

1.04 *Factoring, Factoring, Factoring!*

Of all the topics that you have studied in previous algebra courses, there is no topic more important, there is no topic that you need to review more than the topic of **factoring**. Let's begin with a working definition of factoring. What does it mean to **factor** something? What would you say if you were asked to "**factor the number 15**"? Without hesitation, you would probably answer "**3 times 5**" or "**5 times 3**"! The key word is "**times**." When asked to factor a given number, you naturally answer with a product of two numbers.

DEFINITION

To **FACTOR** means to **EXPRESS AS A PRODUCT!**

Factoring is an important skill that goes all the way back to your first algebra course, and it will continue to be a most important skill, especially in calculus. There are many different types of factoring that become more complex as well as more abstract in the higher math. Some of the exercises in this section will begin with a review of elementary factoring exercises, such as $X^2 - 9$ or $X^2 - 3X$, and then progress into "college algebra" factoring. At the end, there is a preview of factoring as it occurs in calculus. Notice the increase in complexity and abstraction as you "grow" through this "one step at a time."

GUIDELINES TO FACTORING

1. **Common Factor** (Factor Common Factor First!)
2. **Trinomial** (F O I L rearranged to spell F L O I)
3. **Difference of Squares:** $X^2 - Y^2 = (X - Y)(X + Y)$
Diff of Cubes: $X^3 - Y^3 = (X - Y)(X^2 + XY + Y^2)$
Sum of Cubes: $X^3 + Y^3 = (X + Y)(X^2 - XY + Y^2)$
4. **Factoring by Grouping**
5. **Polynomial Factoring** (To be covered in Chapter 3!)
6. **Completing the Square** (Special cases--Optional, if at all!)

In each of the following, factor completely:

1. $x^4 - 81y^4$	2. $x^3 - 8y^3$	3. $x^4 - 17x^2 + 16$
$= (x^2 - 9y^2)(x^2 + 9y^2)$	$= (x - 2y)(x^2 + 2xy + 4y^2)$	$= (x^2 - 1)(x^2 - 16)$
$= (x - 3y)(x + 3y)(x^2 + 9y^2)$		$= (x - 1)(x + 1)(x - 4)(x + 4)$

4. $x^4 + 27xy^3$	5. $x^4 - 64x^2$	6. $x^3 - 5x^2 - 6x$
$= x(x^3 + 27y^3)$	$= x^2(x^2 - 64)$	
$= x(x + 3y)(x^2 - 3xy + 9y^2)$	$= x^2(x - 8)(x + 8)$	

7. $x^3 + 125y^6$	8. $x^3 - 4x^2 + 9x - 36$	9. $x^3 + 4x^2 - 9x - 36$
$= (x + 5y^2)(x^2 - 5xy^2 + 25y^4)$	$= x^2(x - 4) + 9(x - 4)$	
	$= (x - 4)(x^2 + 9)$	

$$10. x^3 - 4x^2 - 9x + 36 \quad 11. x^3 - 9x^2 - 4x + 36 \quad 12. x^4 - 8x^2 - 9$$

$$13. x^4 - 13x^2 + 36$$

$$14. x^4 - 16y^4$$

$$15. x^4 - 16x^2$$

$$16. x^4 - 64x$$

$$17. x^4 + 64x$$

$$18. x^4 + 5x^2 - 36$$

$$19. x^5 - 5x^3 - 36x$$

$$20. 2x^4 - 54x$$

$$21. 3x^{10} + 81xy^6 \\ = 3x \left(\quad \right) \\ = 3x \left(\quad \right) \left(\quad \right)$$

$$22. 5x^5 - 40x^2y^9$$

$$23. x^5 + 9x^4 - x - 9$$

$$24. x^5 + 5x^4 - 81x - 405$$

In the following exercises, notice the patterns that develop.

25a) $x^2 - 9$

26a) $x^2 - 64$

b) $\pi^2 - 9$

b) $\pi^2 - 64$

c) $(\text{Junk})^2 - 9$

c) $(\text{Junk})^2 - 64$

d) $(2X+3Y)^2 - 9$

d) $(2X+3Y)^2 - 64$

27a) $x^2 - 5x + 6$

28a) $x^2 - 5x - 6$

b) $\pi^2 - 5\pi + 6$

b) $\pi^2 - 5\pi - 6$

c) $(\text{Junk})^2 - 5(\text{Junk}) + 6$

c) $(\text{Junk})^2 - 5(\text{Junk}) - 6$

d) $(2X+3Y)^2 - 5(2X+3Y) + 6$

d) $(2X+3Y)^2 - 5(2X+3Y) - 6$

29. $(2X-3Y)^2 - 25$

30. $(2X-3Y)^2 - 169$

$$31. (2X-3Y)^2 - 4(2X-3Y) - 60$$

$$32. (2X-3Y)^2 - 14(2X-3Y) + 40$$

$$33a) X^2 - 10X + 25$$

$$34a) X^2 + 16X + 64$$

$$b) (\text{Junk})^2 - 10(\text{Junk}) + 25$$

$$b) (\text{Junk})^2 + 16(\text{Junk}) + 64$$

$$c) (2X+3Y)^2 - 10(2X+3Y) + 25$$

$$c) (2X+3Y)^2 + 16(2X+3Y) + 64$$

$$35a) X^2 - 5X$$

$$36a) X^2 + 10X$$

$$b) (\text{Junk})^2 - 5(\text{Junk})$$

$$b) (\text{Junk})^2 + 10(\text{Junk})$$

$$c) (2X+3Y)^2 - 5(2X+3Y)$$

$$c) (2X+3Y)^2 + 10(2X+3Y)$$

$$37. (2X-3Y)^2 - 14(2X-3Y) + 49$$

$$38. (2X-3Y)^2 + 10(2X-3Y) + 25$$

$$39. (2X-3Y)^2 + 20(2X-3Y) + 100$$

$$40. (2X-3Y)^2 - 16(2X-3Y) + 64$$

$$41. (2X+3Y)^2 - 5(2X+3Y)$$

$$42. (2X+3Y)^2 + 5(2X+3Y)$$

$$43. (2X+3Y)^2 + (2X+3Y)$$

$$44. (2X+3Y)^2 - (2X+3Y)$$

$$45. (2X+3Y)^4 + 5(2X+3Y)^3$$

$$46. (2X+3Y)^4 - 5(2X+3Y)^3$$

$$47. (X^2-2X)^2 - 7(X^2-2X) - 8$$

$$48. (X^2-5X)^2 + 10(X^2-5X) + 24$$

$$= [(X^2-2X) -] [(X^2-2X) +]$$

$$= (X^2 - 2X -) (X^2 - 2X +)$$

$$= () () () ()$$

$$\text{or } () () ()^2$$

$$49. (X^2-7X)^2 + 16(X^2-7X) + 60$$

$$50. (X^2-5X)^2 - 2(X^2-5X) - 24$$

$$51. (x^2 - 5x)^2 - 36$$

$$52. (x^2 + 5x)^2 - 36$$

$$53. (2x+3y)^4 - 4(2x+3y)^2$$

$$54. (2x+3y)^4 - 9(2x+3y)^2$$

$$\begin{aligned}
 &= (2x+3y)^2 [(\quad)^2 - \quad] \\
 &= (\quad)^2 [(\quad) - \quad] [(\quad) + \quad] \\
 &= (\quad)^2 (\quad) (\quad)
 \end{aligned}$$

$$55. (2x+3y)^5 - 9(2x+3y)^3$$

$$56. (2x+3y)^5 - 16(2x+3y)^3$$

$$57. (2x+3y)^4 - 16$$

$$58. (2x+3y)^4 - 1$$

There are many different variations of factoring by grouping. If there are four terms, sometimes it is "appropriate" to group the first two terms and the last two terms together. Sometimes three of the terms "look good together," and it is appropriate to group three terms together, leaving the other term alone. Sometimes there are more than four terms, and different groupings are "appropriate." What makes a grouping "appropriate" is that the result is factorable. In other words, it works!

$$59. \quad \underbrace{x^2 + 2XY + Y^2} - 49$$

$$= (x+y)^2 - 49$$

$$= [(\quad)] [(\quad)]$$

$$= (\quad \quad \quad)$$

$$60. \quad x^2 + 4XY + 4Y^2 - 25$$

$$61. \quad x^2 - 6XY + 9Y^2 - 36$$

$$62. \quad x^2 - 8XY + 16Y^2 - 81$$

$$63. \quad 25 - \underbrace{x^2 + 2XY - Y^2}$$

$$= 25 - (x^2 - 2xy + y^2)$$

$$= 25 - (\quad)^2$$

$$= [-(\quad)] [+ (\quad)]$$

$$= (\quad \quad \quad) (\quad \quad \quad)$$

$$64. \quad 64 - \underbrace{x^2 + 4XY - 4Y^2}$$

$$65. \quad 16 - \underbrace{x^2 - 10XY - 25Y^2}$$

$$66. \quad 9 - x^2 - 4XY - 4Y^2$$

$$67. \quad x^2 - y^2 + 4y - 4$$

$$68. \quad x^2 - 4y^2 - 4y - 1$$

$$\begin{aligned}
 69. \quad & \underbrace{x^2 + 2xy + y^2} + \underbrace{7x + 7y} + 10 & 70. \quad & \underbrace{x^2 - 2xy + y^2} + \underbrace{5x - 5y} + 6 \\
 & = (x+y)^2 + 7(x+y) + 10 \\
 & = [\quad] [\quad]
 \end{aligned}$$

$$71. \quad \underbrace{x^2 - 4xy + 4y^2} + \underbrace{7x - 14y} + 6 \quad 72. \quad x^2 - 4xy + 4y^2 + 3x - 6y + 2$$

$$73. \quad x^2 - 6xy + 9y^2 - 5x + 15y + 6 \quad 74. \quad x^2 - 6xy + 9y^2 - 8x + 24y + 15$$

$$75. \quad x^2 + 2xy + y^2 + 9x + 9y - 10 \quad 76. \quad x^2 + 2xy + y^2 + 3x + 3y - 10$$

$$77. 4X^2 - 12XY + 9Y^2 - 12X + 18Y + 9$$

$$78. 4X^2 - 12XY + 9Y^2 - 9$$

$$\begin{aligned} 79. & \underbrace{X^7 + 8X^4} - \underbrace{X^3 - 8} \\ &= X^4 (\quad) - 1 (\quad) \\ &= \\ &= \end{aligned}$$

$$80. X^7 - 8X^4 - X^3 + 8$$

$$\begin{aligned} 81. & X^6 - 9X^3 + 8 \\ &= (X^3 \quad)(X^3 \quad) \\ &= \end{aligned}$$

$$82. X^6 + 7X^3 - 8$$

$$83. X^6 - 7X^3 - 8$$

$$84. X^6 + 9X^3 + 8$$

$$85. X^6 - 16X^3 + 64$$

$$86. X^6 + 16X^3 + 64$$

$$87. X^6 - Y^6$$

$$= (\quad)(\quad)$$

$$=$$

$$88. X^6 - 8X^3 - X^3Y^3 + 8Y^3$$

$$89. X^6 - 64$$

$$90. X^6 - 26X^3 - 27$$

It is frequently necessary in higher mathematics (THE Calculus!) to factor expressions that involve fractional and negative exponents. Before attempting this, it will be helpful to clarify procedures of factoring that are often taken for granted. Consider the following examples:

EXAMPLE: Factor $X^{10} + 5X^7$

SOLUTION: The common factor is X^7 (the lowest power of X)

$$X^{10} + 5X^7 = X^7 (X^3 + 5)$$

Notice that you got the X^3 by subtracting the exponents $10 - 7$.

EXAMPLE: Factor $X^3 + 5X^{-2}$

SOLUTION: The common factor is X^{-2} (the lowest power of X)

$$X^3 + 5X^{-2} = X^{-2} (X^5 + 5)$$

Notice that you got the X^5 by subtracting the exponents $3 - (-2)$.

$$= \frac{X^5 + 5}{X^2}$$

In each of the following, factor the common factor:

$$91. \quad x^{\frac{3}{2}} + 8x^{\frac{1}{2}} \quad 92. \quad x^{\frac{5}{2}} + 8x^{\frac{1}{2}} \quad 93. \quad x^{\frac{5}{2}} - 16x^{\frac{1}{2}}$$

$$= x^{\frac{1}{2}}(x^{\frac{3}{2}-\frac{1}{2}} + 8)$$

$$= x^{\frac{1}{2}}(\quad)$$

$$94. \quad (x-y)^{\frac{3}{2}} + 8(x-y)^{\frac{1}{2}} \quad 95. \quad (x-y)^{\frac{5}{2}} - 16(x-y)^{\frac{1}{2}}$$

$$= (x-y)^{\frac{1}{2}}[\quad]$$

$$96. \quad (x-y)^{\frac{5}{2}} + 16(x-y)^{\frac{1}{2}} \quad 97. \quad x^3 - 16x^{-1} \quad 98. \quad x^2 - 27x^{-1}$$

$$= x^{-1}[x^{3-(-1)} - 16]$$

$$= x^{-1}[\quad]$$

$$=$$

$$=$$

$$=$$

$$99. \quad x - 64x^{-2} \quad 100. \quad x - 81x^{-3} \quad 101. \quad x^{\frac{4}{3}} + 17x^{\frac{1}{3}}$$

$$= x^{\frac{1}{3}}(\quad)$$

102. $X^{\frac{7}{3}} - 16 X^{\frac{1}{3}}$

103. $X^{\frac{11}{4}} - 64 X^{\frac{3}{4}}$

104. $X^{\frac{15}{2}} - 27 X^{\frac{9}{2}}$

105. $X^{\frac{1}{2}} + 8 X^{-\frac{1}{2}}$

106. $X^{\frac{3}{2}} - 64 X^{-\frac{1}{2}}$

107. $X^{\frac{7}{3}} - 8 X^{-\frac{2}{3}}$

108. $X^{\frac{5}{2}} + 8 X^{-\frac{1}{2}}$

109. $X^{\frac{4}{3}} - 64 X^{-\frac{2}{3}}$

110. $X^{\frac{10}{3}} - 81 X^{-\frac{2}{3}}$

111. $(2X + 1)^{\frac{3}{2}} + 3X(2X + 1)^{\frac{1}{2}}$

112. $(6X + 5)^{\frac{3}{2}} + 9X(6X + 5)^{\frac{1}{2}}$

$$113. (3X + 5)^{\frac{4}{3}} + 4X(3X + 5)^{\frac{1}{3}}$$

$$114. (X^2 + 4)^{\frac{3}{2}} + 3X^2(X^2 + 4)^{\frac{1}{2}}$$

$$115. (X^2 + 4)^{\frac{1}{2}} + X^2(X^2 + 4)^{-\frac{1}{2}}$$

$$116. (3X + 5)^{\frac{1}{3}} + X(3X + 5)^{-\frac{2}{3}}$$

$$117. (4 - X^3)^{\frac{1}{3}} - X^3(4 - X^3)^{-\frac{2}{3}}$$

$$118. (4 - X^2)^{\frac{1}{2}} - X^2(4 - X^2)^{-\frac{1}{2}}$$

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1. $(x-5y)(x+3y)(x^2+9y^2)$ 2. $(x-2y)(x^2+2xy+4y^2)$
 3. $(x-4)(x+4)(x-1)(x+1)$ 4. $x(x+3y)(x^2-3xy+9y^2)$
 5. $x^2(x-8)(x+8)$ 6. $x(x-6)(x+1)$ 7. $(x+5y^2)(x^2-5xy^2+25y^4)$
 8. $(x-4)(x^2+9)$ 9. $(x+4)(x-3)(x+3)$ 10. $(x-4)(x-3)(x+3)$
 11. $(x-9)(x-2)(x+2)$ 12. $(x-3)(x+3)(x^2+1)$
 13. $(x-3)(x+3)(x-2)(x+2)$ 14. $(x-2y)(x+2y)(x^2+4y^2)$
 15. $x^2(x-4)(x+4)$ 16. $x(x-4)(x^2+4x+16)$ 17. $x(x+4)(x^2-4x+16)$
 18. $(x^2+9)(x-2)(x+2)$ 19. $x(x-3)(x+3)(x^2+4)$ 20. $2x(x-3)(x^2+3x+9)$
 21. $3x(x^3+3y^2)(x^6-3x^3y^2+9y^4)$ 22. $5x^2(x-2y^3)(x^2+2xy^3+4y^6)$
 23. $(k+4)(x-1)(x+1)(x^2+1)$ 24. $(x+5)(x-3)(x+3)(x^2+9)$

- p. 32-44:
- 25a) $(x-3)(x+3)$ d) $(\pi-3)(\pi+3)$ c) $(a-3)(a+3)$
 25d) $(2x+3y-3)(2x+3y+3)$ 26a) $(x-8)(x+8)$ d) $(\pi-8)(\pi+8)$
 26c) $(a-8)(a+8)$ d) $(2x+3y-8)(2x+3y+8)$
 27a) $k-3)(x-2)$ d) $(\pi-3)(\pi-2)$ c) $(a-3)(a-2)$
 27d) $(2x+3y-3)(2x+3y-2)$ 28a) $(x-6)(x+1)$ d) $(\pi-6)(\pi+1)$
 28c) $(a-6)(a+1)$ d) $(2x+3y-6)(2x+3y+1)$
 29. $(2x-3y-5)(2x-3y+5)$ 30. $(2x-3y-13)(2x-3y+13)$
 31. $(2x-3y-10)(2x-3y+6)$ 32. $(2x-3y-10)(2x-3y-4)$
 33a) $(x-5)^2$ d) $(a-5)^2$ c) $(2x+3y-5)^2$
 34a) $(x+8)^2$ d) $(a+8)^2$ c) $(2x+3y+8)^2$
 35a) $x(x-5)$ d) $(a)(a-5)$ c) $(2x+3y)(2x+3y-5)$
 36a) $x(x+10)$ d) $(a)(a+10)$ c) $(2x+3y)(2x+3y+10)$
 37. $(2x-3y-7)^2$ 38. $(2x-3y+5)^2$ 39. $(2x-3y+10)^2$
 40. $(2x-3y-2)^2$ 41. $(2x+3y)(2x+3y-6)$ 42. $(2x+3y)(2x+3y+5)$
 43. $(2x+3y)(2x+3y+1)$ 44. $(2x+3y)(2x+3y-1)$
 45. $(2x+3y)^2(2x+3y+5)$ 46. $(2x+3y)^3(2x+3y-5)$
 47. $(x-4)(x+2)(x-1)^2$ 48. $(x-3)(x-2)(x-4)(x-1)$
 49. $(x-6)(x-1)(x-5)(x-2)$ 50. $(x-6)(x+1)(x-4)(x-1)$
 51. $(x-6)(x+1)(x-2)(x-3)$ 52. $(x+6)(x-1)(x+2)(x+3)$
 53. $(2x+3y)^2(2x+3y-1)(2x+3y+2)$ 54. $(2x+3y)^2(2x+3y-3)(2x+3y+3)$
 55. $(2x+3y)^3(2x+3y-3)(2x+3y+3)$ 56. $(2x+3y)^3(2x+3y-4)(2x+3y+4)$
 57. $(2x+3y-2)(2x+3y+2)(2x+3y)^2+4$ 58. $(2x+3y-1)(2x+3y+1)(2x+3y+1)^2+1$
 59. $(x+y-7)(x+y+7)$ 60. $(x+2y-5)(x+2y+5)$ 61. $(x-2y-6)(x-2y+6)$
 62. $(x-4y-9)(x-4y+9)$ 63. $(5-x+y)(5+x-y)$ 64. $(8-x+2y)(8+x-2y)$
 65. $(4-x-5y)(4+x+5y)$ 66. $(3-x-2y)(3+x+2y)$ 67. $(x-y+2)(x+y-2)$
 68. $(x-2y-1)(x+2y+1)$ 69. $(x+y+2)(x+y+5)$ 70. $(x-y+2)(x-y+3)$
 71. $(x-2y+6)(x-2y+1)$ 72. $(x-2y+2)(x-2y+1)$ 73. $(x-2y-2)(x-2y-3)$
 74. $(x-3y-5)(x-3y-3)$ 75. $(x+y+10)(x+y+1)$ 76. $(x+y+5)(x+y-2)$
 77. $(2x-3y-3)^2$ 78. $(2x-3y-3)(2x-3y+3)$ 79. $(x+2)(x^2+2x+1)(x-1)(x+1)(x^2+1)$
 80. $(x-2)(x^2+2x+1)(x-1)(x+1)(x^2+1)$ 81. $(x-1)(x^2+2x+4)(x+1)(x^2+1)$
 82. $(x+2)(x^2+2x+1)(x-1)(x^2+x+1)$ 83. $(x-2)(x^2+2x+1)(x+1)(x^2+x+1)$

- 1.04 p 22-44: 84. $(x+2)(x^2-2x+4)(x+1)(x^2-x+1)$ 85. $(x-2)(x^2+2x+4)^2$
 86. $(x+2)^2(x^2-2x+4)^2$ 87. $(x-y)(x^2+xy+y^2)(x+y)(x^2-xy+y^2)$
 88. $(x-2)(x^2+2x+4)(x-y)(x^2+xy+y^2)$
 89. $(x-2)(x^2+2x+4)(x+2)(x^2-2x+4)$
 90. $(x-3)(x^2+3x+9)(x+1)(x^2-x+1)$ 91. $x^{1/2}(x+8)$
 92. $x^{1/2}(x^2+8)$ 93. $x^{1/2}(x-4)(x+4)$ 94. $(x-y)^{1/2}(x-y+8)$
 95. $(x-y)^{1/2}(x-y-4)(x-y+4)$ 96. $(x-y)^{1/2}[(x-y)^2+16]$
 97. $\frac{(x-2)(x+2)(x^2+4)}{x}$ 98. $\frac{(x-3)(x^2+3x+9)}{x}$ 99. $\frac{(x-4)(x^2+4x+16)}{x^2}$
 100. $\frac{(x-3)(x+3)(x^2+9)}{x^3}$ 101. $x^{1/3}(x+17)$ 102. $x^{1/3}(x-4)(x+4)$
 103. $x^{1/4}(x-8)(x+8)$ 104. $x^{1/2}(x-3)(x^2+3x+9)$ 105. $\frac{x+8}{x^{1/2}}$
 106. $\frac{(x-8)(x+8)}{x^{1/2}}$ 107. $\frac{(x-2)(x^2+2x+4)}{x^{1/3}}$ 108. $\frac{(x+2)(x^2-2x+4)}{x^{1/2}}$
 109. $\frac{(x-8)(x+8)}{x^{1/3}}$ 110. $\frac{(x-3)(x-3)(x^2+9)}{x^{1/3}}$ 111. $(2x+1)^{1/2}(5x+1)$
 112. $5(6x+5)^{1/2}(3x+1)$ 113. $(3x+5)^{1/3}(7x+5)$ 114. $4(x^2+4)^{1/2}(x^2+1)$
 115. $\frac{2(x^2+2)}{(x^2+4)^{1/2}}$ 116. $\frac{4x+5}{(3x+5)^{1/3}}$ 117. $\frac{2(2-x^2)}{(4-x^2)^{1/3}}$ 118. $\frac{2(2-x^2)}{(4-x^2)^{1/2}}$

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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE