CALCULUS II EXAM 5 A Dr. RAPALJE--SCC at HUNT CLUB

SHOW ALL WORK ON SEPARATE PAPER. Justify and circle all answers.
Where calculators are used, describe procedures, etc.
Formula sheets are allowed.

In 1-2, find the interval of convergence of the power series.
Check endpoints.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{n\, 5^n}$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

In 3-4, find geometric series for the function centered at $c=0$.
Give general form.

3. $f(x) = \dfrac{1}{2x-5}$

4. $f(x) = \dfrac{4}{4+x^2}$

5. Use the definition (derivatives method) to find the Taylor Series for $f(x) = \frac{1}{x}$ centered at $x=1$.

6. Find a Taylor Series for $f(x) = e^{-x^2}$ (center at $x=0$).

In 7-8, find a power series (expanded and general form.)

7. $\displaystyle\int \cos(t^2)\,dt$

8. $\displaystyle\int \frac{\arctan t}{t}\,dt$

Deriving Taylor Series from a Basic List

In the following list, we provide the power series for several elementary functions with the corresponding intervals of convergence.

Power Series for Elementary Functions

Function	*Interval of convergence*
$\dfrac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots + (-1)^n(x-1)^n + \cdots,$	$0 < x < 2$
$\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots,$	$-1 < x < 1$
$\ln x = (x-1) - \dfrac{(x-1)^2}{2} + \dfrac{(x-1)^3}{3} - \dfrac{(x-1)^4}{4} + \cdots + \dfrac{(-1)^{n-1}(x-1)^n}{n} + \cdots,$	$0 < x \le 2$
$e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} + \cdots + \dfrac{x^n}{n!} + \cdots,$	$-\infty < x < \infty$
$\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dfrac{x^9}{9!} - \cdots + \dfrac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots,$	$-\infty < x < \infty$
$\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!} - \cdots + \dfrac{(-1)^n x^{2n}}{(2n)!} + \cdots,$	$-\infty < x < \infty$
$\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dfrac{x^9}{9} - \cdots + \dfrac{(-1)^n x^{2n+1}}{2n+1} + \cdots,$	$-1 \le x \le 1$
$\arcsin x = x + \dfrac{x^3}{2 \cdot 3} + \dfrac{1 \cdot 3 x^5}{2 \cdot 4 \cdot 5} + \dfrac{1 \cdot 3 \cdot 5 x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots + \dfrac{(2n)!\, x^{2n+1}}{(2^n n!)^2 (2n+1)} + \cdots,$	$-1 \le x \le 1$
$(1+x)^k = 1 + kx + \dfrac{k(k-1)x^2}{2!} + \dfrac{k(k-1)(k-2)x^3}{3!} + \dfrac{k(k-1)(k-2)(k-3)x^4}{4!} + \cdots,$	$-1 < x < 1$*

* The convergence at $x = \pm 1$ depends on the value of k.

CALCULUS II EXAM 5A Solutions Dr. Rapalje

1. $\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{n\,5^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{(n+1)\,5^{n+1}} \cdot \frac{n\,5^n}{(x-3)^n}\right| < 1$$

$$= \lim_{n\to\infty}\left|\frac{n}{n+1}\,\frac{(x-3)}{5}\right| < 1$$

$$|x-3| < 5$$
$$-5 < x-3 < 5$$
$$-2 < x < 8$$

Endpts:

$x=-2$

$$\sum_{n=1}^{\infty} \frac{(-1)^n(-5)^n}{n\,5^n} = \sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$$ Diverges

$x=8$

$$\sum_{n=1}^{\infty}\frac{(-1)^n 5^n}{n\,5^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$ Converges

$-2 < x \le 8$

2. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(n+1)!}\cdot\frac{n!}{x^{2n}}\right| < 1$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{n+1}\right| < 1$$

All real x

$-\infty < x < \infty$

3. $f(x) = \frac{-\frac{1}{5}\cdot 1}{-\frac{1}{5}(-5+2x)} = \frac{-\frac{1}{5}}{1-\frac{2x}{5}}$

Geometric with $a = -\frac{1}{5}$, $r = \frac{2x}{5}$

$$= \sum_{n=0}^{\infty} -\frac{1}{5}\left(\frac{2x}{5}\right)^n = \sum_{n=0}^{\infty}\frac{-2^n x^n}{5^{n+1}}$$

4. $f(x) = \frac{4}{4+x^2} = \frac{1}{1+\frac{x^2}{4}}$

$$\sum_{n=0}^{\infty} 1\cdot\left(\frac{-x^2}{4}\right)^n = \sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^{2n}$$

5. $f(x) = x^{-1}$ $\quad f(1) = 1$

$f'(x) = -x^{-2}$ $\quad f'(1) = -1$

$f''(x) = 2x^{-3}$ $\quad f''(1) = 2$

$f'''(x) = -3\cdot 2\cdot x^{-4}$ $\quad f'''(1) = -3!$

$f^{n}(x) = \frac{(-1)^n n!}{x^{n+1}}$ $\quad f^{n}(1) = (-1)^n n!$

$$f(x) = 1 - 1(x-1) + \frac{\cancel{2}}{\cancel{2!}}(x-1)^2 - \frac{\cancel{3!}}{\cancel{3!}}(x-1)^3 + \ldots$$

$$= \sum_{n=0}^{\infty}(-1)^n(x-1)^n$$

6. $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

$$e^{-x^2} = \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!}$$

$$e^{-x^2} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}$$

7. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$

$$\cos t^2 = 1 - \frac{t^4}{2!} + \frac{t^8}{4!} - \frac{t^{12}}{6!} + \ldots = \sum_{n=0}^{\infty}\frac{t^{4n}}{(2n)!}$$

$$\int \cos t^2\,dt = t - \frac{t^5}{5\cdot 2!} + \frac{t^9}{9\cdot 4!} - \frac{t^{13}}{13\cdot 6!} + \ldots = \sum_{n=0}^{\infty}\frac{t^{4n+1}(-1)^n}{(4n+1)(2n)!} + C$$

8. $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)}$$

$$\frac{\arctan t}{t} = 1 - \frac{t^2}{3} + \frac{t^4}{5} - \frac{t^6}{7} \ldots$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{2n+1}$$

$$\int\frac{\arctan t}{t}\,dt = t - \frac{t^3}{3\cdot 3} + \frac{t^5}{5\cdot 5} - \frac{t^7}{7^2}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)^2} + C$$