1.08 *Equations and Properties of Equations*
Linear, Absolute Value, Quadratic Fractional, and Literal

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Perhaps the premiere task in all of mathematics is the solving of equations. There are many, many types of equations to be solved, from simple **linear equations** (such as 2X = 6) in a first year algebra course to **differential equations** (equations involving "derivatives") in higher mathematics courses. The solution to an equation is the set of all replacement values of the variable for which the equation is true. If an equation is true for all values of the variable, then the equation is called an **identity**. If the equation is true for some, but not all, values of the variable then the equation is called a **conditional equation**. If the equation is **never** true for any value of the variable, then the equation is called a **contradiction**, and there is **no solution**. "No solution" is frequently represented by the **empty set**, "**{ }**" or the greek letter phi "**Φ**".

PROPERTIES OF EQUATIONS

Methods of solving equations are as varied as the types of equations to be solved. However varied the strategies may be, all must be executed according to and without violating several properties of equations:

1. **REFLEXIVE PROPERTY** **a = a.** Any number is equal to itself.

2. **SYMMETRIC PROPERTY** **If a = b, then b = a.** The order in which the equality is given does not matter. For example, you can say "X=4" or "4=X", the meaning is the same--the value of X is 4.

3. **TRANSITIVE PROPERTY** **If a = b and b = c, then a = c.** The word "trans" means "across." If you can get from point "a" to "b", and then from "b" to "c", then you can get from "a" <u>across</u> "b" to "c."

4. ADDITION PROPERTY **If a = b, then a + c = b + c**
If a = b, then a - c = b - c.

The same number may be added (or subtracted) from both sides of an equation.

5. MULTIPLICATION PROPERTY **If a=b, then ac = bc**
If a=b and c≠0, then a/c = b/c.

Both sides of an equation may be multiplied or divided by the same non-zero number.

LINEAR EQUATIONS

A linear equation in one variable is an equation in which the highest degree of the variable is one (no variable squared, cubed, or higher terms). We usually think of an equation being "linear" as opposed to being "quadratic". If it is a linear conditional equation, there will be only one solution. This section provides an opportunity to distinguish between conditional equations, identities, and contradictions.

EXAMPLE 1. Solve for X:

$5(3X-4) - X(X-5) = X(5-X)$

$15x-20-x^2+5x = 5x-x^2$

$-x^2+20x-20 = -x^2+5x$

$15x = 20$

$x = \frac{20}{15} = \frac{4}{3}$

Conditional Eq.

EXAMPLE 2. Solve for X:

$5(3X-4) - X(X-5) = 4X(5-X)+3X^2$

$15x-20-x^2+5x = 20x-4x^2+3x^2$

$-x^2+20x-20 = -x^2+20x$

$-20 = 0$

No Solution

Contradiction

EXAMPLE 3. Solve for X:

$5(3X-4) - X(X-5) = X(20-X) - 20$

$15x-20-x^2+5x = 20x-x^2-20$

$-x^2+20x-20 = -x^2+20x-20$

$0 = 0$

All Real x

Identity

EXAMPLE 4. Solve for X:

$5(3X-4) - X(X-5) = X(15-X)-20$

$15x-20-x^2+5x = 15x-x^2-20$

$-x^2+20x-20 = -x^2+15x-20$

$5x = 0$

$x = 0$

Conditional Eq.

EXERCISES: Solve the equations for X. Identify which are contradictions, identities, or conditional equations.

1. $4(X+3) = 6(2X-5) - 2X$
2. $6(X+3) = 3(2X-3) + 27$
3. $6(X+3) = 3(6-2X) + 4X$
4. $6(X+3) - 3(5-2X) = 12X$
5. $6(X+3) - 3(6-2X) = 12X$
6. $X(X-6) = 4 - X(2-X)$
7. $X(X-2) = 4 - X(2-X)$
8. $X(3X-8) = 12X - 3X(4-X)$

ABSOLUTE VALUE EQUATIONS

The absolute value of a number refers to the "size" of a number or the "magnitude" of a number without regard to whether the number is positive or negative. You remember that the absolute value of a number cannot be negative. The following formal definition of absolute value may at first appear to contradict this last statement.

$$\textit{DEFINITION:}\quad |X| = X \text{ if } X \geq 0$$
$$= -X \text{ if } X < 0$$

Does it appear that in the second part of the definition $|X| = -X$, that the absolute value of X equals a "negative"? What you must remember is that in the second part of the definition, X is itself negative, that the absolute value of X is actually the negative of the negative, which is positive! This formal definition of absolute value of X confirms the fact that there are generally two cases to consider--there are two solutions to be found.

Consider the simple example, **|X| = 3**. Obviously, the solutions are **X = 3** and **X = -3**. Likewise **|Junk| = 3** has two solutions: **Junk = 3** and **Junk = -3**. This thought introduces the next examples, in which the variable, instead of being **"X"** or **"Junk"**, is **"2X - 5"**:

EXAMPLE 1. Solve for X:

|2X - 5| = 3

Solution:

2X - 5 = 3 or **2X - 5 = -3**

2X = 8   **2X = 2**

X = 4   **X = 1**

Check: (just for fun)

X=4: $|2(4)-5| = 3$, $|3| = 3$

X=1: $|2(1)-5| = 3$, $|-3| = 3$

EXAMPLE 2. Solve for X:

|2X - 5| = -3

Solution: **No Solution**, since abs. value cannot equal a negative.

Notice that in Example 2, because the absolute value equals a negative number, there are not two solutions to solve. In fact, if you try to solve two cases as in Example 2, you missed the problem completely. Whenever an absolute value of any variable equals a negative number, there is **no solution**!

EXAMPLE 3. Solve for X:

$$|2X - 5| = |X - 10|$$

Since there are two absolute values in each of these examples, it might appear that there should be two cases for each for a total of 2x2 = 4 cases to solve. The four cases are as follows:

Case I: Positive = Positive

$$(2X - 5) = (X - 10)$$

Case II: Positive = Negative

$$(2X - 5) = -(X - 10)$$

Case III: Negative = Negative

$$-(2X - 5) = -(X - 10)$$

Case IV: Negative = Positive

$$-(2X - 5) = (X - 10)$$

However, before solving all four cases, notice that Case III is actually Case I, where both sides of the equation were multiplied by -1. Also, Case IV is the same as Case II, with both sides multiplied by -1. Therefore, you need only solve Cases I and II.

Solution:

Case I:

$$2X - 5 = X - 10$$
$$X = -5$$

Case II:

$$2X - 5 = -(X - 10)$$
$$2X - 5 = -X + 10$$
$$3X = 15$$
$$X = 5$$

Check: (just for fun!) X = -5

$$|2(-5)-5| = |(-5)-10|$$
$$|-15| = |-15|$$

X = 5

$$|2(5)-5| = |5-10|$$
$$|5| = |-5|$$

The solution for Example 3 is X = -5, 5.

EXAMPLE 4. Solve for X:

$$|2X - 5| = |2X - 15|$$

Case I: $2X - 5 = 2X - 15$

$-5 = -15$

No Solution for Case I

Case II: $2X - 5 = -(2X - 15)$

$2X - 5 = -2X + 15$

$4X = 20$

$X = 5$

The solution is only X = 5.

SUMMARY

I. For $c \geq 0$, $|aX + b| = c$ has two cases to solve:

$aX + b = c$ or $aX + b = -c$

[Note: If c=0, the two cases are the same!]

II. For $c < 0$, $|aX + b| = c$ has **No Solution!**

III. $|aX + b| = |cX + d|$ has two cases to solve:

$aX + b = cX + d$ or $aX + b = -(cX + d)$

EXERCISES:

1. $|2X - 7| = 5$
2. $|2X - 7| = -5$
3. $|3X + 6| = -18$
4. $|3X + 6| = 18$

5. $|3X - 5| = 5$

6. $|3X - 5| = 10$

7. $|4X - 12| = 0$

8. $|4X + 12| = 0$

9. $|2X - 3| = |X + 6|$

10. $|2X + 3| = |4X - 9|$

11. $|3X - 4| = |12 - X|$

12. $|3X + 4| = |12 - X|$

13. $|2X + 4| = |12 - 2X|$

14. $|3X - 5| = |5 + 3X|$

15. $|2X - 3| = |3 - 2X|$

16. $|8 - X| = |8 + X|$

17. $|X + 4| = |4 - X|$

18. $|8 - X| = |X - 8|$

19. $|7 - 3X| = |2X + 3|$

20. $|5X - 12| = |3 - 2X|$

QUADRATIC EQUATIONS

A **quadratic equation** is an equation in which there is a variable raised to the second power, in the form $aX^2+bX+c=0$. As you have already learned, the best way to solve a quadratic equation is by **setting the equation equal to zero** and **factoring**. Unfortunately, not all quadratic expressions can be factored. In cases where factoring is not possible, other methods must be used--either **completing the square** or the **quadratic formula.**

EXAMPLE 1. Solve for X:

$$X^2 + 21X = 100$$
$$X^2 + 21X - 100 = 0$$
$$(X + 25)(X - 4) = 0$$
$$X = -25 \text{ or } X = 4$$

EXAMPLE 2. Solve for X:

$$(X-3)(X-4) = 2$$
$$X^2 - 7X + 12 = 2$$
$$X^2 - 7X + 10 = 0$$
$$(X - 5)(X - 2) = 0$$
$$X = 5 \text{ or } X = 2$$

EXERCISES. Solve for X by the method of factoring.

1. $X^2 + X = 12$
2. $X^2 - 18 = 3X$
3. $X(X-4) = -2X + 8$
4. $2X^2 = 3 + 5X$
5. $2X^2 = 5X - 3$
6. $(X - 4)^2 = 2X$

Unfortunately, factoring doesn't always work!! In cases in which the equation cannot be factored, other methods, **completing the square** or the **quadratic formula**, must be used. The **quadratic formula always works**, whether the equation factors or not! The completing the square method is important because it is by completing the square that the quadratic formula is derived. Completing the square has other applications in the next chapter and also in higher mathematics.

We begin with some perfect square equations. Perfect square equations (see the next exercises) can be solved by simply taking the square root of both sides of the equation. When you take the square root of both sides, you must include a "±" (that is, "+" or "-") in order to get both solutions of the equation.

EXERCISES. Solve the following perfect square equations:

1. $X^2 = 9$ $X = \pm$ ____
2. $X^2 = 25$ $X =$ ________
3. $X^2 = 121$ $X =$ ________
4. $X^2 = 169$ $X =$ ________

5. $X^2 = 20$
   $X = \pm\sqrt{20}$
   $X = \pm$ ____
6. $X^2 = 50$
   $X =$ ________
   $X =$ ________
7. $X^2 = 72$
   $X =$ ________
   $X =$ ________
8. $X^2 = 300$
   $X =$ ________
   $X =$ ________

9. $(X+2)^2 = 9$
   $X+2 = \pm 3$
   $X = -2 \pm 3$
   $X = -2+3$ or $-2-3$
   $X =$ ____ or ____
10. $(X+2)^2 = 25$
   $X+2 = \pm 5$
   $X =$ ________
   $X =$ ____ or ____
11. $(X-3)^2 = 121$

12. $(X-3)^2 = 169$

13. $(X-5)^2 = 100$

14. $(X+7)^2 = 81$

15. $(X-5)^2 = 7$

$X-5 = \pm\sqrt{7}$

$X =$ ________

16. $(X+5)^2 = 7$

______ $= \pm\sqrt{\ }$

$X =$ ________

17. $(X-8)^2 = 13$

18. $(X-5)^2 = 20$

19. $(X+5)^2 = 60$

20. $(X-8)^2 = 27$

21. $X^2 - 4X + 4 = 25$

$(X - 2)^2 = 25$

$X - 2 = \pm 5$

$X =$ _________

$X =$ _____ or _____

22. $X^2 - 6X + 9 = 36$

23. $X^2 + 6X + 9 = 144$

24. $X^2 - 10X + 25 = 100$

25. $X^2 - 14X + 49 = 5$

26. $X^2 + 8X + 16 = 13$

27. $X^2 + 8X + 16 = 20$

28. $X^2 - 12X + 36 = 50$

More often than not, the equation that is given is not a perfect square equation, as in the previous exercises. In order to "build" perfect square equations, it will help to recognize some **perfect square trinomials.** As examples, consider:

$$X^2 + 2X + 1 = (X + 1)^2 \qquad X^2 - 2X + 1 = (X - 1)^2$$
$$X^2 + 4X + 4 = (X + 2)^2 \qquad X^2 - 4X + 4 = (X - 2)^2$$
$$X^2 + 6X + 9 = (X + 3)^2 \qquad X^2 - 6X + 9 = (X - 3)^2$$
$$X^2 + 8X + 16 = (X + 4)^2 \qquad X^2 - 8X + 16 = (X - 4)^2$$
$$X^2 + 10X + 25 = (X + 5)^2 \qquad X^2 - 10X + 25 = (X - 5)^2$$

In the following exercises, what constant term is needed to "complete the square"? [Hint: Figure out the b) part first.]

29a) X^2+2X+______
b) $(X +$ _____$)^2$

30a) X^2+4X+______
b) $(X +$ _____$)^2$

31a) X^2+6X+______
b) $(X +$ _____$)^2$

32a) X^2-8X+______
b) $(X -$ _____$)^2$

33a) X^2-10X+______
b) $(X -$ _____$)^2$

34a) X^2-12X+______
b) $(X -$ _____$)^2$

35. Did you see a pattern? Take b)_______ **of the middle term and** a)_______ **it. Now try some harder ones.**

36. $X^2 + 16X +$ _____	**37.** $X^2 + 24X +$ _____	**38.** $X^2 + 40X +$ _____
39. $X^2 - 30X +$ _____	**40.** $X^2 - 80X +$ _____	**41.** $X^2 + 5X +$ _____
42. $X^2 - 5X +$ _____	**43.** $X^2 + 9X +$ _____	**44.** $X^2 + 13X +$ _____
45. $X^2 + X +$ _____	**46.** $X^2 - X +$ _____	**47.** $X^2 + bX +$ _____
48. $X^2 - bX +$ _____	**49.** $X^2 - \pi X +$ _____	**50.** $X^2 + \pi X +$ _____

RULE: When completing the square for $X^2 + bX +$ _____, take half the coefficient of X and square.

EXAMPLE 3. Solve for X:

$X^2 + 6X - 7 = 0$	Add +7 to both sides to express in the form $X^2 + 6X +$ ____.
$X^2 + 6X +$ ____ $= 7 +$ ____	Take half of 6 and square to get 9. Add +9 to both sides of equation.
$X^2 + 6X + 9 = 7 + 9$	Rewrite with perfect square on left.
$(X + 3)^2 = 16$	Take square root of both sides. **(Don't forget the ± sign!)**
$X + 3 = \pm 4$	Add -3 to both sides.
$X = -3 \pm 4$	
$X = -3 + 4$ or $X = -3 - 4$	
$X = 1$ or $X = -7$	

Check by factoring: $X^2 + 6X - 7 = 0$
$(X - 1)(X + 7) = 0$
$X = 1$; $X = -7$

[Note: Factoring is easier, but it doesn't always work!]

EXAMPLE 4. Solve for X:

$X^2 + 6X - 8 = 0$	Add +8 to both sides of equation.
$X^2 + 6X + \_\_\_\_ = 8 + \_\_\_\_$	Take half of 6 and square to get 9. Add +9 to both sides of equation.
$X^2 + 6X + \underline{\ 9\ } = 8 + \underline{\ 9\ }$	Rewrite with perfect square on left.
$(X + 3)^2 = 17$	Take square root of both sides. **(Don't forget the ± sign!)**
$X + 3 = \pm\sqrt{17}$	Add -3 to both sides.
$X = -3 \pm \sqrt{17}$	Answer does not simplify.

Note: Because of the radical, this problem cannot be solved by factoring.

EXERCISES. Solve the equations by method of completing the square.

1. $X^2 + 6X - 9 = 0$

$X^2 + 6X + \_\_\_ = 9 + \_\_\_$

$(X + \quad)^2 = \_\_\_\_\_\_$

$X + \_\_\_\_ = \pm \_\_\_\_$

$X = \_\_\_ \pm \_\_\_$

$X = \_\_\_\_\_\_\_\_\_\_$

2. $X^2 + 6X + 5 = 0$

$X^2 + 6X + \_\_\_ = -5 + \_\_\_$

$(X + \quad)^2 = \_\_\_\_\_\_$

$X + \_\_\_\_ = \pm \_\_\_\_$

$X = \_\_\_ \pm \_\_\_$

$X = \_\_\_\_\_\_\_\_\_\_$

3. $X^2 - 6X - 9 = 0$

4. $X^2 - 2X - 8 = 0$

5. $X^2 + 2X - 48 = 0$

6. $X^2 - 12X - 64 = 0$

7. $X^2 - 10X - 15 = 0$

8. $X^2 - 8X + 8 = 0$

In 9 - 14, watch out for complex numbers.

9. $X^2 + 4X + 5 = 0$

10. $X^2 - 6X + 13 = 0$

11. $X^2 - 10X + 50 = 0$

12. $X^2 + 8X + 52 = 0$

13. $\mathbf{X^2 + 8X + 40 = 0}$

14. $\mathbf{X^2 - 6X + 36 = 0}$

If the coefficient of X^2 is not **1**, then divide both sides of the equation by that coefficient.

15. $\mathbf{2X^2 + 7X + 6 = 0}$

$$X^2 + \frac{7}{2}X + 3 = 0$$

Half of $\frac{7}{2}$ *is* $\frac{7}{4}$, *and* $(\frac{7}{4})^2$ *is* $\frac{49}{16}$

$$X^2 + \frac{7}{2}X + \qquad = -3 +$$

$$(X + \frac{7}{4})^2 = -3 + \frac{49}{16}$$

$$(X + \frac{7}{4})^2 = \frac{-48 + 49}{16}$$

$$(X + \frac{7}{4})^2 = \frac{1}{16}$$

$$X + \frac{7}{4} = \pm \frac{1}{4}$$

$$X = -\frac{7}{4} \pm \frac{1}{4}$$

X = ______ or ______

X = ______ or ______

16. $\mathbf{2X^2 - 9X + 10 = 0}$

Quadratic Formula

The general form of the quadratic equation is $aX^2+bX+c = 0$, where a, b, and c represent real numbers. This equation can be solved by completing the square, in a manner similar to the last two exercises.

$$aX^2 + bX + c = 0$$ Subtract c from both sides.

$$aX^2 + bX = -c$$ Divide both sides by a, $(a \neq 0)$.

$$X^2 + \frac{b}{a}X = -\frac{c}{a}$$ Complete the square.

Half of $\frac{b}{a}$ *is* $\frac{b}{2a}$, *and* $(\frac{b}{2a})^2$ *is* $\frac{b^2}{4a^2}$

$$X^2 + \frac{b}{a}X + (\qquad) = -\frac{c}{a} + (\qquad)$$

$$X^2 + \frac{b}{a}X + (\frac{b^2}{4a^2}) = -\frac{c}{a} + (\frac{b^2}{4a^2})$$ Add $\frac{b^2}{4a^2}$ to both sides.

Find LCD = $4a^2$ on right side.

$$(X + \frac{b}{2a})^2 = -\frac{c}{a}\cdot\frac{4a}{4a} + \frac{b^2}{4a^2}$$

$$(X + \frac{b}{2a})^2 = \frac{-4ac + b^2}{4a^2}$$

Take square root of both sides.

$$X + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Denom $\mathbf{4a^2}$ is a perfect square.

$$X + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

Add $-\frac{b}{2a}$ to both sides.

$$X = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

LCD = **2a** on right side.

$$\boxed{X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

World Famous Quadratic Formula!!

The quadratic formula, derived on the previous page, may be used to solve <u>any</u> quadratic equation. It is usually best to use factoring if the equation factors and use the quadratic formula otherwise. However, completing the square may be easier than the quadratic formula, particularly if **a=1** and **b is even**. ***Please note that the <u>use</u> of the quadratic formula on the next pages is <u>much</u> easier than the derivation on the previous page.***

EXAMPLE 5. Solve for X:

$$X^2 + 6X - 2 = 0$$

a=1 b=6 c=-2

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-6 \pm \sqrt{36 - 4(1)(-2)}}{2(1)}$$

$$= \frac{-6 \pm \sqrt{36 + 8}}{2}$$

$$= \frac{-6 \pm \sqrt{44}}{2}$$

$$= \frac{-6 \pm 2\sqrt{11}}{2}$$

$$= -3 \pm \sqrt{11}$$

EXAMPLE 6. Solve for X:

$$X^2 + 5 = 6X$$

Set equal to zero: $X^2 - 6X + 5 = 0$

a=1 b=-6 c=5

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{6 \pm \sqrt{36 - 4(1)(5)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{36 - 20}}{2}$$

$$= \frac{6 \pm \sqrt{16}}{2}$$

← NOTE: Whenever $b^2 - 4ac$ = perfect square, the problem <u>could</u> have been factored!!

$$= \frac{6 \pm 4}{2}$$

$$= \frac{10}{2} = 5 \text{ or } \frac{2}{2} = 1$$

1. $X^2 - X - 3 = 0$

2. $2X^2 = 3 + 7X$

3. $X^2 - 4X - 6 = 0$

4. $X^2 = 4X + 2$

5. $2X(X - 4) = -7$

6. $3X^2 + 2(3 + X) = 4 - 6X$

Watch out for complex numbers:

7. **$X(X + 6) + 25 = 0$**

8. **$X^2 = 2(3X - 5)$**

9. **$4X(X + 3) = -13$**

10. **$9X^2 = 4(3X - 2)$**

11. $2X(2 - X) = 3$

12. $2X(X + 2) = -5$

13. $4X(X + 5) = -27$

14. $3X^2 = 2(X - 1)$

FRACTIONAL EQUATIONS

When solving a fractional equation, first find the least common denominator (LCD). Then multiply both sides of the equation by the LCD, and divide out all denominator factors. However, if you multiply both sides of an equation by a variable, you must check the answers to be sure no denominators equal zero. If an answer that you get ever makes a denominator equal zero, then that answer must be rejected. The answer thus obtained was not a "legal" answer. Like evidence that is illegally obtained and cannot be allowed in court, such answers must be thrown out, and if no other solution can be found, there is no solution for the problem. In such cases, the answer is "No Solution" or the empty set, which is denoted by "ϕ" or "{ }".

PRINCIPLE

Whenever an equation is solved by multiplying both sides of that equation by a variable, the solution must be checked to be sure no denominators equal zero.

Before beginning the exercises, one more principle will be useful in this section. This is the **definition of equality of fractions.** Two fractions, $\frac{a}{b}$ *and* $\frac{c}{d}$, are equal if and only if

$\mathbf{a \cdot d = b \cdot c}$.

$$\frac{a}{b} = \frac{c}{d} \text{ means that } a \cdot d = b \cdot c$$

Solve the equations. Be sure to check all denominators.

EXAMPLE 1

$$\frac{4}{X} = \frac{X-2}{2} \quad (x \neq 0)$$

$4 \cdot 2 = x(x-2)$

$x^2 - 2x = 8$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$x = 4$ or $x = -2$

EXERCISES 1.

$$\frac{4}{X} = \frac{X+2}{2}$$

2.

$$\frac{X}{X+4} = \frac{6}{X-4}$$

EXAMPLE 2 LCD = 6

$$6 \cdot \frac{X(X-1)}{6} + 6 \cdot \frac{X}{3} = 1 \cdot 6$$

$x^2 - x + 2x = 6$

$x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

$x = -3$ or $x = 2$

3.

$$\frac{X(X+1)}{6} - \frac{X}{3} = 1$$

EXAMPLE 3 LCD = $(x-1)(x-5)$

$$(x-1)(x-5)\frac{X}{X-1} + (x-1)(x-5)\frac{2}{X-5} = (x-1)(x-5)\frac{-4}{(X-5)(X-1)}$$

$x \neq 5$, $x \neq 1$

$x(x-5) + 2(x-1) = -4$

$x^2 - 5x + 2x - 2 = -4$

$x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

$x = 2$ ~~$x = 1$~~

Reject! Denom $\neq 0$.

4.

$$\frac{X}{X-1} + \frac{5}{X-5} = \frac{-1}{(X-5)(X-1)}$$

5. $\dfrac{5}{X-2} - \dfrac{5}{X+2} = 4$

6. $\dfrac{3}{X-3} - \dfrac{7}{X+3} = 2$

EXAMPLE 4 $\dfrac{1}{X^2-4X+3} - \dfrac{1}{X^2+4X-5} = \dfrac{1}{X^2+2X-15}$

LCD = $(x-3)(x-1)(x+5)$

$x \neq 3, 1, -5$

$$(x-3)(x-1)(x+5)\frac{1}{(x-3)(x-1)} - (x-3)(x-1)(x+5)\frac{1}{(x+5)(x-1)} = (x-3)(x-1)(x+5)\frac{1}{(x+5)(x-3)}$$

$$x+5 - x+3 = x-1$$

$$8 = x-1$$

$$9 = x$$

7. $\dfrac{1}{X^2-4X+3} - \dfrac{2}{X^2+4X-5} = \dfrac{4}{X^2+2X-15}$

8. $$\frac{X}{X^2-X-6} - \frac{1}{X^2-5X+6} = \frac{2}{X^2-4}$$

9. $$\frac{X}{X^2-X-6} + \frac{3}{X^2-5X+6} + \frac{2}{X^2-4} = 0$$

10. $$\frac{X+2}{X^2-X-6} - \frac{X}{X^2-4} = \frac{1}{X^2-5X+6}$$

LITERAL EQUATIONS

It is frequently necessary to manipulate formulas and solve for one variable in terms of constants and other variables in the formula. These formulas are called **literal equations**, perhaps because there are so many "litters" (joke!) involved. You will probably recognize some of the formulas that are used here, since many of them come from science, business, geometry, and other areas of life. Other formulas have been made up especially for practice in this section.

The procedure for solving literal equations is similar to that of solving other equations. Usually, you must separate all variable terms on one side of the equation, and non-variable terms on the other side. The key step is to factor the variable as a common factor, so as to get the variable in one place, then divide both sides of the equation by the resulting factor. This may leave a strange-looking fraction, and the answers will probably be very abstract. Just do what you know is correct algebraically and have confidence in your work!

EXAMPLE 1. Solve for X:

$\mathbf{aXY + bX = cY + d}$ Variable (X) terms are all on left side.

$\mathbf{X(aY + b) = cY + d}$ Factor out the variable (X).

$X = \dfrac{cY + d}{aY + b}$ Divide both sides by **aY + b**.

EXAMPLE 2. Solve for Y:

$\mathbf{aXY + bX = cY + d}$ Get variable (Y) terms on left side.

$\mathbf{aXY - cY = d - bX}$ Get "non-Y" terms on right side.

$\mathbf{Y(aX - c) = d - bX}$ Factor out the variable (Y).

$Y = \dfrac{d - cX}{aX - c}$ (or $\dfrac{cX - d}{c - aX}$) Divide both sides by **aX - c**.

→ If you mult numer. & denom. by (−1).

ANSWERS 1.08

p. 111: 1. Conditional equation, 7; 2. Identity, all reals; 3. Conditional equation, 0; 4. Contradiction, No solution; 5. Identity, all reals; 6. Conditional equation, -1; 7. Contradiction, no sol; 8. Conditional equation, 0.

p. 114-116:

1. 6,1; 2. No sol; 3. No sol; 4. 4,-8; 5. 0, 10/3; 6. 5, -5/3; 7. 3; 8. -3; 9. 9,-1; 10. 6,1; 11. 4,-4; 12. 2,-8; 13. 2; 14. 0; 15. All reals; 16. 0; 17. 0; 18. All reals; 19. 4/5, 10; 20. 15/7, 3.

p. 117: 1. -4,3; 2. 6,-3; 3. 4,-2; 4. -1/2, 3; 5. 3/2, 1; 6. 8,2.

p. 118-121:

1. ± 3; **2.** ± 5; **3.** ± 11; **4.** ± 13; **5.** $\pm 2\sqrt{5}$; **6.** $\pm 5\sqrt{2}$; **7.** $\pm 6\sqrt{2}$; **8.** $\pm 10\sqrt{3}$; **9.** 1,-5; **10.** 3,-7; **11.** 14,-8; **12.** 16,-10; **13.** 15,-5; **14.** 2,-16; **15.** $5\pm\sqrt{7}$; **16.** $-5\pm\sqrt{7}$; **17.** $8\pm\sqrt{13}$; **18.** $5\pm 2\sqrt{5}$; **19.** $-5\pm 2\sqrt{15}$; **20.** $8\pm 3\sqrt{3}$; **21.** 7,-3; **22.** 9,-3; **23.** 9,-15; **24.** 15,-5; **25.** $7\pm\sqrt{5}$; **26.** $-4\pm\sqrt{13}$; **27.** $-4\pm 2\sqrt{5}$; **28.** $6\pm 5\sqrt{2}$; **29**a)1 b)1; **30**a)4 b)2; **31**a)9, b)3; **32**a)16, b)4; **33**a)25, b)5; **34**a)36, b)6; **35**b)half a) square; **36.** 64; **37.** 144; **38.** 400; **39.** 225; **40.** 1600; **41.** 25/4 or 6.25; **42.** 25/4 or 6.25; **43.** 81/4 or 20.25; **44.** 169/4; **45.** 1/4 or 0.25; **46.** 1/4 or 0.25; **47.** $b^2/4$; **48.** $b^2/4$; **49.** $\pi^2/4$; **50.** $\pi^2/4$.

p. 122-124:

1. $-3\pm 3\sqrt{2}$; **2.** -1,-5; **3.** $3\pm 3\sqrt{2}$; **4.** 4,-2; **5.** -8,6; **6.** 16,-4; **7.** $5\pm 2\sqrt{10}$; **8.** $4\pm 2\sqrt{2}$; **9.** $-2\pm i$; **10.** $3\pm 2i$; **11.** $5\pm 5i$; **12.** $-4\pm 6i$; **13.** $-4\pm 2i\sqrt{6}$; **14.** $3\pm 3i\sqrt{3}$; **15.** -3/2, -2; **16.** 5/2, 2,

p. 126-129:

1. $\frac{1 \pm \sqrt{13}}{2}$ **2.** $\frac{7 \pm \sqrt{73}}{4}$ **3.** $2 \pm \sqrt{10}$ **4.** $2 \pm \sqrt{6}$

5. $\frac{4 \pm \sqrt{2}}{2}$ **6.** $\frac{-4 \pm \sqrt{10}}{3}$ **7.** $-3\pm 4i$; **8.** $3\pm i$;

9. $\frac{-3 \pm 2i}{2}$ **10.** $\frac{2 \pm 2i}{3}$ **11.** $\frac{2 \pm i\sqrt{2}}{2}$ **12.** $\frac{-2 \pm i\sqrt{6}}{2}$

13. $\frac{-5 \pm i\sqrt{2}}{2}$ **14.** $\frac{1 \pm i\sqrt{5}}{3}$.

p. 131-133:

1. -4,2; **2.** 12,-2; **3.** 3,-2; **4.** 2,-2; **5.** 3,-3; **6.** -6,4; **7.** No sol (Reject 3); **8.** 1,4; **9.** 0,-3; **10.** No sol (Rej 3).

p. 135-137:

1. $\dfrac{d - b}{a - c}$ 2. $\dfrac{ab + cd}{a + c}$ 3. $\dfrac{y - b}{m}$ 4. $\dfrac{y - a + mb}{m}$

5. $\dfrac{C - BY}{A}$ 6. $\dfrac{2A}{b}$ 7. $\dfrac{3V}{\pi r^2}$ 8. $\sqrt{\dfrac{3V}{\pi h}}$

9. $\dfrac{2A}{B + b}$ 10. $\dfrac{2A - bh}{h}$ or $\dfrac{2A}{h} - b$ 11. $\dfrac{5}{9}(F - 32)$

12. $\dfrac{9}{5}C - 32$ or $\dfrac{9C + 160}{5}$ 13. AY 14.

15. $\dfrac{XY - PbX}{P}$ 16. $\dfrac{XY - Pa}{PX}$ 17. $\dfrac{Pa}{Y - Pb}$ 18. $\dfrac{Pa + PbX}{X}$

19. $\dfrac{UF}{U - F}$ 20. $\dfrac{YZ}{Z + Y}$ 21. $\dfrac{3X \pm \sqrt{9X^2 + 8X - 48}}{4}$

22. $-3X \quad \dfrac{1 \pm \sqrt{9X^2 - 6X - 47}}{4}$ 23. $\dfrac{-3X + 1 \pm \sqrt{9X^2 - 30X + 1}}{2X}$

24. $\dfrac{3X \pm \sqrt{13X^2 - 24X}}{2X}$