CALCULUS II EXAM 3B NAME Dr. RAPALJE

Calculators, Formula Sheets allowed.
Please turn in this test with your work sheets
Show all work on separate paper.

1. Give a set of parametric equations in standard form for a straight line passing through $(3,-7)$ and $(-3,-3)$.

2. Give the x,y equations:

a) $x = 4\cos^2 t$
$y = 2\sin t$

b) $x = t^2 + t$
$y = t^2 - t$

3. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of the parameter:

$x = 2 + \sec\theta \qquad y = 1 + 2\tan\theta.$ Evaluate them at $t = \frac{\pi}{6}$.

4. Find the arc length:

a) $x = 2t^2 \qquad y = 2t^3$
from $t = 0$ to $t = 2$.

b) $y = \sqrt{36 - x^2}$, $0 \le x \le 4$.

5. Find the curvature $K(x)$ and radius of curvature $R(x)$ for $x = \theta - \sin\theta \qquad y = 1 - \cos\theta$

6. Find the point of maximum curvature for $y = e^x$

7. Find the center of curvature for $y = x^2$ at $(1,1)$

CALCULUS II EXAM 3B Solutions — Dr. RAPALJE

1. $(3,-7)$ $(-3,-3)$

$m = \frac{-3+7}{-3-3} = \frac{-4}{6} = -\frac{2}{3}$

Let $b=2$, $a=-3$

$k = \frac{1}{\sqrt{a^2+b^2}} = \frac{1}{\sqrt{13}}$

$x = x_0 + akt \quad y = y_0 + bkt$

$x = 3 - \frac{3}{\sqrt{13}}t \quad y = -7 + \frac{2}{\sqrt{13}}t$

or

$x = -3 - \frac{3}{\sqrt{13}}t \quad y = -3 + \frac{2}{\sqrt{13}}t$

2a) $x = 4\cos^2 t$, $y = 2\sin t$

$\cos^2 t = \frac{x}{4}$

$\sin^2 t = \frac{y^2}{4}$

$\frac{x}{4} + \frac{y^2}{4} = 1$

$x + y^2 = 4$

b) $x = t^2 + t$, $y = t^2 - t$

$x - y = 2t \quad t = \frac{x-y}{2}$

$x = \frac{(x-y)^2}{4} + \frac{x-y}{2}$

$4x = x^2 - 2xy + y^2 + 2x - 2y$

$0 = x^2 - 2xy + y^2 - 2x - 2y$

3. $x = 2 + \sec\theta \quad y = 1 + 2\tan\theta$

$\frac{dx}{d\theta} = \sec\theta\tan\theta \quad \frac{dy}{d\theta} = 2\sec^2\theta$

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2\sec^2\theta}{\sec\theta\tan\theta} = 2\sec\theta\cot\theta =$

$= 2\frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta} = 2\csc\theta = 4$ at $\frac{\pi}{6}$

$\frac{d}{d\theta}\left(\frac{dy}{dx}\right) = -2\csc\theta\cot\theta =$

$\frac{d^2y}{dx^2} = \frac{-2\csc\theta\cot\theta}{\sec\theta\tan\theta} = \frac{-2\frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta}}{\frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta}}$

$= -2\frac{\cos\theta}{\sin^2\theta}\cdot\frac{\cos^3\theta}{\sin\theta} = -2\cot^3\theta$

at $\frac{\pi}{6} \quad = -2(\sqrt{3})^3 = -2\cdot 3\sqrt{3} = -6\sqrt{3}$

4a) $x = 2t^2 \quad y = 2t^3 \quad 0 \le t \le 2$

$x' = 4t \quad y' = 6t^2$

$s = \int_0^2 \sqrt{16t^2 + 36t^4}\,dt$

$= \int_0^2 \sqrt{4t^2(4+9t^2)}\,dt$

$= \int_0^2 2t\sqrt{4+9t^2}\,dt$ let $u = 4+9t^2$, $du = 18t\,dt$, $\frac{du}{9} = 2t\,dt$

$= \int_4^{40} u^{\frac{1}{2}}\frac{du}{9}$

$= \frac{1}{9}\frac{2}{3}u^{3/2}\Big|_4^{40} = \frac{2}{27}\left[40^{3/2} - 4^{3/2}\right]$

$= \frac{2}{27}\left(40\sqrt{40} - 4\sqrt{4}\right)$

$= \frac{16}{27}\left(10\sqrt{10} - 1\right)$

4b) $y = \sqrt{36 - x^2} \quad 0 \le x \le 4$

$y' = \frac{-x}{\sqrt{36-x^2}}$

$s = \int_0^4 \sqrt{1 + \frac{x^2}{36-x^2}}\,dx$

$= \int_0^4 \sqrt{\frac{36-x^2+x^2}{36-x^2}}\,dx$

$= \int_0^4 \frac{6}{\sqrt{36-x^2}}\,dx = 6\arcsin\frac{x}{6}\Big|_0^4$

$= 6\arcsin\frac{2}{3}$

5. $x = \theta - \sin\theta \quad y = 1 - \cos\theta$

$x' = 1 - \cos\theta \quad y' = \sin\theta$

$x'' = \sin\theta \quad y'' = \cos\theta$

$k = \frac{(1-\cos\theta)(\cos\theta) - \sin\theta(\sin\theta)}{\left((1-\cos\theta)^2 + \sin^2\theta\right)^{3/2}}$

$= \frac{\cos\theta - \cos^2\theta - \sin^2\theta}{\left[1 - 2\cos\theta + \cos^2\theta + \sin^2\theta\right]^{3/2}} = \frac{\cos\theta - 1}{(2-2\cos\theta)^{3/2}}$

$R = \frac{(2-2\cos\theta)^{3/2}}{|\cos\theta - 1|}$

6. $y = e^x$

$y' = e^x$

$y'' = e^x$

$$K = \frac{e^x}{(1+e^{2x})^{3/2}}$$

$$K' = \frac{(1+e^{2x})^{3/2} e^x - e^x \frac{3}{2}(1+e^{2x})^{1/2} e^{2x} \cdot 2}{(1+e^{2x})^3}$$

$$= \frac{e^x (1+e^{2x})^{1/2}\left[1+e^{2x} - 3e^{2x}\right]}{(1+e^{2x})^3}$$

$$= \frac{e^x(1-2e^{2x})}{(1+e^{2x})^{5/2}} = 0 \text{ at } 1-2e^{2x} = 0$$

$$1 = 2e^{2x}$$

$$e^{2x} = \frac{1}{2}$$

$$2x = \ln\frac{1}{2}$$

$$x = \frac{1}{2}\ln\frac{1}{2}$$

$$= -\frac{1}{2}\ln 2$$

$$= \ln 2^{-1/2}$$

$$= \ln\frac{1}{\sqrt{2}}$$

$y = e^x$,

$y = e^{\ln 2^{-1/2}} = 2^{-1/2}$

$y = \frac{1}{\sqrt{2}}$

$\left(\ln\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

7. $y = x^2$

$y' = 2x \quad m_T = 2 \quad m_N = -\frac{1}{2}$

$y'' = 2$

If $x = 1/2$

$$K = \frac{2}{(1+4x^2)^{3/2}} = \frac{2}{5^{-3/2}}$$

$$R = \frac{5^{3/2}}{2}$$

$y - 1 = m(x-1)$ = Eq. Normal Line.

$y - 1 = -\frac{1}{2}(x-1)$

$(y-1)^2 = \frac{1}{4}(x-1)^2$

$(x-1)^2 + (y-1)^2 = \frac{125}{4}$ Circle

$(x-1)^2 + \frac{1}{4}(x-1)^2 = \frac{125}{4}$

$\frac{5}{4}(x-1)^2 = \frac{125}{4}$

$(x-1)^2 = 25$

$x - 1 = \pm 5$

$x = 1 \pm 5$

$x = 6$ $\quad (x = -4)$

$y - 1 = -\frac{1}{2}(x-1)$

$= -\frac{1}{2}(-5)$

$y = 5/2 + 1 = (7/2)$