2.03 *The Parabola*

Dr. Robert J. Rapalje

There are several different ways to explain and graph parabolas. Probably the most sophisticated method is presented in analytic geometry, where a parabola is defined as the set of all points at an equal distance from a given point (which is the focus of the parabola) and a given line. The equation of such a parabola is in the form $y = ax^2 + bx + c$ or $x = ay^2 + by + c$.

A second approach, the one you probably used in lower math classes, is to graph an equation in the form $y = ax^2 + bx + c$ or $x = ay^2 + by + c$ by plotting enough points to get an idea what the graph looks like.

After completing Section 2.02 on "Graphing by Translation," it seemed a good idea to explain graphing of parabolas by "translating" the well-known parabolas $y = x^2$ or $x = y^2$, by shifting these graphs up, down, right, left, or inverting the graphs. In this section, equations in general form $y = ax^2 + bx + c$ are converted to the "translated" form $y - k = a(x - h)^2$ by the method of completing the square. In this form, the vertex at (h,k) is easily identified. However, the completing the square process can get downright ugly!

Then I realized that there is an easier way to find the vertex of $y = ax^2 + bx + c$. The vertex will ALWAYS be at $x = \frac{-b}{2a}$. **In my opinion, this is the preferred method of graphing parabolas.** For a detailed explanation on this method, please click here to see the Math in Living Color section that pertains to this topic. I actually recommend that you use the $x = \frac{-b}{2a}$ method, together with the methods of the **graphing calculator**, to solve the problems in this section, instead of the method of completing the square. Please see also **Calculator Workshop Notes for the TI83/84**.

The **focus** (pun!) of this section on non-linear graphs will be the **parabola.**

1. From the previous section, the equation **$Y = (X-h)^2 + k$** or **$Y-k = (X-h)^2$** represents a **a)**___________ that opens **b)**_________, with vertex at **c)**________. The graph of **$Y = -(X-h)^2 + k$** or **$Y-k = -(X-h)^2$** opens **d)**_________, with vertex at **e)**________.

Complete tables and graph the following exercises:

2. **$Y = X^2$**

X	Y
0	
1	
2	
-1	
-2	

3. **$Y = 2X^2$**

X	Y
0	
1	
2	
-1	
-2	

4. **$Y = 4X^2$**

X	Y
0	
1	
2	
-1	
-2	

5. **$Y = -4X^2$**

X	Y
0	
1	
2	
-1	
-2	

6. **$Y = \frac{1}{2}X^2$**

X	Y
0	
1	
2	
-1	
-2	

7. **$Y = \frac{1}{4}X^2$**

X	Y
0	
1	
2	
-1	
-2	

8. **$Y = -\frac{1}{2}X^2$**

X	Y
0	
1	
2	
-1	
-2	

9. **$Y = -\frac{1}{4}X^2$**

X	Y
0	
1	
2	
-1	
-2	

10. In general, for the equation $\mathbf{Y = aX^2}$, if the value of "a" is positive, the graph opens **a)** ___________, and if the value of "a" is negative, then the graph opens **b)** ___________.

11. In general, for the equation $\mathbf{Y = aX^2}$, the larger the value of $|a|$, the **a) (more steep, less steep)** the graph. If $|a| < 1$, then the graph is **b) (more steep, less steep)** than the graph of $\mathbf{Y = X^2}$.

You may have guessed already that equations of parabolas do not usually come in the "standard forms" given thus far. When the equations are not in "standard form," the method of **completing the square** will be used to rewrite the equation in standard form, from which the vertex and graph can easily be determined. Remember, when completing the square, take **half the coefficient of x, and square.**

In each of the following, what constant term must be added to each of the following in order to "complete the square"?

12a) $X^2 + 4X +$ ____ **13a)** $X^2 + 6X +$ ____ **14a)** $X^2 - 10X +$ ____

b) $= (X +$ ____$)^2$ **b)** $= (X +$ ____$)^2$ **b)** $= (X -$ ____$)^2$

15a) $X^2 - 14X +$ ____ **16a)** $X^2 - 20X +$ ____ **17a)** $X^2 + 16X +$ ____

b) $= (\qquad)^2$ **b)** $= (\qquad)^2$ **b)** $= (\qquad)^2$

18a) $X^2 + 40X +$ ____ **19a)** $X^2 + 26X +$ ____ **20a)** $X^2 + 5X +$ ____

b) = **b)** = **b)** =

21a) $X^2 + 9X +$ ____ **22a)** $X^2 - 13X +$ ____ **23a)** $X^2 + X +$ ____

b) = **b)** = **b)** =

24. Now, if you are given the equation of a parabola such as $\mathbf{Y = X^2 + 4X + 3}$, assuming that the coefficient of X^2 is 1, you must take half the X coefficient (half of 4 is 2) and square (to obtain 4). Then add this quantity (in this case 4) and subtract the same quantity from the right side of the equation, as shown:

$\mathbf{Y = X^2 + 4X + 3}$ (Half and square = 4)

$\mathbf{Y +}$ ____ $\mathbf{= X^2 + 4X +}$ ____ $\mathbf{+ 3}$ (Add -3 each side)

$\mathbf{Y}$ ______ $\mathbf{= (X +}$ ____$)^2$

Vertex is at (____, ____)

In each of the following, complete the square as above and find the vertices of the parabolas.

25. $Y = X^2 + 4X - 2$ **26.** $Y = X^2 - 4X - 5$ **27.** $Y = X^2 - 6X + 5$

28. $Y = X^2 + 6X - 2$ **29.** $Y = X^2 + 8X + 5$ **30.** $Y = X^2 - 8X + 5$

31. $Y = X^2 - 4X - 2$ **32.** $Y = X^2 + 4X - 5$ **33.** $Y = X^2 + 6X + 5$

If the coefficient of X^2 is not 1, then whatever it is, factor it out of the X^2 and the X terms ONLY, as illustrated in this example: Leave the blank spaces to complete the square in the next step.

34. $Y = 2X^2 + 8X + 18$

$Y +$ ____ $= 2(X^2 + 4X +$ ____$) + 18$ (Half & square = 4. You must put 4 on second line, but you really added **2·4=8** to each side, so add **8** in the blank on the left side.)

$Y +$ ____ $= 2(X +$ ____$)^2$

Last, add -18 to each side.

Y______ = $2(X +$ ____$)^2$. **The vertex is at (____, ____).**

35. $Y = 2X^2 + 4X - 6$ **36.** $Y = 3X^2 - 6X + 5$

37. $Y = 3X^2 - 18X + 24$ **38.** $Y = 4X^2 - 8X$ **39.** $Y = 4X^2 + 40X$

If the coefficient of X^2 is negative, then factor out the negative.

40. $Y = -2X^2 + 8X + 10$

Y ______ = $-2(X^2 - 4X +$ ____) + 10 (Half & square = 4, so you put 4 in second blank. But you really added **$-2 \cdot 4 = -8$,** so you need to add **-8** to the left side.)

Y ______ = $-2(X -$ ____$)^2$ Last, add **-10** to each side.
The vertex is at (____, ____).

41. $Y = -2X^2 + 4X - 6$

42. $Y = -3X^2 - 6X + 5$

43. $Y = -3X^2 - 18X + 24$

44. $Y = -4X^2 - 8X$

45. $Y = -4X^2 + 40X$

46. $Y = 4X^2 + 4X + 12$

Y ______ = $4(X^2 + 1X +$ ____) + 12

Y ______ = $4(\qquad)^2$.

Vertex: (____, ____)

47. $Y = 2X^2 - 2X - 8$

48. $Y = -4X^2 + 12X - 4$

49. $Y = -4X^2 - 20X + 6$

50. Now consider the graph of the equation **$X = Y^2$.** Complete the table of values as in the previous section, and draw the graph.

X	Y

51. Complete the table and graph the equation **$X = Y^2 + 4$**.

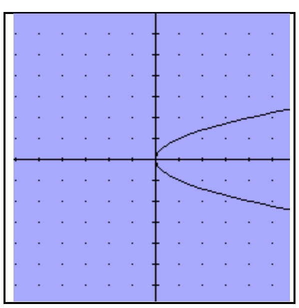
X	Y

52. Complete the table and graph the equation $X = Y^2 - 4$.

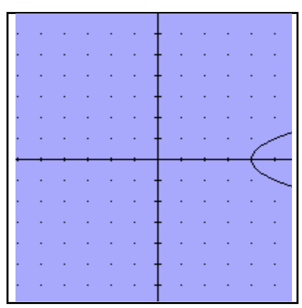
X	Y

The correct graphs for these equations are given for your convenience on the next page.

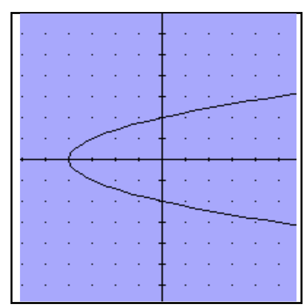
50. $X = Y^2$



51. $X = Y^2 + 4$



52. $X = Y^2 - 4$



How do the graphs of $\mathbf{X = Y^2 + 4}$ and $\mathbf{X = Y^2 - 4}$ compare to the graph of $\mathbf{X = Y^2}$? Did you notice that the "+4" just moves the graph right 4 units, and the "-4" moves the graph left 4 units. Except for the position of the graphs, these are all really the "same" graph. Except for the direction of these graphs, they are all really the "same" graphs as $\mathbf{Y = X^2}$, $\mathbf{Y = X^2 + 4}$, and $\mathbf{Y = X^2 - 4}$.

Can you speculate about the graphs of the equations? Graph these equations by translation.

53. $\mathbf{X = (Y-2)^2}$

54. $\mathbf{X = (Y+2)^2}$

Compare the graphs of $\mathbf{X = (Y-2)^2}$ and $\mathbf{X = (Y+2)^2}$ to the graph of $\mathbf{X = Y^2}$. Can you speculate about the effect of the "Y-2" and the "Y+2" on the graphs of these equations?

55. Complete the following statements:

a. The graph $\mathbf{X = Y^2 + 4}$ is shifted __________________.

b. The graph $\mathbf{X = Y^2 - 4}$ is shifted __________________.

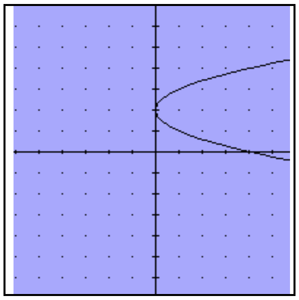
c. The graph $\mathbf{X = (Y-2)^2}$ is shifted __________________.

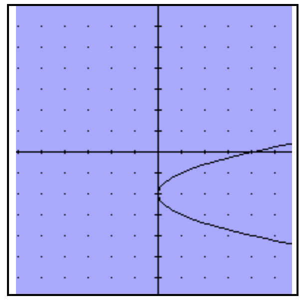
d. The graph $\mathbf{X = (Y+2)^2}$ is shifted __________________.

e. The graph $\mathbf{X - 4 = (Y-2)^2}$ is shifted _________ and _________.

f. The graph $\mathbf{X + 4 = (Y+2)^2}$ is shifted _________ and _________.

The correct graphs of $\mathbf{X = (Y-2)^2}$ and $\mathbf{X = (Y+2)^2}$ are as follows:





In each of the following, complete the square as above and find the vertices of the parabolas.

56. $X = Y^2 + 4Y - 2$

X____ = Y^2 + 4Y + (___) - 2

X____ = (Y + ___)^2

Vertex is at (____,____)

(Remember, give X-coord. first!)

57. $X = Y^2 - 4Y - 5$

58. $X = Y^2 - 8Y + 5$

59. $X = Y^2 - 4Y - 2$

60. $X = 2Y^2 + 8Y + 18$

X____ = 2(Y² + 4Y + ____) + 18 (Half & square = 4. You must put 4 on second line, but you really add **2·4=8** to left side.)

X____ = 2(Y + ____)² **The vertex is at (____, ____).**

61. $X = 2Y^2 + 4Y - 6$

62. $X = 3Y^2 - 6Y + 5$

63. $X = 4Y^2 - 8Y$

64. $X = 4Y^2 + 40Y$

65. $X = -2Y^2 + 8Y + 10$

66. $X = -3Y^2 - 6Y + 5$

67. $X = 4Y^2 + 4Y + 12$

X____ = $4(Y^2 + 1Y +$ ____) + 12

X____ = $4(\qquad)^2$.

Vertex: (____, ____)

68. $X = 2Y^2 - 3Y + 1$

In 69-76, find the vertex and sketch the graph:

69. $Y = X^2 - 6X + 5$

70. $Y = -X^2 - 4X$

71. $X = -Y^2 - 6Y - 5$

72. $X = Y^2 + 8Y + 12$

73. $Y = -4X^2 + 8X - 4$

74. $Y = -2X^2 - 4X + 6$

75. $X = 4Y^2 + 6Y + 3$

76. $X = -4Y^2 + 4Y - 3$

APPLICATIONS

Every equation in the form $\mathbf{Y = aX^2 + bX + c}$, for $a \neq 0$, may be graphed as a parabola opening **upward if a>0, or downward if a<0.** Moreover, each graph has a **vertex** that can be determined by the process of completing the square. For parabolas that open upward, the **vertex** represents the **minimum point** on the graph, the **minimum value of Y.** For parabolas that open downward, the **vertex** represents the **maximum point** on the graph, the **maximum value of Y.**

For those who may now be wondering, "What good is all of this?", here are some very nice applications in algebra, geometry, and business. Consider the following examples.

EXAMPLE 1: The cost equation C for a business to manufacture and sell X widgets is $\mathbf{C = X^2 - 12X + 100}$ (C is the cost in dollars and X is the number of widgets produced).

a) Notice that the graph of this cost equation is a parabola opening upward, and therefore, there is a **minimum** point. Complete the square in order to find this minimum point.

b) Find the value of X that will minimize the cost. Find the minimum cost.

SOLUTION: a)

$$C = X^2 - 12X + 100$$

$$C\underline{\qquad} = (X^2 - 12X + \underline{\qquad}) + 100$$

$$C\underline{\;+\;36\;} = (X^2 - 12X + \underline{\;36\;}) + 100$$

$\mathbf{C\underline{\ +\ 36\ }\ -\ 100 = (X^2 - 12X + \underline{\ 36\ })}$

$\mathbf{C - 64 = (X - 6)^2}$

Minimum point (6, 64).

b) **Minimum cost occurs when 6 units are produced, and the minimum cost of production is $64.**

EXAMPLE 2: The revenue equation R for a business that is marketing widgets is $\mathbf{R = 100 + 12X - X^2}$ (R is the revenue in dollars from the sale of X widgets).

a) Notice that the graph of this revenue equation is a parabola opening **downward**, and therefore, there is a **maximum** point. Complete the square in order to find this maximum point.

b) Find the value of X that will maximize the revenue. Find the maximum revenue.

SOLUTION: a) $\mathbf{R = -X^2 + 12X + 100}$

$\mathbf{R\underline{\qquad} = -(X^2 - 12X + \underline{\qquad}) + 100}$

$\mathbf{R\underline{\ -\ 36\ } = -(X^2 - 12X + \underline{\ 36\ }) + 100}$

$\mathbf{R\underline{\ -\ 36\ }\ -\ 100 = (X^2 - 12X + \underline{\ 36\ })}$

$\mathbf{R - 136 = (X - 6)^2}$

Maximum point (6, 136).

b) **Maximum revenue occurs when 6 units are produced, and the maximum revenue is $136.**

EXAMPLE 3: A man has 200 feet of fence. He wishes to enclose a rectangular area along a river by fencing three sides, using the river as one length of the rectangle.

200-2X

X **X**

River

a) Find the equation for the enclosed area **A** in terms of the width **X**.

b) Notice that the equation is that of a parabola opening down. Complete the square to find the "vertex" or the "maximum point" of the "parabola."

c) For what value of X does the maximum occur, and what is the maximum area that can be obtained.

SOLUTION: a) $A = X(200-2X)$ or $A = -2X^2 + 200X$

b) $A = -2(X^2 - 100X \quad)$

$A \underline{\qquad} = -2(X^2 - 100X + \underline{\qquad})$

$A \underline{- 5000} = -2(X^2 - 100X + \underline{\ 2500\ })$

$A - 5000 = -2(X - 50)^2$ Vertex at (50, 5000)

c) **Maximum area occurs when width X = 50, length = 100, Maximum Area = L·W = 5000 sq ft.**

EXERCISES:

1. The cost equation C for producing X units of a product is given by $C = X^2 - 50X + 1000$. What value of X minimizes the cost? Find the minimum cost of production.

2. The cost equation C for producing X units of a product is given by $C = 5X^2 - 50X + 1000$. What value of X minimizes the cost? Find the minimum cost of production.

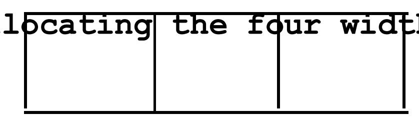
3. The revenue equation R for a product is $R = 1000 + 50X - X^2$. For what value of X is the revenue maximized? Find the maximum revenue.

4. The revenue equation R for a product is $R = 1000 + 50X - 5X^2$. For what value of X is the revenue maximized? Find the maximum revenue.

5. A woman has 400 feet of fence. She wishes to enclose a rectangular area along a river by fencing three sides, using the river as one length of the rectangle. Find the dimensions of the rectangle that will result in a maximum area. What is the maximum area obtained?

6. EXTRA CHALLENGE: A farmer wishes to enclose a rectangular area with two dividers down the center as indicated in the figure. If she has 800 feet of fence, what width and length will result in maximum area? Find the maximum area obtained.

Hint: Let X = each width. Find each length by taking half of the fence left after allocating the four widths.



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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE