MAC 2311 EXAM 5B **R^2** **NAME ______________________________**

Show all work on separate paper. Turn in ALL worksheets. Give all irrational answers in exact (radical) form. When you use the calculator, say so, and explain what you did.

In 1 – 5, find the derivatives and simplify. Show all work!

1. $y = \ln\,(x + \sqrt{4 + x^2})$

2. $y = \ln \sqrt{\dfrac{4 + x^2}{x}}$

3. $y = x^2 e^{-5x}$

4. $y = x^{\sin 2x}$ (Use logarithms)

5. $y = \ln \dfrac{\sqrt{x^2 + 1}}{x(2x^3 - 1)^2}$ (Use log differentiation. You need NOT find LCD, etc.)

In 6 – 8, evaluate the integrals. Show all work by algebraic techniques!

6. $\displaystyle\int \frac{x}{\sqrt{x^2 + 4}}\, dx$

7. $\displaystyle\int \frac{x}{x^2 + 4}\, dx$

8. $\displaystyle\int e^x \sqrt{9 - e^x}\, dx$

9. Solve the equations. Give answer in exact form and also calculate decimal approximations.

   a) $3^{2x-5} = 75$

   b) $\ln(2x - 5) = 3$

10. Derive formulas for the following by using logarithmic and implicit differentiation. (As in #4.)

    a) Find the derivative of $y = a^x$, where a is a constant.

    b) Find the derivative of $y = x^x$, where *x* (of course!) is the variable.

11. Solve the differential equation: $y\,y' = \sin x$.

12. Solve the differential equation: $y' = 3y$ given the condition that y(1) = 20.

13a) How much money must be invested now at 6% annual rate compounded semiannually, in order to amount to $100,000 in 20 years? (See formulas below.)

b) How much money must be invested now at 6% annual rate compounded continuously, in order to amount to $100,000 in 20 years?

14. A certain type of bacteria increases continuously at a rate proportional to the number present. If there are 500 present at a given time, and 1000 present 2 hours later, how many will there be 5 hours from the initial time given?

15a) What is the underlying mathematical assumption for the exponential growth formula $y = y_0 e^{kt}$?

b) Give the major fallacy of population growth and decay solutions.

c) Beginning with $\frac{dy}{dx} = ky$, show that the solution of this differential equation is $y = y_0 e^{kt}$.

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \qquad A = Pe^{rt} \qquad y = y_0 e^{kt}$$

MAC 2311 EXAM 5B Solutions

1. $y=\ln(x+\sqrt{4+x^2})$

$y'=\frac{1}{x+\sqrt{4+x^2}}\cdot\left(1+\frac{1}{2}(4+x^2)^{-1/2}\cdot 2x\right)$

$=\frac{1}{x+\sqrt{4+x^2}}\cdot\left(1+\frac{x}{\sqrt{4+x^2}}\right)$

$=\frac{1}{\cancel{x+\sqrt{4+x^2}}}\cdot\frac{\cancel{\sqrt{4+x^2}+x}}{\sqrt{4+x^2}}$

$=\frac{1}{\sqrt{4+x^2}}$

2. $y=\ln\sqrt{\frac{4+x^2}{x}}=\ln\left(\frac{4+x^2}{x}\right)^{1/2}$

$y=\frac{1}{2}\left[\ln(4+x^2)-\ln x\right]$

$y'=\frac{1}{2}\left[\frac{1}{4+x^2}\cdot 2x-\frac{1}{x}\right]$

$=\frac{1}{2}\left[\frac{2x}{4+x^2}\frac{x}{x}-\frac{1}{x}\frac{(4+x^2)}{(4+x^2)}\right]$

$=\frac{1}{2}\left[\frac{2x^2-4-x^2}{x(4+x^2)}\right]=\frac{(x^2-4)}{2x(x^2+4)}$

3. $y=x^2e^{-5x}$ (Product Rule)

$y'=x^2\cdot e^{-5x}(-5)+e^{-5x}\cdot 2x$

$=xe^{-5x}(-5x+2)$

4. $y=x^{\sin 2x}\Rightarrow \ln y=\ln x^{\sin 2x}$

$\ln y=\sin 2x\cdot\ln x$

$\frac{1}{y}y'=\sin 2x\cdot\frac{1}{x}+\ln x\cdot\cos 2x\cdot 2$

$y'=y\left(\frac{\sin 2x}{x}+2\cos 2x\ln x\right)$

$y'=x^{\sin 2x}\left(\frac{\sin 2x+2x\cos 2x\ln x}{x}\right)$

5. $y=\ln\frac{\sqrt{x^2+1}}{x(2x^3-1)^2}$

$y=\frac{1}{2}\ln(x^2+1)-\ln x-2\ln(2x^3-1)$

$y'=\frac{1}{\cancel{2}(x^2+1)}\cdot\cancel{2}x-\frac{1}{x}-2\cdot\frac{1}{2x^3-1}\cdot 6x^2$

$y'=\frac{x}{x^2+1}-\frac{1}{x}-\frac{12x^2}{2x^3-1}$

6. $\int\frac{x}{\sqrt{x^2+4}}dx$ let $u=x^2+4$, $du=2x\,dx$, $\frac{du}{2}=x\,dx$

$=\int\frac{\frac{du}{2}}{u^{1/2}}$

$=\frac{1}{2}\int u^{-1/2}du$

$=\frac{1}{2}\frac{2}{1}u^{1/2}+C$

$=(x^2+4)^{1/2}+C$

7. $\int\frac{x}{x^2+4}dx$

$\int\frac{\frac{du}{2}}{u}$

$=\frac{1}{2}\int\frac{1}{u}du$

$=\frac{1}{2}\ln u+C$

$=\frac{1}{2}\ln(x^2+4)+C$

8. $\int e^x\sqrt{9-e^x}\,dx$ let $u=9-e^x$, $du=-e^x dx$, $-du=e^x dx$

$\int u^{1/2}(-du)$

$-\frac{2u^{3/2}}{3}+C$

$-\frac{2}{3}(9-e^x)^{3/2}+C$

9a) $3^{2x-5}=75$

$\ln 3^{2x-5}=\ln 75$

$(2x-5)\ln 3=\ln 75$

$2x\ln 3-5\ln 3=\ln 75$

$\frac{2x\ln 3}{2\ln 3}=\frac{\ln 75+5\ln 3}{2\ln 3}$

$x\approx 4.465$

b) $\ln(2x-5)=3$

$e^3=2x-5$

$e^3+5=2x$

$x=\frac{e^3+5}{2}\approx 12.543$

10a) $y=a^x$

$\ln y=\ln a^x$

$\ln y=x\ln a$

$\frac{1}{y}y'=\ln a$

$y'=y\ln a$

$y'=a^x\ln a$

b) $y=x^x$

$\ln y=\ln x^x$

$\ln y=x\ln x$

$\frac{1}{y}y'=x\cdot\frac{1}{x}+\ln x\cdot 1$

$y'=y(1+\ln x)=x^x(1+\ln x)$

11. $yy'=\sin x$

$y\frac{dy}{dx}=\sin x$

$\int y\,dy=\int\sin x\,dx$

$\frac{y^2}{2}=-\cos x+C$

$y^2=-2\cos x+C$

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12. $y' = 3y \quad y(1) = 20$

$\frac{dy}{dx} = 3y$

$\frac{dy}{y} = 3dx$

$\ln y = 3x + c$

$e^{\ln y} = e^{(3x+c)}$

$y = e^{3x} \cdot e^{c}$

$y = c_1 e^{3x}$

$20 = c_1 e^{3}$

$c_1 = \frac{20}{e^3}$

$y = \frac{20}{e^3} e^{3x}$

$y = 20e^{3x-3}$

13a) $A = P\left(1+\frac{r}{n}\right)^{nt}$

$100{,}000 = P\left(1+\frac{.06}{2}\right)^{2\cdot 20}$

$100{,}000 = P(1.03)^{40}$

$P = \frac{100{,}000}{(1.03)^{40}} = \$30{,}655.68$

b) $A = Pe^{rt}$

$100{,}000 = Pe^{(.06)(20)}$

$P = \frac{100{,}000}{e^{1.2}} = \$30{,}119.42$

14. $y = y_0 e^{kt}$ $\quad y_0 = 500$, $y = 1000$ at $t = 2$

$1000 = 500e^{2k}$

$2 = e^{2k}$

$\ln 2 = \ln e^{2k}$

$\ln 2 = 2k$

$k = \frac{\ln 2}{2} \approx .3465735903\ldots$

$\boxed{STO} \rightarrow k$

$y = 500\,e^{(5k)}$

$y \approx 2828.427$

2828 or 2829

15a) Mathematical assumption is that the growth rate of the population varies directly as the population. (i.e., the more you have the more you get!)

b) Major fallacy – that the growth rate "k" is a constant!

c) $\frac{dy}{d} = ky$

$\frac{dy}{y} = k\,dt$

$\int \frac{dy}{y} = \int k\,dt$

$\ln y = kt + c$

$e^{\ln y} = e^{(kt+c)}$

$y = e^{kt} \cdot e^{c}$

$y = c_1 e^{kt}$

when $t=0$, $y = y_0$, so $c_1 = y_0$

$y_0 = c_1 e^{0}$

$y = y_0 e^{kt}$