3.08 *Determinants and Cramer's Rule*

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A **matrix** is an array of numbers, usually enclosed within brackets. The following are all examples of matrices, which have a variety of applications in mathematics:

$$\begin{bmatrix} 3 & 2 \\ 6 & -2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & -3 \\ 4 & -3 & 5 \end{bmatrix} \quad \left[\begin{array}{ccc|c} 8 & 2 & 0 & 1 \\ -3 & 0 & 4 & 6 \\ 3 & 9 & -2 & 4 \end{array}\right] \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right]$$

Notice that matrices may consist of a number of rows and columns. If the number of rows in the matrix equals the number of columns, then it is called a **square matrix.**

The **determinant** of the 2 by 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined:

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. Notice the pattern $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Calculate the determinants:

1. $\begin{vmatrix} 6 & 3 \\ 2 & 5 \end{vmatrix}$

2. $\begin{vmatrix} 6 & 3 \\ -2 & 5 \end{vmatrix}$

3. $\begin{vmatrix} 5 & 2 \\ 5 & -2 \end{vmatrix}$

4. $\begin{vmatrix} -7 & 9 \\ 5 & 3 \end{vmatrix}$

5. $\begin{vmatrix} -13 & -2 \\ -8 & 3 \end{vmatrix}$

6. $\begin{vmatrix} 5 & -6 \\ 5 & -6 \end{vmatrix}$

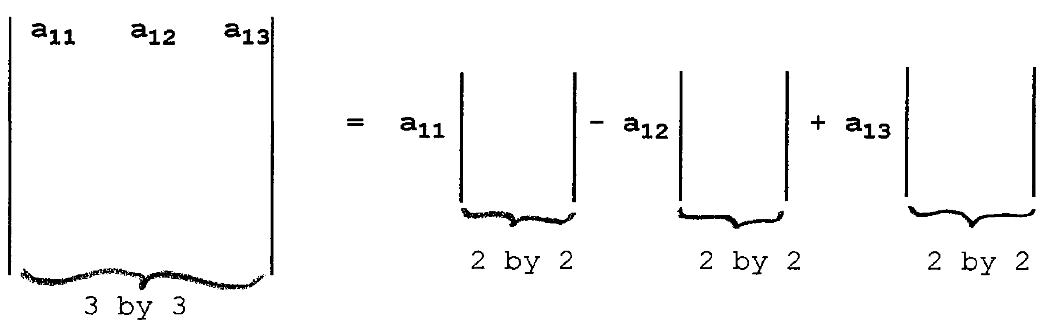
By the way, determinants can be defined <u>only</u> for square matrices that are 2 by 2 or larger.

The 3 by 3 determinant can be calculated by the following definition:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = + a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{32} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\begin{aligned} &= + a_{11}(a_{22}\, a_{33} - a_{23}\, a_{32}) \\ &\qquad - a_{12}(a_{21}\, a_{33} - a_{23}\, a_{31}) \\ &\qquad\qquad + a_{13}(a_{21}\, a_{32} - a_{22}\, a_{31}) \end{aligned}$$

Before panicking at the sight of this, notice some simple patterns. Notice that this expansion of the 3 by 3 determinant is in **three** parts with alternating signs, "+, −, +". The coefficients of these terms consist of $\mathbf{a_{11}}$, $\mathbf{-a_{12}}$, and $\mathbf{a_{13}}$ respectively--that is, the first row of the determinant.



The 2 by 2 determinants are called the **<u>minor determinants</u>** or the **<u>minors</u>** of the larger 3 by 3 determinant.

Now look at each minor to see where it is in the 3 by 3 determinant, especially in relation to the coefficient:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = + a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\begin{vmatrix} \mathbf{a_{11}} & a_{12} & a_{13} \\ a_{21} & \mathbf{a_{22}} & \mathbf{a_{23}} \\ a_{31} & \mathbf{a_{32}} & \mathbf{a_{33}} \end{vmatrix} \qquad \begin{vmatrix} a_{11} & \mathbf{a_{12}} & a_{13} \\ \mathbf{a_{21}} & a_{22} & \mathbf{a_{23}} \\ \mathbf{a_{31}} & a_{32} & \mathbf{a_{33}} \end{vmatrix} \qquad \begin{vmatrix} a_{11} & a_{12} & \mathbf{a_{13}} \\ \mathbf{a_{21}} & \mathbf{a_{22}} & a_{23} \\ \mathbf{a_{31}} & \mathbf{a_{32}} & a_{33} \end{vmatrix}$$

a_{11} minor **a_{12} minor** **a_{13} minor**

Here is a handy way to expand a 3 by 3 determinant by minors of the first row. Begin with a_{11} as the first coefficient. Mark out the row and column containing a_{11} as shown below:

$$\begin{vmatrix} \mathbf{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The remaining numbers form the 2 by 2 minor determinant: $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$.

Next, alternate signs, so use **$-a_{12}$** as the next coefficient. Mark out the row and column containing **a_{12}** as shown below:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The remaining numbers form the 2 by 2 minor determinant: $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$.

Finally, alternating signs back to positive, use the **a_{13}** as the last coefficient. Mark out the row and column containing **a_{13}** as shown below:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The remaining numbers form the 2 by 2 minor determinant: $\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$.

EXAMPLE: Calculate by expanding by minors:

$$\begin{vmatrix} 2 & 3 & 1 \\ 0 & 6 & 4 \\ 3 & -2 & 5 \end{vmatrix} = +2\begin{vmatrix} 6 & 4 \\ -2 & 5 \end{vmatrix} - 3\begin{vmatrix} 0 & 4 \\ 3 & 5 \end{vmatrix} + 1\begin{vmatrix} 0 & 6 \\ 3 & -2 \end{vmatrix}$$

$$= 2[30-(-8)] - 3[0-12] + 1[0-18]$$

$$= 2(38) - 3(-12) + 1(-18)$$

$$= 76 + 36 - 18$$

$$= 94$$

EXERCISES: Evaluate the determinants.

1.

$$\begin{vmatrix} 3 & 2 & 4 \\ 4 & 2 & -3 \\ -5 & 1 & 8 \end{vmatrix}$$

2.

$$\begin{vmatrix} 3 & 2 & 4 \\ -5 & 1 & 8 \\ 4 & 2 & -3 \end{vmatrix}$$

3.

$$\begin{vmatrix} 3 & -2 & -4 \\ -5 & 1 & 8 \\ 4 & 2 & -3 \end{vmatrix}$$

4.

$$\begin{vmatrix} 3 & -2 & -4 \\ -5 & 1 & 8 \\ -4 & -2 & 3 \end{vmatrix}$$

How does #2 differ from #1??

How does #4 differ from #3??

5.

$$\begin{vmatrix} 2 & -3 & -4 \\ 1 & 0 & 5 \\ -1 & 3 & 6 \end{vmatrix}$$

6.

$$\begin{vmatrix} 2 & -4 & -3 \\ 1 & 5 & 0 \\ -1 & 6 & 3 \end{vmatrix}$$

7.

$$\begin{vmatrix} 4 & 1 & -3 \\ 2 & -2 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

8.

$$\begin{vmatrix} 4 & 1 & -3 \\ 2 & -2 & 1 \\ 2 & 4 & 6 \end{vmatrix}$$

How does #6 differ from #5??
How does #8 differ from #7??

9.

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -3 & 2 & 1 \end{vmatrix}$$

10.

$$\begin{vmatrix} 4 & -2 & 1 \\ 4 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix}$$

11.

$$\begin{vmatrix} 4 & -2 & 1 \\ 5 & 1 & 3 \\ 0 & 0 & 0 \end{vmatrix}$$

12.

$$\begin{vmatrix} 1 & 2 & 0 \\ -3 & 1 & 0 \\ 4 & -5 & 0 \end{vmatrix}$$

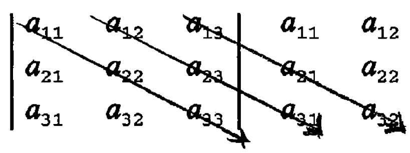
How are # 9 and #10 similar??

How are #11 and #12 similar??

Another method for evaluating a determinant is to re-write the first two columns to the right of the determinant as shown below:

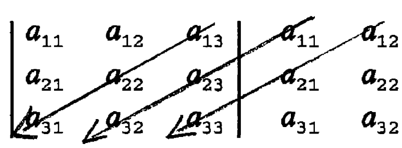
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

Next draw the following three diagonal lines:



which gives: $\mathbf{a_{11}\ a_{22}\ a_{33} + a_{12}\ a_{23}\ a_{31} + a_{13}\ a_{21}\ a_{32}}$.

Then draw three diagonal lines going "backward" as indicated below:



These give the "negative" terms:

$$\mathbf{- a_{13}\ a_{22}\ a_{31} - a_{11}\ a_{23}\ a_{32} - a_{12}\ a_{21}\ a_{33}}\ .$$

The final result is

$$\mathbf{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}}.$$

13.

$$\begin{vmatrix} 2 & -1 & 3 \\ -3 & 2 & 4 \\ 1 & 3 & 5 \end{vmatrix} \begin{matrix} 2 & -1 \\ -3 & 2 \\ 1 & 3 \end{matrix}$$

= 20 - 4 - 27 - 6 - 24 - 15

= ___________.

14.

$$\begin{vmatrix} 3 & 2 & -1 \\ 2 & 3 & 5 \\ -2 & 4 & 3 \end{vmatrix}$$

15.

$$\begin{vmatrix} 1 & -4 & 2 \\ 3 & 2 & 1 \\ -1 & 3 & 5 \end{vmatrix}$$

16.

$$\begin{vmatrix} 2 & 1 & -1 \\ 3 & 5 & -2 \\ -2 & 0 & 6 \end{vmatrix}$$

The most common methods of solving systems of linear equations (the **elimination method**, the **substitution method**, and the **graphical method**) were described in the previous section. Another method known as **Cramer's Rule** is an excellent application of **determinants**, an application that lends itself to the technology of the day.

According to **Cramer's Rule**, the following pair of equations

$$\mathbf{a_1X + b_1Y = c_1}$$
$$\mathbf{a_2X + b_2Y = c_2}$$

can be solved by simply calculating three determinants. The values of X and Y are "determined" (is this a pun?) so that each has the denominator $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$. To find the numerator of X, replace the X-coefficients with $\mathbf{c_1}$ and $\mathbf{c_2}$, respectively. To find the numerator of Y, replace the Y-coefficients with $\mathbf{c_1}$ and $\mathbf{c_2}$, respectively, as follows:

$$X = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad Y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

EXAMPLE 1: Solve by Cramer's Rule.

5X + 3Y = 14
9X + 4Y = 7

SOLUTION:

$$X = \frac{\begin{vmatrix} 14 & 3 \\ 7 & 4 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 9 & 4 \end{vmatrix}} = \frac{56 - 21}{20 - 27} \qquad Y = \frac{\begin{vmatrix} 5 & 14 \\ 9 & 7 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 9 & 4 \end{vmatrix}} = \frac{35 - 126}{20 - 27}$$

$$= \frac{35}{-7} = -5 \qquad = \frac{-91}{-7} = 13$$

(-5, 13)

EXAMPLE 2: Solve by Cramer's Rule.

5Y - 3X = 34
X = 7 - 2Y

First, you must rewrite the equations in standard form.

-3X + 5Y = 34
X + 2Y = 7

SOLUTION:

$$X = \frac{\begin{vmatrix} 34 & 5 \\ 7 & 2 \end{vmatrix}}{\begin{vmatrix} -3 & 5 \\ 1 & 2 \end{vmatrix}} = \frac{68 - 35}{-6 - 5} = \frac{33}{-11} = -3 \qquad Y = \frac{\begin{vmatrix} -3 & 34 \\ 1 & 7 \end{vmatrix}}{\begin{vmatrix} -3 & 5 \\ 1 & 2 \end{vmatrix}} = \frac{-21 - 34}{-6 - 5} = \frac{-55}{-11} = 5$$

(-3, 5)

EXAMPLE 3: Solve by the Cramer's Rule.

3X + 5Y = 2
6X + 10Y = -2

SOLUTION:

$$X = \frac{\begin{vmatrix} 2 & 5 \\ -2 & 10 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 6 & 10 \end{vmatrix}} = \frac{20 - (-10)}{30 - 30} \qquad Y = \frac{\begin{vmatrix} 3 & 2 \\ 6 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 6 & 10 \end{vmatrix}} = \frac{-6 - 12}{30 - 30}$$

Denominator Determinants = 0

Notice that in this case the denominator determinant is zero. Obviously, you can't divide by zero, and therefore, **<u>Cramer's Rule does NOT always apply</u>!!** Whenever the "denominator determinant" is zero, there will be a "parallel lines" or "same line" situation, and you must use another method to solve the problem. The **elimination method** may be used to show that this is the case of **two parallel lines.**

EXERCISES: In #1-12, solve the systems of equations by Cramer's Rule, if it applies.

1. $3X + 7Y = 6$
   $2X + 3Y = -1$

2. $-3X + 7Y = 4$
   $2X - 3Y = -6$

3. $9X - 4Y = 2$
   $2X + 5Y = -29$

4. $50X - 9Y = 1$
   $7X - 2Y = -8$

5. $2X - 6Y = 12$
   $-X + 3Y = -6$

6. $X = 3Y + 18$
   $6Y - 2X = 36$

7. $5X - 4Y = 22$
   $Y = -4X + 5$

8. $-8X + 6Y = 32$
   $X = 2Y + 6$

Cramer's Rule may also be applied to systems of three equations and three unknowns (and higher!), which will be studied in the next section. Extending to three variables requires the use of 3 by 3 determinants as follows.

Given:

$$\begin{aligned} a_{11}X + a_{12}Y + a_{13}Z &= a \\ a_{21}X + a_{22}Y + a_{23}Z &= b \\ a_{31}X + a_{32}Y + a_{33}Z &= c \end{aligned}$$

The denominator determinant is $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

To find the numerator of X, replace the X-coefficients with **a, b,** and **c.** Likewise, to find the numerator of Y, replace the Y-coefficients with **a, b,** and **c;** to find the numerator of Z, replace the Z-coefficients with **a, b,** and **c,** as follows:

$$X = \frac{\begin{vmatrix} a & a_{12} & a_{13} \\ b & a_{22} & a_{23} \\ c & a_{32} & a_{33} \end{vmatrix}}{D} \qquad Y = \frac{\begin{vmatrix} a_{11} & a & a_{13} \\ a_{21} & b & a_{23} \\ a_{31} & c & a_{33} \end{vmatrix}}{D} \qquad Z = \frac{\begin{vmatrix} a_{11} & a_{12} & a \\ a_{21} & a_{22} & b \\ a_{31} & a_{32} & c \end{vmatrix}}{D}$$

Of course, as before, if **D = 0**, then **Cramer's Rule does not apply**, and you must use the method of the next section. For now, an example and a very few exercises will be sufficient.

EXAMPLE 4: Solve the system using Cramer's Rule.

$$\begin{aligned} 3X + Y + Z &= 8 \\ 2X + 2Y - Z &= 10 \\ X - 3Y + 2Z &= -4 \end{aligned} \qquad D = \begin{vmatrix} 3 & 1 & 1 \\ 2 & 2 & -1 \\ 1 & -3 & 2 \end{vmatrix} = 3(4-3) -1(4+1) +1(-6-2)$$

$$X = \frac{\begin{vmatrix} 8 & 1 & 1 \\ 10 & 2 & -1 \\ -4 & -3 & 2 \end{vmatrix}}{D} \qquad Y = \frac{\begin{vmatrix} 3 & 8 & 1 \\ 2 & 10 & -1 \\ 1 & -4 & 2 \end{vmatrix}}{D} \qquad Z = \frac{\begin{vmatrix} 3 & 1 & 8 \\ 2 & 2 & 10 \\ 1 & -3 & -4 \end{vmatrix}}{D}$$

$$= \frac{8(4-3)-1(20-4)+1(-30+8)}{-10} = \frac{3(20-4)-8(4+1)+1(-8-10)}{-10} = \frac{3(-8+30)-1(-8-10)+8(-6-2)}{-10}$$

$$\mathbf{X} = \frac{8 - 16 - 22}{-10} = \mathbf{3} \qquad \mathbf{Y} = \frac{48 - 40 - 18}{-10} = \mathbf{1} \qquad \mathbf{Z} = \frac{66 + 18 - 64}{-10} = \mathbf{-2}$$

In each of the following exercises, use Cramer's Rule to solve the system of equations.

1. $$\begin{aligned} 2X + 3Y - 3Z &= 9 \\ 5X - 2Y - 8Z &= 6 \\ 4X - Y - 5Z &= -1 \end{aligned}$$

2. $$\begin{aligned} 3X + Y + Z &= 8 \\ 2X + 2Y - Z &= 10 \\ X - 3Y + 2Z &= -4 \end{aligned}$$

3. $$\begin{aligned} X - 5Y + Z &= 14 \\ -2X + Y + 2Z &= -6 \\ 4X + 4Y - Z &= 3 \end{aligned}$$

4. $$\begin{aligned} 3X + 2Y + Z &= 23 \\ 2X + Y + Z &= 11 \\ -X + 3Y + Z &= -10 \end{aligned}$$

5. $$\begin{aligned} 3X - 5Y &= 1 \\ 4X + 3Z &= 0 \\ 3Y + 2Z &= 2 \end{aligned}$$

6. $$\begin{aligned} 3X + 2Y &= -2 \\ 2Y - 3Z &= 1 \\ X - 2Y + 2Z &= 4 \end{aligned}$$

ANSWERS 3.08

p. 444: **1.** 24; **2.** 36; **3.** -20; **4.** -66; **5.** -55; **6.** 0..

p.448-452: **1.** 79; **2.** -79 (2nd and 3rd row interchanged);
3. -35; **4.** 35 (3rd row multiplied by -1);
5. -9; **6.** 9 (2nd and 3rd column interchanged);
7. -55; **8.** -110 (3rd row multiplied by 2);
9. 0; **10.** 0 (2 rows identical, determinant = 0);
11. 0; **12.** 0 (row or column of 0s, determinant = 0);
13. -56; **14.** -79; **15.** 93; **16.** 36.

p. 455: **1.** (-5,3); **2.** (-6,-2); **3.** (-2,-5); **4.** (2,11);
5. Cramer's Rule does not apply;
6. Cramer's Rule does not apply;
7. (2,-3); **8.** (-10,-8).

p.457-458: **1.** (-6,2,-5); **2.** (3,1,-2); **3.** (3,-2,1); **4.** (9,3,-10);
5. (-3,-2,4); **6.** (2,-4,-3).

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