4.04 *Solving Exponential Equations and Logarithmic Equations*

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ANSWERS TO ALL EXERCISES ARE INCLUDED AT THE END OF THIS PAGE

One very important application of logarithms is solving exponential equations--that is, equations that have variables in the exponents. Note, however, that sometimes **(see #1-6 in the exercises that follow)** it is possible to solve **exponential equations** by inspection when the base numbers of the equation are all powers of a common number. Otherwise, when base numbers of the equation are not powers of the same number, you may take the loarithm (any base, usually **base "e"** or **base "10"**) of both sides of the equation. Then, using the laws of logarithms, you may convert from an **exponential equation** to an **linear equation** which, with the help of a calculator, can easily be solved. The exercises will begin with equations that can be solved by inspection (because of a "common base number"), and then move into other equations in which the use of logarithms is required.

In exercises 1 - 20, solve the equations. Notice that in the first six exercises, the base numbers can be expressed as a power of some common number.

1. $4^X = 8^{X-2}$

$(2^2)^X = (2^3)^{X-2}$

$2^{2X} = 2^{3X-6}$

$2X = 3X - 6$

2. $8^X = 4^{X-2}$

3. $27^{X+2} = 9^{2X-1}$

4. $27^{X} = 81^{X+1}$

5. $8^{X-3} = 32^{X+1}$

6. $16^{X-5} = 8^{2X-1}$

Obviously, these are "special" (contrived!) problems, carefully selected so they can be solved by inspection. You could easily solve the equation $\mathbf{2^X = 16}$ (by inspection X = 4), but how do you solve $\mathbf{3^X = 16}$ which does not come out even? The procedure is to take the **ln** of both sides as follows:

7. $3^X = 16$

$\ln 3^X = \ln 16$ Take the **ln** of both sides!

$X \ln 3 = \ln 16$ By law of logarithms. Now it becomes a linear equation. Divide both sides by ln 3.

$\frac{X \ln 3}{\ln 3} = \frac{\ln 16}{\ln 3}$ Use a calculator to obtain the value.

X = ________

8. Although you already know by inspection that the answer is X=4, use the method of logarithms to compute:

$2^X = 16$

$\ln 2^X = \ln 16$ Take the **ln** of both sides!

X ( ) = ( ) By law of logarithms.

X = Use a calculator to obtain the value.

X = ________

9. $7^X = 49$

10. $3^X = 243$

11. $7^X = 98$

12. $3^X = 100$

13. $4^X = 3^{(X+2)}$

$\ln 4^X = \ln 3^{(X+2)}$

$X \ln 4 = (X + 2) \ln 3$

$X \ln 4 = X \ln 3 + 2 \ln 3$

$X \ln 4 - X \ln 3 = 2 \ln 3$

$X (\ln 4 - \ln 3) = 2 \ln 3$

X = ______________

= ______________

14. $8^X = 5^{(X+2)}$

15. $5^{X} = 8^{(X-2)}$

16. $10^{X} = 5^{(X-2)}$

17. $4^{(X-1)} = 27^{(X+1)}$

18. $7^{(2X-5)} = 10^{(X+4)}$

19. $9^{(X-1)} = 27^{(X+1)}$

20. $4^{(X+2)} = 32^{(X-4)}$

To solve logarithmic equations, use the laws of logarithms and the definition of logs to convert the equation to exponential form. Remember that any value of X that results in the log of a negative number must be rejected.

21. $\log_{10}(X) + \log_{10}(X + 3) = 1$

$$\log_{10} X(X + 3) = 1$$

$$10^1 = X(X + 3)$$

$$10 = X^2 + 3X$$

$$0 = X^2 + 3X - 10$$

0 = (   )(   )

X = ______ X = ______

[Note: You must reject X = -5, since in the original equation it results in the log of a negative number!]

22. $\log_{10}(X) + \log_{10}(X - 3) = 1$

23. $\log_{10}(X) + \log_{10}(X - 15) = 2$

24. $\log_{3}(X) + \log_{3}(X + 8) = 2$

25. $\log_{4}(X) + \log_{4}(X + 6) = 2$

26. $\log_3(X + 4) = \log_3(X) + 2$

$$\log_3(X + 4) - \log_3(X) = 2$$

$$\log_3 \frac{X + 4}{X} = 2$$

$$3^2 = \frac{X + 4}{X}$$

$$9X = X + 4$$

27. $\log_5(X + 4) = \log_5(X) + 1$

28. $\log_5(X) = \log_5(X + 4) + 2$

29. $\log_3(X - 2) = \log_3(X + 2) - 2$

30. $\log_3(X - 2) = \log_3(X + 2) + 2$

31. $\log_b(X + 4) - \log_b(X - 2) = \log_b(X)$

$$\log_b \frac{(X + 4)}{(X - 2)} = \log_b(X)$$

$$\frac{(X + 4)}{(X - 2)} = (X)$$ **[If $\log_b A = \log_b B$, then $A = B$]**

$\mathbf{X + 4 = X(X - 2)}$ [You solve it!]

[Hint: Reject X = -1.]

32. $\log_b(X + 6) - \log_b(X - 4) = \log_b(X)$

33. $\log_b(X - 6) - \log_b(X) = \log_b(X - 4)$

34. $\log_b(X^2 + 3) - \log_b(X) = \log_b(X + 7) - \log_b 2$

35. $\log_b(X - 2) + \log_b(X + 2) = \log_b(3) + \log_b(X)$

36. $\log_b(X + 4) + \log_b(X + 2) = \log_b(4) + \log_b(X + 8)$

In 37 - 46, use calculators to find the logs of other bases:

37. $\log_3 30 = X$

$3^X = 30$

[Take ln of both sides]

$\ln 3^X = \ln 30$

$X \ln 3 = \ln 30$

[Divide both sides by ln 3]

X = ___________

X = ___________

38. $\log_5 30 = X$

39. $\log_7 50 = X$

40. $\log_8 100 = X$

41. **Did you discover a pattern (short-cut or formula) in the previous exercises? Are all the steps illustrated in #37 necessary? Write a formula for $\log_b N$:**

$\log_b N =$ ____________

42. **Prove:** $\log_b N = \dfrac{\ln N}{\ln b}$

Let $Y = \log_b N$ [Now, proceed as in #37]

In 43 - 46, instead of using "ln N" and "ln b", try using "$\log_{10}$ N" and "$\log_{10}$ b". Compare results to #37-40.

43. $\log_3 30 = \dfrac{\log_{10} 30}{\log_{10} 3}$

= __________

44. $\log_5 30 =$

45. $\log_7 50 =$

46. $\log_8 100 =$

ANSWERS -- 4.04

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1. 6 2. −4 3. 8 4. −4 5. −7 6. $-\frac{17}{2}$ 7. 2.52

8. 4 9. 2 10. 5 11. 2.36 12. 4.19 13. 7.64 14. 6.85

15. 8.85 16. −4.64 17. −2.45 18. 11.92 19. −5 20. 8

21. 2 (Reject −5) 22. 5 (Reject −2) 23. 20 (Reject −5)

24. 1 (Reject −9) 25. 2 (Reject −8) 26. $\frac{1}{2}$ 27. 1

28. ∅ (Reject $-\frac{25}{6}$) 29. $\frac{5}{2}$ 30. ∅ (Reject $-\frac{5}{2}$) 31. 4 (Reject −1)

32. 6 (Reject −1) 33. ∅ (Reject 2, 3) 34. 1, 6

35. 4 (Reject −1) 36. 4 (Reject −6) 37. 3.10 38. 2.11

39. 2.01 40. 2.21 41. See 42 43. 3.10 44. 2.11 45. 2.01 46. 2.21

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