CALCULUS I EXAM 6 Sections 5.7 – 6.3 (Larson/Hostetler)

Show all work on separate paper. CALCULATORS ALLOWED.
You may keep this copy of the test and your answers. (One problem free)

In 1–4, find $\frac{dy}{dx}$ and simplify:

1. $y = \ln\left(x + \sqrt{4+x^2}\right)$

2. $y = \ln\sqrt{\frac{4-x^2}{x}}$

3. $y = xe^{x^2-5x}$

4. $y = x^{\sin 2x}$

In 5–7, perform the integration:

5. $\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})}$ 6. $\int e^{2x}\sqrt{1+e^{2x}}\,dx$ 7. $\int \frac{2e^{x}-2e^{-x}}{e^{x}+e^{-x}}\,dx$

8. Use the fact that $\ln 2 \approx .7$ and $\ln 3 \approx 1.1$ to find approximations without calculators for:
   a) $\ln 72$ b) $\ln \sqrt[3]{12}$

9. Find $\frac{dF}{dx}$ for $F(x) = \int_0^x (t^2 - 4t + 3)\,dt$

10. Show the equivalence of $\int \sec x\,dx = \ln|\sec x + \tan x| + c$ and $\int \sec x\,dx = -\ln|\sec x - \tan x| + c$
(You need not verify the integrals!)

11. Find $f^{-1}(x)$ for $f(x) = \frac{-2}{\sqrt{x^2-9}}$. Give domain and range of f and f^{-1}!

12. Find the tangent line and normal line to the graph $y = 2e^{-3x}$ at $(0,1)$

13. Prove that if $y = a^x$, then $\frac{dy}{dx} = a^x \ln a$. (Hint: You must take the ln of both sides. Use of formula sheet formula is not a proof!!)

14. ~~Integrate~~: $\int \frac{e^{-x}}{1+e^{-x}}\,dx$. Show that the result is equivalent to $x - \ln(e^x+1) + c$.

15. In 1960, the population of a town was 2500, and in 1970 it was 3500. The population increases at a rate proportional to the existing population (i.e., $y = y_0e^{kt}$). Find the equation of growth, and the population in the year 2000.

CALCULUS I EXAM 6A Solutions

1. $y = \ln\left(x + \sqrt{4+x^2}\right)$

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{4+x^2}} \cdot \left[1 + \tfrac{1}{2}(4+x^2)^{-1/2} \cdot 2x\right]$$
$$= \frac{1}{x+\sqrt{4+x^2}} \cdot \left[1 + \frac{x}{\sqrt{4+x^2}}\right]$$
$$= \frac{1}{x+\sqrt{4+x^2}} \cdot \frac{\sqrt{4+x^2}+x}{\sqrt{4+x^2}}$$
$$= \frac{1}{\sqrt{4+x^2}}$$

2. $y = \ln\sqrt{\dfrac{4-x^2}{x}}$

$$y = \tfrac{1}{2}\left[\ln(4-x^2) - \ln x\right]$$
$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{4-x^2}(-2x) - \frac{1}{x}\right]$$
$$= \frac{1}{2}\left[\frac{-2x^2-4+x^2}{x(4-x^2)}\right]$$
$$= -\frac{1}{2}\left(\frac{x^2+4}{x(4-x^2)}\right)$$

Better yet: $\dfrac{x^2+4}{2x(x^2-4)}$

3. $y = x^2 e^{-5x}$

$$\frac{dy}{dx} = x^2e^{-5x}(-5) + e^{-5x}\cdot 2x$$
$$= e^{-5x}(-5x^2+2x)$$

or $xe^{-5x}(2-5x)$

4. $y = x^{\sin 2x}$

$$\ln y = \ln x^{\sin 2x}$$
$$\ln y = (\sin 2x)(\ln x)$$
$$\frac{1}{y}y' = \sin 2x \cdot \frac{1}{x} + \ln x(\cos 2x)\cdot 2$$
$$y' = x^{\sin 2x}\left(\frac{\sin 2x + 2x\ln x\cos 2x}{x}\right)$$

5. $\displaystyle\int \frac{dx}{\sqrt{x}(1+\sqrt{x})}$ Let $u = 1+\sqrt{x}$, $du = \tfrac{1}{2}x^{-1/2}dx$, $2du = \dfrac{dx}{\sqrt{x}}$

$$= \int \frac{2\,du}{u}$$
$$= 2\ln u + C$$
$$= 2\ln(1+\sqrt{x}) + C$$

6. $\displaystyle\int e^{2x}\sqrt{1+e^{2x}}\,dx$ Let $u = 1+e^{2x}$, $du = 2e^{2x}dx$, $\dfrac{du}{2} = e^{2x}dx$

$$= \int u^{1/2}\frac{du}{2}$$
$$= \frac{2u^{3/2}}{3\cdot 2} + C$$
$$= \frac{1}{3}\left(1+e^{2x}\right)^{3/2} + C$$

7. $\displaystyle\int \frac{2e^x - 2e^{-x}}{e^x+e^{-x}}\,dx$ Let $u = e^x+e^{-x}$, $du = (e^x - e^{-x})dx$

$$= \int \frac{2\,du}{u} = 2\ln u + C$$
$$= 2\ln(e^x+e^{-x}) + C$$

8a) $\ln 72 = \ln 9\cdot 8$
$= \ln 3^2 + \ln 2^3$
$= 2\ln 3 + 3\ln 2$
$= 2(1.1) + 3(.7)$
$= 2.2 + 2.1 = 4.3$

b) $\ln\sqrt[3]{12}$
$= \frac{1}{3}\ln 2^2\cdot 3$
$= \frac{1}{3}\left[2\ln 2 + \ln 3\right]$
$= \frac{1}{3}\left[1.4 + 1.1\right]$
$= \frac{2.5}{3} = .833$

9. $F = \displaystyle\int_0^x (t^2-4t+3)\,dt$

$$\frac{dF}{dx} = x^2-4x+3$$

10. $-\ln|\sec x - \tan x| = \ln\dfrac{1}{|\sec x - \tan x|}\cdot\dfrac{|\sec x+\tan x|}{|\sec x+\tan x|}$

$$= \ln\frac{|\sec x+\tan x|}{\sec^2 x - \tan^2 x}$$
$$= \ln\frac{|\sec x+\tan x|}{1}$$

11. $f(x) = \dfrac{-2}{\sqrt{x^2-9}}$ D: $x>3$ R: $y<0$ Right half (Lower half)

$$y = \frac{-2}{\sqrt{x^2-9}}$$
$$y^2 = \frac{4}{x^2-9}$$
$$x^2y^2 - 9y^2 = 4$$
$$x^2y^2 = 4+9y^2$$
$$x^2 = \frac{9y^2+4}{y^2}$$
$$x = \pm\frac{\sqrt{9y^2+4}}{y}$$

$x = -\dfrac{\sqrt{9y^2+4}}{y}$ Use the upper left half.

$f^{-1}(x) = -\dfrac{\sqrt{9x^2+4}}{x}$

D: $x<0$ (left only)

R: $f^{-1}>3$

12. $y = 2e^{-3x}$, $(0,2)$

$$\frac{dy}{dx} = -6e^{-3x}$$

$m = -6$ yint $=2$.

TANGENT LINE: $y = -6x+2$

NORMAL LINE: $m = \frac{1}{6}$ yint $=2$.

$y = \frac{1}{6}x+2$

13. $y = a^x$

$$\ln y = \ln a^x$$
$$\ln y = x\ln a$$
$$\frac{1}{y}\frac{dy}{dx} = \ln a$$
$$\frac{dy}{dx} = y\ln a = a^x\ln a$$

14. $\displaystyle\int \frac{e^{-x}}{1+e^{-x}}\,dx$ Let $u = 1+e^{-x}$, $du = -e^{-x}dx$

$$= \int -\frac{du}{u} = -\ln(1+e^{-x}) + C$$
$$= \ln\frac{1}{1+e^{-x}} + C$$
$$= \ln\frac{e^x\cdot 1}{e^x[1+e^{-x}]} + C$$
$$= \ln\frac{e^x}{e^x+1} + C = \ln e^x - \ln(e^x+1) + C$$
$$= x - \ln(e^x+1) + C$$

15. $y = y_0e^{kt}$

$y = 2500e^{kt}$

$3500 = 2500e^{10k}$

$1.4 = \frac{7}{5} = e^{10k} = (e^k)^{10}$

$e^k = (1.4)^{1/10}$

or $k = \frac{1}{10}\ln(1.4) \approx .0336$

$y = 2500e^{\frac{\ln 1.4}{10}t}$ or $y = 2500(e^k)^t$

$y = 2500(1.4)^{t/10}$

Either way, when $t = 40$

$y = 2500e^{40\left(\frac{\ln 1.4}{10}\right)} \approx 9604$ OR $y = 2500(1.4)^4 \approx 9604$

(Use calculator value – DON'T ROUND OFF)