^2Y}} \cdot \frac{\sqrt[3]{3XY^2}}{\sqrt[3]{3XY^2}} = \_\_\_\_\_\_$

$=$

$=$

20. $\frac{12XY^2}{\sqrt[3]{3XY^5}} \cdot \frac{\sqrt[3]{9X^2Y}}{\sqrt[3]{9X^2Y}} = \_\_\_\_\_\_$

$=$

$=$

21. $\dfrac{12X^4Y^2}{\sqrt[3]{4X^5Y}}$

22. $\dfrac{12X^4Y^2}{\sqrt[3]{2XY^7}}$

[Note: Missing factors for #21: $2XY^2$; #22: $4X^2Y^2$; #23: $5XY^2$; #24: $25X^2Y$.]

23. $\dfrac{40XY^2}{\sqrt[3]{25X^2Y^4}}$

24. $\dfrac{40X^4Y^2}{\sqrt[3]{5X^7Y^8}}$

25. $\dfrac{12X^4Y^2}{\sqrt[5]{4X^2Y^3}}$

26. $\dfrac{12X^4Y^2}{\sqrt[5]{8X^8Y^2}}$

27. $\dfrac{12X^4Y^2}{\sqrt[5]{16X^4Y^6}}$

28. $\dfrac{12X^4Y^2}{\sqrt[5]{2X^8Y^{12}}}$

When the denominator of the fraction involves **binomial** radical expressions, such as $\frac{17}{6 - \sqrt{2}}$, a special procedure is used. Multiplying the numerator and denominator by $6 + \sqrt{2}$ will eliminate the radicals from the denominator. For the fraction $\frac{6}{\sqrt{6} + \sqrt{2}}$, multiply numerator and denominator by $\sqrt{6} - \sqrt{2}$. In general, whatever the binomial denominator may be, you multiply the numerator and denominator by the same quantity as the denominator but with the opposite sign in the middle. This is called the **conjugate** of the denominator.

In each of the following exercises, rationalize the denominators and reduce each fraction to lowest terms:

1. $\frac{17}{6 - \sqrt{2}}$

2. $\frac{6}{\sqrt{6} + \sqrt{2}}$

3. $\frac{15}{\sqrt{5} + 5\sqrt{2}}$

4. $\frac{20}{3\sqrt{6} + 2}$

5. $\dfrac{12}{4 + 2\sqrt{3}}$

6. $\dfrac{12}{6 - 3\sqrt{3}}$

7. $\dfrac{15}{2\sqrt{6} - 3\sqrt{2}}$

8. $\dfrac{6}{3\sqrt{2} + 4\sqrt{3}}$

9. $\dfrac{\sqrt{27}}{2\sqrt{6} - 3\sqrt{3}}$

10. $\dfrac{\sqrt{12}}{6\sqrt{2} + \sqrt{6}}$

11. $\dfrac{3 + \sqrt{6}}{3 - \sqrt{6}}$

12. $\dfrac{3 - \sqrt{6}}{3 + \sqrt{6}}$

13. $\frac{3 + \sqrt{3}}{6 + 2\sqrt{3}}$

14. $\frac{2\sqrt{2} - 1}{3 - 6\sqrt{2}}$

15. $\frac{4\sqrt{5} + 5\sqrt{2}}{3\sqrt{2} - 2\sqrt{5}}$

16. $\frac{3\sqrt{5} - 5\sqrt{6}}{5\sqrt{6} - 3\sqrt{5}}$

17. $\frac{4\sqrt{10} - 5\sqrt{6}}{3\sqrt{2} - 2\sqrt{5}}$

18. $\frac{3\sqrt{10} - 2\sqrt{6}}{4\sqrt{10} + 5\sqrt{6}}$

[In 19-22, leave numerators in factored form!]

19. $\dfrac{X - Y}{X\sqrt{Y} - Y\sqrt{X}}$

20. $\dfrac{X^2Y - Y^2X}{X\sqrt{Y} - Y\sqrt{X}}$

21. $\dfrac{X^2 - Y^2}{X\sqrt{X} + Y\sqrt{Y}}$

22. $\dfrac{X^2 - Y^2}{X\sqrt{Y} - Y\sqrt{X}}$

23. $\dfrac{X\sqrt{Y} + Y\sqrt{X}}{X\sqrt{Y} - Y\sqrt{X}}$

24. $\dfrac{X\sqrt{Y} - Y\sqrt{X}}{X\sqrt{Y} + Y\sqrt{X}}$

25. $\dfrac{h}{\sqrt{X + h} - \sqrt{X}}$

26. $\dfrac{h}{\sqrt{X + h} + \sqrt{X}}$

27. $\dfrac{h}{\sqrt{X} + \sqrt{X + h}}$

28. $\dfrac{h}{\sqrt{X} - \sqrt{X + h}}$

It is frequently necessary in higher mathematics to **rationalize the numerator of a fraction.** This is exactly the same process, except you multiply the numerator and denominator by the **conjugate of the numerator.**

In #29 - 36, rationalize the numerators.

29. $\dfrac{3 - \sqrt{3}}{3 - 6\sqrt{3}}$

30. $\dfrac{3 + \sqrt{3}}{6 - 2\sqrt{3}}$

31. $\dfrac{3\sqrt{10} - 2\sqrt{6}}{4\sqrt{10} + 5\sqrt{6}}$

32. $\dfrac{4\sqrt{5} + 5\sqrt{2}}{3\sqrt{2} - 2\sqrt{5}}$

33. $\dfrac{X\sqrt{X} + Y\sqrt{Y}}{X - Y}$

34. $\dfrac{X\sqrt{Y} - Y\sqrt{X}}{X\sqrt{Y} + Y\sqrt{X}}$

35. $\dfrac{\sqrt{X} - \sqrt{X + h}}{h}$

36. $\dfrac{\sqrt{X} + \sqrt{X + h}}{h}$

FRACTIONAL EXPONENTS

The evaluation of expressions with fractional exponents was introduced in **Section 1.02**. Remember that numerical expressions can be evaluated by use of a calculator, or by applying the definition of fractional exponents:

> ***Definition:***
>
> $$X^{\frac{1}{b}} = \sqrt[b]{X}$$
>
> $$X^{\frac{a}{b}} = (\sqrt[b]{X})^{a}$$
>
> $$X^{\frac{a}{b}} = \sqrt[b]{X^{a}}$$

According to this definition: $9^{\frac{1}{2}} = \sqrt{9} = 3$,

$$27^{\frac{1}{3}} = \sqrt[3]{27} = 3 \text{ ,}$$

$$9^{\frac{3}{2}} = (\sqrt{9})^{3} = 27 \text{ ,}$$

$$27^{\frac{2}{3}} = (\sqrt[3]{27})^{2} = 9 \text{ .}$$

While expressions such as these may be easier to compute with a calculator (and certainly require less thought), it is also necessary to be familiar with the definition and the applications to radical expressions. For many of these applications, it will be helpful to be familiar with the definition in reverse:

> ***Definition (reversed):***
>
> $$\sqrt[b]{X} = X^{\frac{1}{b}}$$
>
> $$(\sqrt[b]{X})^{a} = X^{\frac{a}{b}}$$
>
> $$\sqrt[b]{X^{a}} = X^{\frac{a}{b}}$$

As a review of fractional exponents, in each of the following, find the value of the expression first by the definition. Then confirm the answer using a calculator.

Definition: | Calculator:

1. $49^{\frac{1}{2}} = \sqrt{(\quad)}$ $\qquad$ $49^{\frac{1}{2}} =$

$=$

2. $16^{\frac{1}{4}} =$ $\qquad$ $16^{\frac{1}{4}} =$

$=$

3. $16^{\frac{3}{4}} = (\sqrt[4]{(\quad)})^3$ $\qquad$ $16^{\frac{3}{4}} =$

$=$

4. $125^{\frac{2}{3}} =$ $\qquad$ $125^{\frac{2}{3}} =$

$=$

5. $32^{-\frac{3}{5}} =$ $\qquad$ $32^{-\frac{3}{5}} =$

$=$

6. $32^{-\frac{4}{5}} =$ $\qquad$ $32^{-\frac{4}{5}} =$

$=$

Definition:	Calculator:
7. $-32^{\frac{2}{5}} =$	$-32^{\frac{2}{5}} =$
$=$	
8. $-32^{\frac{3}{5}} =$	$-32^{\frac{3}{5}} =$
$=$	
9. $(-32)^{\frac{2}{5}} =$	$(-32)^{\frac{2}{5}} =$ X
$=$	
10. $(-32)^{\frac{4}{5}} =$	$(-32)^{\frac{4}{5}} =$ X
$=$	
11. $(-32)^{\frac{3}{5}} =$	$(-32)^{\frac{3}{5}} =$ X
$=$	
12. $(-32)^{-\frac{3}{5}} =$	$(-32)^{-\frac{3}{5}} =$ X
$=$	

Many Calculators are not valid for negative base numbers raised to fractional powers.

13. $(-16)^{\frac{3}{4}} =$

$=$

$\left[(-16)^{\frac{3}{4}} = \times\right.$

14. $-16^{\frac{3}{4}} =$

$=$

$-16^{\frac{3}{4}} =$

It is frequently possible, when the radicand is a perfect power, to **reduce the order of the radical.** For example consider $\sqrt[6]{8}$.

$$\begin{aligned}\sqrt[6]{8} &= \sqrt[6]{2^3}\\ &= 2^{\frac{3}{6}}\\ &= 2^{\frac{1}{2}}\\ &= \sqrt{2}\end{aligned}$$

As this illustration demonstrates, it is sometimes possible to reduce a **sixth root (order 6)** to a **square root (order 2).** Recall the earlier problem $\sqrt[3]{X^{12}}$ which equals X^4 by **dividing exponents?** Compare this problem $\sqrt[3]{X^{12}}$ to the new problem $\sqrt[12]{X^3}$.

$$\begin{aligned}\sqrt[12]{X^3} &= X^{\frac{3}{12}}\\ &= X^{\frac{1}{4}}\\ &= \sqrt[4]{X}\end{aligned}$$

And in $\sqrt[12]{X^3}$, as before, you are **dividing exponents,** in reverse, and in so doing, you **reduce the order of the radical** from 12 to 4.

In each of the following exercises, reduce the order of the radical. Simplify each radical expression completely.

1. $\sqrt[12]{X^6}$ 2. $\sqrt[12]{X^4}$ 3. $\sqrt[12]{X^8}$ 4. $\sqrt[12]{X^9}$

5. $\sqrt[12]{X^{18}}$ 6. $\sqrt[12]{X^{20}}$ 7. $\sqrt[12]{8}$ 8. $\sqrt[12]{125}$

9. $\sqrt[12]{27}$ 10. $\sqrt[12]{81}$ 11. $\sqrt[12]{16}$ 12. $\sqrt[12]{25}$

13. $\sqrt[6]{125}$ 14. $\sqrt[6]{25}$ 15. $\sqrt[12]{64}$ 16. $\sqrt[9]{64}$

17. $\sqrt[4]{64}$ 18. $\sqrt[9]{8}$ 19. $\sqrt[10]{32}$ 20. $\sqrt[6]{81}$

NOTE: To reduce the order in each of the following radicands, you must find a "**common power**" as illustrated by #21.

21. $\sqrt[12]{125X^6Y^3} = \sqrt[12]{(5X^2Y)^3}$

$= \sqrt[4]{(\quad)}$

22. $\sqrt[12]{27X^3Y^3} =$

23. $\sqrt[12]{27X^3Y^6} =$

24. $\sqrt[12]{49X^4Y^2} =$

25. $\sqrt[12]{81X^4Y^8} =$

26. $\sqrt[12]{16X^8Y^4} =$

27. $\sqrt[12]{64X^{12}Y^6} =$

28. $\sqrt[12]{125X^6Y^{12}} =$

What is the meaning of a "root of a root"? For example, what is the meaning of $\sqrt[3]{\sqrt{2}}$, or $\sqrt[4]{\sqrt[3]{2}}$, or in general, $\sqrt[m]{\sqrt[n]{X}}$. The last expression:

$$\sqrt[m]{\sqrt[n]{X}} = \sqrt[m]{X^{\frac{1}{n}}}$$

$$= (X^{\frac{1}{n}})^{\frac{1}{m}}$$

$$= X^{\frac{1}{mn}}$$

$$= \sqrt[mn]{X}$$

In other words, **when you take the root of a root, you multiply the indices of the radicals.** As examples, $\sqrt[3]{\sqrt{2}} = \sqrt[6]{2}$ and $\sqrt[4]{\sqrt[3]{2}} = \sqrt[12]{2}$.

29. $\sqrt[3]{\sqrt[5]{2}}$ 30. $\sqrt[3]{\sqrt[3]{3}}$ 31. $\sqrt[5]{\sqrt{X}}$ 32. $\sqrt[4]{\sqrt[5]{Y}}$

33. $\sqrt[3]{\sqrt[5]{X^3}}$ 34. $\sqrt[6]{\sqrt[3]{X^2}}$ 35. $\sqrt[15]{\sqrt{Y^5}}$ 36. $\sqrt{\sqrt[9]{X^3}}$

37. $\sqrt{\sqrt[3]{Y^{12}}}$ 38. $\sqrt[6]{\sqrt[3]{Y^{12}}}$ 39. $\sqrt{\sqrt{XY}}$ 40. $\sqrt[4]{\sqrt{Y}}$

41. $\sqrt[3]{\sqrt[5]{27}}$

42. $\sqrt[5]{\sqrt[3]{32}}$

43. $\sqrt{\sqrt[5]{49}}$

44. $\sqrt[3]{\sqrt{125}}$

45. $\sqrt[3]{\sqrt[4]{\sqrt[5]{2}}}$

46. $\sqrt[3]{\sqrt[3]{\sqrt[3]{3}}}$

47. $\sqrt{\sqrt{\sqrt{2}}}$

48. $\sqrt[5]{\sqrt{\sqrt[3]{X}}}$

49. $\sqrt[3]{\sqrt[4]{\sqrt[5]{125}}}$

50. $\sqrt[3]{\sqrt{\sqrt[3]{25}}}$

51. $\sqrt[6]{\sqrt[5]{\sqrt{X^{20}}}}$

52. $\sqrt[5]{\sqrt{\sqrt[3]{X^{10}}}}$

Is it possible to multiply a square root times a cube root? Is it possible to compute $\sqrt{2} \cdot \sqrt[3]{3}$? In general, is it possible to multiply roots that do not have the same index? It might seem that the answer is "No!" However, if both radical expressions can be converted to radicals with a **"common order,"** then these can be multiplied. Consider the examples:

$$\sqrt{2} \cdot \sqrt[3]{3} = 2^{\frac{1}{2}} \cdot 3^{\frac{1}{3}} = 2^{\frac{3}{6}} \cdot 3^{\frac{2}{6}} = \sqrt[6]{2^3} \cdot \sqrt[6]{3^2} = \sqrt[6]{8} \cdot \sqrt[6]{9} = \sqrt[6]{72}$$

$$\sqrt[4]{3} \cdot \sqrt[3]{2} = 3^{\frac{1}{4}} \cdot 2^{\frac{1}{3}} = 3^{\frac{3}{12}} \cdot 2^{\frac{4}{12}} = \sqrt[12]{3^3} \cdot \sqrt[12]{2^4} = \sqrt[12]{27} \cdot \sqrt[12]{16} = \sqrt[12]{432}$$

In each of the following, multiply the radical expressions (with different orders) by first finding a common order.

53.

$$\sqrt{3} \cdot \sqrt[3]{2} = (\quad)^{\frac{1}{2}} \cdot (\quad)^{\frac{1}{3}} = (\quad)^{\frac{3}{6}} \cdot (\quad)^{\frac{2}{6}} = \sqrt[6]{(\qquad)} \cdot \sqrt[6]{(\qquad)} = \sqrt[6]{(\quad)} \cdot \sqrt[6]{(\quad)} = \sqrt[6]{(\qquad)}$$

54.

$$\sqrt[3]{3} \cdot \sqrt[4]{2} = (\quad)^{\frac{1}{3}} \cdot (\quad)^{\frac{1}{4}} = (\quad)^{\frac{(\ )}{12}} \cdot (\quad)^{\frac{(\ )}{12}} = \sqrt[12]{(\qquad)} \cdot \sqrt[12]{(\qquad)} = \sqrt[12]{(\quad)} \cdot \sqrt[12]{(\quad)} = \sqrt[12]{(\qquad)}$$

55. $\sqrt[3]{2} \cdot \sqrt[4]{3} =$

56. $\sqrt{5} \cdot \sqrt[3]{2} =$

[Hint: In #57, the common order is "4"; in #58 it is "6".]

57. $\sqrt{2} \cdot \sqrt[4]{3}$ =

58. $\sqrt[6]{5} \cdot \sqrt[3]{2}$ =

59. $\sqrt[6]{2} \cdot \sqrt{3}$ =

60. $\sqrt[4]{5} \cdot \sqrt[8]{2}$ =

61. $\sqrt[3]{X} \cdot \sqrt[4]{Y}$ =

62. $\sqrt[6]{X} \cdot \sqrt{Y}$ =

63. $\sqrt[6]{X} \cdot \sqrt[3]{Y}$ =

64. $\sqrt[6]{X} \cdot \sqrt[4]{Y}$ =

If the base numbers are the same, then the problem can be simplified by "adding the exponents."

65. $\sqrt[3]{X} \cdot \sqrt[4]{X} = X^{\frac{1}{3}} \cdot X^{\frac{1}{4}}$
$= X^{(\frac{1}{3} + \frac{1}{4})}$
$= X^{(\frac{(\ )}{12} + \frac{(\ )}{12})}$
$= X^{\frac{(\ )}{12}}$
$= \sqrt[12]{(\qquad)}$

66. $\sqrt[4]{X} \cdot \sqrt[5]{X} =$

67. $\sqrt{X} \cdot \sqrt[6]{X} =$

68. $\sqrt{X} \cdot \sqrt[12]{X} =$

69. $\sqrt[3]{X} \cdot \sqrt[6]{X} =$

70. $\sqrt[5]{X} \cdot \sqrt{X} =$

71. $\sqrt[6]{X^5} \cdot \sqrt[3]{X^2} =$

72. $\sqrt[3]{X^2} \cdot \sqrt[4]{X^3} =$

1.06

p. 67–76: 1. $5x^3$ 2. $7x^6$ 3. $4x^8$ 4. $5x^{50}$

5. $5x^2$ 6. $2x^4$ 7. $3x^9$ 8. $4x^{17}$

9. $2x^4$ 10. $3x^3$ 11. $2x^4$ 12. $2x^{12}$

13. $5x^3\sqrt{5}$ 14. $4x^6\sqrt{3x}$ 15. $6x^4\sqrt{2x}$

16. $5x^3\sqrt{2x}$ 17. $5x^4y^4\sqrt{3y}$ 18. $2x^5y^3\sqrt{10x}$

19. $7x^3y^6\sqrt{2xy}$ 20. $10x^7y^{12}\sqrt{3xy}$ 21. $3x^2y^3\sqrt[3]{2y}$

22. $2x^2y^4\sqrt[3]{2x}$ 23. $2xy^2\sqrt[3]{9x^2y^2}$ 24. $2xy^4\sqrt[3]{10xy^2}$

25. $2x^2y\sqrt[4]{2y^2}$ 26. $2xy^4\sqrt[4]{3x}$ 27. $3x^2y^2\sqrt[4]{2xy^2}$

28. $3xy^3\sqrt[4]{5x^2y^2}$ 29. $2x^3y\sqrt[5]{3x^2y^4}$ 30. $2x^5y^2\sqrt[5]{2y^3}$

31. $24-12\sqrt{2}$ 32. $60-40\sqrt{2}$ 33. $18\sqrt{2}$ 34. $31\sqrt{3}$

35. $5\sqrt[3]{2}$ 36. $-9\sqrt[3]{3}$ 37. $26\sqrt[3]{5}$ 38. $7\sqrt[3]{4}$

39. $5\sqrt[4]{2}$ 40. $13\sqrt[3]{5}$ 41. $38x^2y^3\sqrt{6x}$

42. $-2x^4y^3\sqrt{5xy}$ 43. $7x^4y^2\sqrt[3]{2xy^2}$ 44. $38xy\sqrt[3]{2x}$

45. 0 46. $12x^2y^3\sqrt{5x}$ 47. $\sqrt{35}$ 48. $\sqrt{33}$

49. $2\sqrt{15}$ 50. $3\sqrt{10}$ 51. $\sqrt[3]{35}$ 52. $\sqrt[3]{33}$

53. $2\sqrt[3]{9}$ 54. $5\sqrt[3]{2}$ 55. $7\sqrt{55}$ 56. $11\sqrt{15}$

57. $23\sqrt{6}$ 58. $17\sqrt{10}$ 59. $46\sqrt{3}$ 60. $62\sqrt{5}$

1.06

p. 67–76:

61. $5\sqrt[3]{14}$ 62. $7\sqrt[3]{10}$ 63. $5\sqrt[3]{9}$

64. $3\sqrt[3]{175}$ 65. $5\sqrt[3]{42}$ 66. $11\sqrt[3]{10}$

67. $72\sqrt{5}$ 68. $36\sqrt{15}$ 69. $210\sqrt{30}$

70. $80\sqrt[3]{26}$ 71. $420\sqrt[3]{10}$ 72. $48\sqrt[3]{35}$

73. $32\sqrt{15}-48\sqrt{5}$ 74. $24\sqrt{2}+20\sqrt{3}$

75. $160\sqrt{6}+360\sqrt{3}$ 76. $30\sqrt{10}-240\sqrt{3}$

77. $92+48\sqrt{6}$ 78. $-48-7\sqrt{6}$ 79. $60-27\sqrt{2}$

80. $-90-35\sqrt{3}$ 81. $168-72\sqrt{5}$ 82. $146+80\sqrt{3}$

83. 48 84. 90 85. $36+24\sqrt[3]{15}+4\sqrt[3]{225}$

86. $16\sqrt[3]{36}-40\sqrt[3]{6}+25$ 87. 62 88. 130

89. $162\sqrt{2}-132\sqrt{3}$ 90. $1284\sqrt{6}+1815\sqrt{3}$

91. $\frac{2\sqrt{2}+\sqrt{3}}{4}$ 92. $\frac{\sqrt{3}-\sqrt{6}}{5}$ 93. $6\sqrt{3}-\sqrt{2}$

94. $\frac{3\sqrt{6}+5\sqrt{2}}{3}$ 95. $7-3\sqrt{5}$ 96. $\frac{26+15\sqrt{3}}{6}$

p. 77–79:

1. $3\sqrt{2}$ 2. $4\sqrt{5}$ 3. $\frac{10\sqrt{6}}{3}$ 4. $\frac{3\sqrt{10}}{5}$

5. $\sqrt{2}$ 6. $\frac{6\sqrt{5}}{5}$ 7. $\frac{4\sqrt{5x}}{5x}$ 8. $\frac{2\sqrt{5x}}{5x}$

9. $\frac{5\sqrt{2x}}{4x^2}$ 10. $\frac{2\sqrt{3x}}{3x^2}$ 11. $3\sqrt[3]{4}$ 12. $3\sqrt[3]{2}$

13. $2\sqrt[3]{9}$ 14. $2\sqrt[3]{3}$ 15. $2\sqrt[3]{5}$ 16. $2\sqrt[3]{25}$

17. $5\sqrt[3]{7}$ 18. $5\sqrt[3]{49}$ 19. $\frac{2\sqrt[3]{3xy^2}}{xy}$ 20. $4\sqrt[3]{9x^2y}$

p 77–79:

21. $6x^2y\sqrt[3]{2xy^2}$ 22. $\frac{6x^3\sqrt[3]{4x^2y^2}}{y}$

23. $8\sqrt[3]{5xy^2}$ 24. $\frac{8x\sqrt[3]{25x^2y}}{y}$

25. $6x^3y\sqrt[5]{8x^3y^2}$ 26. $6x^3y\sqrt[3]{4x^2y^3}$

27. $6x^3\sqrt[5]{2xy^4}$ 28. $\frac{6x^2\sqrt[4]{4x^2y^3}}{y}$

p. 80–85:

1. $\frac{6+\sqrt{2}}{2}$ 2. $\frac{3(\sqrt{6}-\sqrt{2})}{2}$ 3. $\frac{5\sqrt{2}-\sqrt{5}}{3}$ 4. $\frac{2(3\sqrt{6}-2)}{5}$

5. $6(2-\sqrt{3})$ 6. $4(2+\sqrt{3})$ 7. $\frac{5(2\sqrt{6}+3\sqrt{2})}{2}$ 8. $\frac{4\sqrt{3}-3\sqrt{2}}{[illegible]}$

9. $-3(2\sqrt{2}+3)$ 10. $\frac{2\sqrt{6}-\sqrt{2}}{11}$ 11. $5+2\sqrt{6}$ 12. $5-2\sqrt{6}$

13. $\frac{1}{2}$ 14. $-\frac{1}{3}$ 15. $-11\sqrt{10}-35$ 16. -1

17. $-12\sqrt{5}+15\sqrt{3}-20\sqrt{2}+5\sqrt{30}$ 18. $\frac{90-23\sqrt{15}}{5}$

19. $\frac{x\sqrt{y}+y\sqrt{x}}{xy}$ 20. $(x\sqrt{y}+y\sqrt{x})$ 21. $\frac{(x\sqrt{x}-y\sqrt{y})(x+y)}{x^2+xy+y^2}$

22. $\frac{(x+y)(x\sqrt{y}+y\sqrt{x})}{xy}$ 23. $\frac{x+2\sqrt{xy}+y}{x-y}$ 24. $\frac{x-2\sqrt{xy}+y}{x-y}$

25. $\sqrt{x+h}+\sqrt{x}$ 26. $\sqrt{x+h}-\sqrt{x}$ 27. $\sqrt{x+h}-\sqrt{x}$

28. $-\sqrt{x+h}-\sqrt{x}$ 29. $\frac{-2}{3+5\sqrt{3}}$ 30. $\frac{1}{2(2-\sqrt{3})}$

31. $\frac{33}{90+23\sqrt{15}}$ 32. $\frac{15}{11\sqrt{10}-35}$ 33. $\frac{x^2+xy+y^2}{x\sqrt{x}-y\sqrt{y}}$

34. $\frac{x-y}{x+2\sqrt{xy}+y}$ 35. $\frac{-1}{\sqrt{x}+\sqrt{x+h}}$ 36. $\frac{1}{\sqrt{x+h}-\sqrt{x}}$

p. 86–89:

1. 7 2. 2 3. 8 4. 25 5. $\frac{1}{8}$ or 0.125

6. $\frac{1}{16}$ or 0.0625 7. −4 8. −8 9. 4 10. 16

11. −8 12. $-\frac{1}{8}$ or −0.125 13. No real solution 14. −8

p. 90–96:

1. $\sqrt{x}$ 2. $\sqrt[3]{x}$ 3. $\sqrt[3]{x^2}$ 4. $\sqrt[4]{x^3}$ 5. $x\sqrt{x}$ 6. $x\sqrt[3]{x^2}$

7. $\sqrt[4]{2}$ 8. $\sqrt[4]{5}$ 9. $\sqrt[4]{3}$ 10. $\sqrt[3]{3}$ 11. $\sqrt[3]{2}$ 12. $\sqrt[6]{5}$

13. $\sqrt{5}$ 14. $\sqrt[3]{5}$ 15. $\sqrt{2}$ 16. $\sqrt[3]{4}$ 17. $2\sqrt{2}$ 18. $\sqrt[3]{2}$

19. $\sqrt{2}$ 20. $\sqrt[3]{9}$ 21. $\sqrt[4]{5x^2y}$ 22. $\sqrt[4]{3xy}$ 23. $\sqrt[4]{3xy^2}$

24. $\sqrt[6]{7x^2y}$ 25. $\sqrt[3]{3xy^2}$ 26. $\sqrt[3]{2x^2y}$ 27. $x\sqrt{2y}$ 28. $y\sqrt[6]{5x^2}$

29. $\sqrt[6]{2}$ 30. $\sqrt[9]{3}$ 31. $\sqrt[10]{x}$ 32. $\sqrt[10]{y}$ 33. $\sqrt[5]{x}$ 34. $\sqrt[9]{x}$

35. $\sqrt[6]{y}$ 36. $\sqrt[8]{x}$ 37. y^2 38. $\sqrt[3]{y^2}$ 39. $\sqrt[4]{xy}$ 40. $\sqrt[8]{y}$

41. $\sqrt[5]{3}$ 42. $\sqrt[3]{2}$ 43. $\sqrt[5]{7}$ 44. $\sqrt{5}$ 45. $\sqrt[16]{2}$ 46. $\sqrt[27]{3}$

47. $\sqrt[8]{2}$ 48. $\sqrt[24]{x}$ 49. $\sqrt[24]{5}$ 50. $\sqrt[9]{5}$ 51. $\sqrt[3]{x}$ 52. $\sqrt[3]{x}$

53. $\sqrt[6]{108}$ 54. $\sqrt[12]{648}$ 55. $\sqrt[12]{432}$ 56. $\sqrt[6]{500}$ 57. $\sqrt[4]{12}$

58. $\sqrt[6]{20}$ 59. $\sqrt[6]{54}$ 60. $\sqrt[8]{50}$ 61. $\sqrt[12]{x^4y^3}$ 62. $\sqrt[6]{xy^3}$

63. $\sqrt[6]{xy^2}$ 64. $\sqrt[12]{x^2y^3}$ 65. $\sqrt[4]{x^7}$ 66. $\sqrt[4]{x^9}$ 67. $\sqrt[9]{x^2}$

68. $\sqrt[12]{x^7}$ 69. $\sqrt{x}$ 70. $\sqrt[9]{x^7}$ 71. $x\sqrt{x}$ 72. $x\sqrt[15]{x^5}$