CALCULUS II EXAM 1C (Chap 6-7) Dr. RAPALJE

Show all work on separate paper. Calculators, gold sheets, trig sheets are allowed.

1. $\int_1^3 \frac{e^{3/x}}{x^2}\,dx$

2. Find $\frac{dy}{dx}$ for $y = x \arcsin x + \sqrt{1-x^2}$

3. Prove that if $y = \arctan x$, then $\frac{dy}{dx} = \frac{1}{1+x^2}$. (Derive the formula!)

4. $\int \frac{x^3}{\sqrt{4-x^4}}\,dx$

5. $\int \frac{x^3}{4-x^4}\,dx$

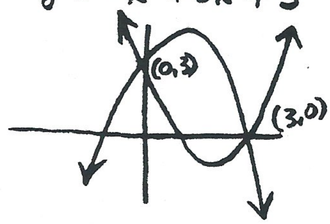
6. $\int \frac{x}{\sqrt{4-x^4}}\,dx$

7. $\int \frac{2x}{x^2+8x+25}\,dx$

8. $\int \frac{\operatorname{arcsec} x}{x\sqrt{x^2-9}}\,dx$

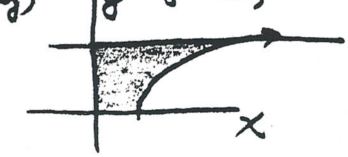
In 9-10, find the area between the graphs as indicated:

9. $y = x^2 - 4x + 3$
$y = -x^2 + 2x + 3$



10. Set up to find the area using
a) vertical slices (dx)
b) horizontal slices (dy)
c) Solve easiest way.

$x = y^2 + 1$
$y = 2$, $x = 0$, $y = 0$



In 11-12, set up for disk and shell methods. Solve easiest way.

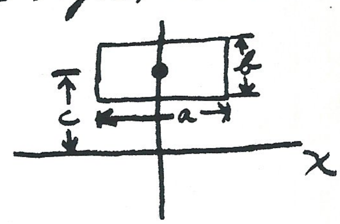
11. $y = \frac{1}{x^2}$, x axis, $x = 1$, $x = 4$ rotated about x axis.

12. $y = \frac{1}{x^2}$, x axis, $x = 1$, $x = 4$ rotated about y axis.

13. Set up only to find the center of mass $(\bar{x}, \bar{y})$ of the region in the first quadrant bounded by $y = x^3$, $y = 0$, and $x = 3$.

14. Set up only to find the center of mass $(\bar{x}, \bar{y})$ of the region in the first quadrant bounded by $y = x^3$ and $y = x$.

15. Use the Theorem of Pappus to find the volume obtained by rotating the rectangle shown about the x axis. (Assume $c > \frac{b}{2}$)



16. Find the length of the arc $y = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$, $1 \le x \le 2$

17. The arc $y = \sqrt{a^2 - x^2}$ is rotated about the x axis. Calculate the surface area generated using methods of this section, and thus verify that surface area of a sphere is $4\pi a^2$.

CALCULUS II EXAM 1C Solutions Dr. RAPALJE

1. $\int_1^3 \frac{e^{3/x}}{x^2}\,dx$ Let $u = 3x^{-1}$

$du = -3x^{-2}dx$

$\frac{du}{-3} = \frac{dx}{x^2}$

$x=1 \Rightarrow u=3$

$x=3 \Rightarrow u=1$

$= \int e^u \frac{du}{-3}$

$= -\frac{1}{3} e^u \Big|_3^1$

$= \frac{1}{3} e^u \Big|_1^3 = \frac{1}{3}(e^3 - e)$

2. $y = x \arcsin x + \sqrt{1-x^2}$

$\frac{dy}{dx} = x \frac{1}{\sqrt{1-x^2}} + \arcsin x \cdot 1 + \frac{1}{2}(1-x^2)^{-1/2}(-2x)$

$= \arcsin x$

3. $y = \arctan x \Leftrightarrow x = \tan y$

$1 = \sec^2 y \frac{dy}{dx}$

$[1 + \tan^2 y = \sec^2 y]$

$\frac{dy}{dx} = \frac{1}{\sec^2 y}$

$= \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

4. $\int \frac{x^3\,dx}{\sqrt{4-x^4}}$

Let $u = 4-x^4$

$du = -4x^3dx$

$\frac{du}{-4} = x^3dx$

$= -\frac{1}{4}\int \frac{du}{u^{1/2}}$

$= -\frac{1}{4}\int u^{-1/2}du = \frac{-1}{4}\cdot\frac{2}{1}u^{1/2} + c$

$= -\frac{1}{2}\sqrt{4-x^4} + c$

5. $\int \frac{x^3\,dx}{4-x^4}$

$= -\frac{1}{4}\int \frac{du}{u}$

$= -\frac{1}{4}\ln u + c$

$= -\frac{1}{4}\ln|4-x^4| + c$

6. $\int \frac{x}{\sqrt{4-x^4}}\,dx$ Let $u = x^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$= \frac{1}{2}\int \frac{du}{\sqrt{4-u^2}}$

$= \frac{1}{2}\arcsin\frac{u}{2} + c$

$= \frac{1}{2}\arcsin\left(\frac{x^2}{2}\right) + c$

7. $\int \frac{2x}{x^2+8x+16+9}\,dx$ (8)

Let $u = x+4$

$du = dx$

$x = u-4$

$= \int \frac{2x\,dx}{(x+4)^2+9}$

$= \int \frac{2(u-4)\,du}{u^2+9}$

$= \int \frac{2u\,du}{u^2+9} - 8\int \frac{du}{u^2+9}$

$= \ln(u^2+9) - \frac{8}{3}\arctan\frac{u}{3} + c$

$= \ln(x^2+8x+25) - \frac{8}{3}\arctan\frac{x+4}{3} + c$

8. $\int \frac{\operatorname{arcsec} x\,dx}{x\sqrt{x^2-1}}$

Let $u = \operatorname{arcsec} x$

$du = \frac{1}{x\sqrt{x^2-1}}dx$

$= \int u\,du = \frac{u^2}{2} + c$

$= \frac{1}{2}(\operatorname{arcsec} x)^2 + c$

9. $\int(\text{upper} - \text{Lower})\,dx$

$= \int_0^3 (-x^2+2x+3) - (x^2-4x+3)\,dx$

$= \int_0^3 (-2x^2+6x)\,dx$

$= -\frac{2x^3}{3} + 3x^2\Big|_0^3 = -\frac{2}{3}(27) + 27 = 9$

10 a) $\int_0^1 2\,dx + \int_1^5 (2-\sqrt{x-1})\,dx$

(8)

b) $\int_0^2 (y^2+1)\,dy$

$x = y^2+1$

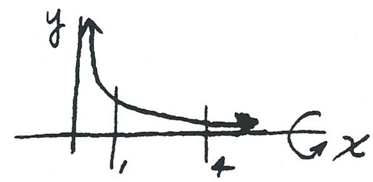
$y^2 = x-1$

$y = \sqrt{x-1}$

c) $= \frac{y^3}{3} + y\Big|_0^2$

$= \frac{8}{3} + 2 = \frac{14}{3}$

11. $y = \frac{1}{x^2}$



a) Disk

$\pi\int_1^4 \left(\frac{1}{x^2}\right)^2 dx$

c) $\pi\int_1^4 x^{-4}dx = \frac{\pi x^{-3}}{-3}\Big|_1^4 = -\frac{\pi}{3}\left(\frac{1}{64} - 1\right) = \frac{21\pi}{64}$

SHELL b) $2\pi\int_0^{1/16}(4-1)y\,dy + 2\pi\int_{1/16}^1 \left(\frac{1}{\sqrt{y}} - 1\right)y\,dy$

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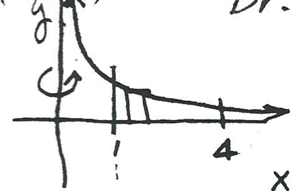
12. $y = \frac{1}{x^2}$,



b) SHELL:

$$2\pi \int_1^4 x f(x)\,dx$$

$$= 2\pi \int_1^4 x \cdot \frac{1}{x^2}\,dx$$

c) $$= 2\pi \int_1^4 \frac{1}{x}\,dx$$

$$= 2\pi \ln x \Big|_1^4 = 2\pi(\ln 4 - \ln 1)$$

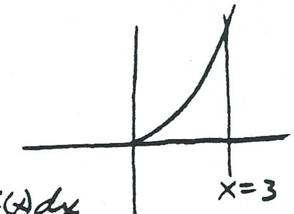
$$= 2\pi \ln 4$$

$x = \frac{1}{\sqrt{y}}$

a) DISK:

$$\pi \int_0^{1/16} (4^2 - 1^2)\,dy + \pi \int_{1/16}^{1} \left[\left(\frac{1}{\sqrt{y}}\right)^2 - 1^2\right] dy$$

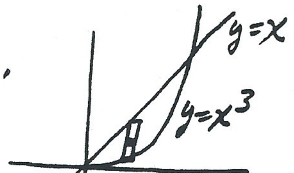
13. $y = x^3$, $y = 0$, $x = 3$



$$\bar{x} = \frac{\int x f(x)\,dx}{\int f(x)\,dx} \qquad \bar{y} = \frac{\frac{1}{2}\int [f(x)]^2\,dx}{\int f(x)\,dx}$$

$$\bar{x} = \frac{\int_0^3 x^4\,dx}{\int_0^3 x^3\,dx} \qquad \bar{y} = \frac{\frac{1}{2}\int_0^3 x^6\,dx}{\int_0^3 x^3\,dx}$$

14.



$$\bar{x} = \frac{\int_0^1 x(x - x^3)\,dx}{\int_0^1 (x - x^3)\,dx}$$

$$\bar{y} = \frac{\int_0^1 \left(\frac{x + x^3}{2}\right)(x - x^3)\,dx}{\int_0^1 (x - x^3)\,dx}$$

15. Area = ab

radius of path = c

Perimeter path = $2\pi c$.

V = (Perimeter path)(Area)

$$= 2\pi c \cdot ab$$

$$= 2\pi abc$$

16. $y = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$

$$\frac{dy}{dx} = \frac{1}{2}x^2 - \frac{1}{2}x^{-2}$$

$$= \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$$

$$= \frac{x^4 - 1}{2x^2}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^8 - 2x^4 + 1}{4x^4}}$$

$$= \sqrt{\frac{4x^4 + x^8 - 2x^4 + 1}{4x^4}}$$

$$= \sqrt{\frac{x^8 + 2x^4 + 1}{4x^4}}$$

$$= \frac{x^4 + 1}{2x^2}$$

$$s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_1^2 \frac{x^4 + 1}{2x^2}\,dx$$

$$= \frac{1}{2}\int_1^2 (x^2 + x^{-2})\,dx$$

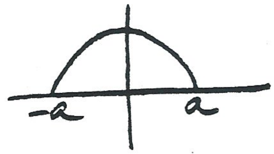
$$= \frac{1}{2}\left[\frac{x^3}{3} + \frac{x^{-1}}{-1}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{8}{3} - \frac{1}{2}\right] - \frac{1}{2}\left[\frac{1}{3} - 1\right]$$

$$= \frac{1}{2}\,\frac{13}{6} - \frac{1}{2}\left(-\frac{4}{6}\right)$$

$$= \frac{17}{12}$$

17. $$S.A. = 2\pi \int f(x)\,ds$$



$$= 2\pi \int_{-a}^{a} \sqrt{a^2 - x^2}\,ds$$

$y = \sqrt{a^2 - x^2}$

$$\frac{dy}{dx} = \frac{1}{2}(a^2 - x^2)^{-1/2}(-2x)$$

$$= \frac{-x}{\sqrt{a^2 - x^2}}$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \sqrt{1 + \frac{x^2}{a^2 - x^2}}\,dx$$

$$= \sqrt{\frac{a^2 - x^2 + x^2}{a^2 - x^2}}\,dx$$

$$= \frac{a}{\sqrt{a^2 - x^2}}\,dx$$

By symmetry:

$$S.A. = 2\pi \int_{-a}^{a} \sqrt{a^2 - x^2}\,ds \text{ or } 4\pi \int_0^a \sqrt{a^2 - x^2}\,ds$$

$$= 4\pi \int_0^a \cancel{\sqrt{a^2 - x^2}}\,\frac{a}{\cancel{\sqrt{a^2 - x^2}}}\,dx$$

$$= 4\pi a \int_0^a dx = 4\pi a\, x\Big|_0^a = 4\pi a^2$$