MS 229 EXAM 4 B Chap. 18: 1–6 Dr. Rapalje

1. Find $\frac{\partial z}{\partial x}$ if $z = \arctan\left(\frac{y}{x}\right)$

2. Find $\frac{\partial w}{\partial x}$ if $w^2 - 3xw + x^2y = 4$

3. Find $\frac{\partial z}{\partial t}$ if $z = e^x \sin xy$ where $x = \ln t$, $y = 2st$.
Evaluate when $s = t = 1$.

4. Find $d_\theta f(x,y)$ where $f(x,y) = e^x \cos y$ at the point $(0, \pi/4)$.
Then determine the value(s) of $\theta$ which make(s) $d_\theta f$ a maximum.

5. At a certain instant a right circular cone has a radius of 2 inches and an altitude of 3 inches. At this instant the radius is decreasing at a rate of 5 in/sec and the volume is increasing at the rate of $4\pi$ $\text{in}^3/\text{sec}$. How fast is the altitude changing at this moment?

6. Find the equation of the normal line to the surface $z = \frac{x+y}{xy}$ at the point $(1,1,2)$.

7. Find the equation of the tangent plane to $y^{1/2} + z^{1/2} = 7$ at the point $(3, 16, 9)$.

8. Find a vector perpendicular to the plane containing $\vec{u} = 3\vec{j} + 2\vec{k}$ and $\vec{v} = 2\vec{i} - 3\vec{j}$.

Thought for today: For God hath not called us unto uncleanness, but unto holiness.
Let us therefore come boldly unto the throne of grace, that we may obtain mercy and find grace to help in time of need.
I Thes 4:7 and Heb 4:16

MS 229 EXAM 4 Solutions. (I hope!)

1. $z = \arctan(y/x)$

$\frac{\partial z}{\partial x} = \frac{1}{\left(\frac{y}{x}\right)^2 + 1}\left(-\frac{y}{x^2}\right)$

$= \frac{x^2}{x^2+y^2} \cdot \frac{-y}{x^2}$

$= \frac{-y}{x^2+y^2}$

2. $w^2 - 3xw + x^2y = 4$

$2w\frac{\partial w}{\partial x} - 3x\frac{\partial w}{\partial x} - 3w + 2xy = 0$

$\frac{\partial w}{\partial x}(2w - 3x) = 3w - 2xy$

$\frac{\partial w}{\partial x} = \frac{3w-2xy}{2w-3x}$

3. $z = e^x \sin xy$

$x = \ln t$

$y = 2st$

$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$

$\frac{\partial z}{\partial x} = e^x y \cos xy + e^x \sin xy$

$\frac{\partial x}{\partial t} = 1/t \qquad \frac{\partial y}{\partial t} = 2s$

$\frac{\partial z}{\partial y} = x e^x \cos xy$

$\boxed{\frac{\partial z}{\partial t} = e^x(y\cos xy + \sin xy)\frac{1}{t} + 2sxe^x\cos xy}$

when $s = t = 1$, $x = 0$, $y = 2$

$\frac{\partial z}{\partial t} = 1(2\cos 0 + \sin 0)\cdot 1 + 2\cdot 0$

$= \boxed{2}$

4. $\vec{a} = (\cos\theta)\vec{i} + (\sin\theta)\vec{j} \quad f(x,y) = e^x \cos y$

$\frac{\partial f}{\partial x} = e^x \cos y = \frac{\sqrt{2}}{2}$ at $(0, \pi/4)$

$\frac{\partial f}{\partial y} = -e^x \sin y = -\frac{\sqrt{2}}{2}$ at $(0, \pi/4)$

$d_\theta f(x,y) = \frac{\sqrt{2}}{2}\cos\theta - \frac{\sqrt{2}}{2}\sin\theta$

$k(\theta) =$ or $\frac{\sqrt{2}}{2}(\cos\theta - \sin\theta)$

$k'(\theta) = \frac{\sqrt{2}}{2}(-\sin\theta - \cos\theta) = 0$

at $\sin\theta = -\cos\theta$

$\tan\theta = -1$

$\theta = 3\pi/4,\ 7\pi/4$

$k''(\theta) = \frac{\sqrt{2}}{2}(-\cos\theta + \sin\theta)$

$k''(\theta) < 0$ at $\boxed{\frac{7\pi}{4}}$ only.

5. $V = \frac{1}{3}\pi r^2 h$, when $r = 2, h = 3, \frac{dr}{dt} = -5$ in/sec

$\frac{dV}{dt} = \frac{1}{3}\pi\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right) \qquad \frac{dV}{dt} = 4\pi$ in³/sec

$4\pi \text{ in}^3/\text{sec} = \frac{1}{3}\pi\left(4\text{in}^2\frac{dh}{dt} + 12\text{ in}^2 \cdot -5\text{ in/sec}\right)$

$12 \text{ in}^3/\text{sec} = 4\text{in}^2\frac{dh}{dt} - 60\text{ in}^3/\text{sec}$

$72 \text{ in}^3/\text{sec} = 4\text{in}^2\frac{dh}{dt}$

$\frac{dh}{dt} = \boxed{18 \text{ in/sec}}$ increasing.

6. $z = \frac{x+y}{xy} = y^{-1} + x^{-1}$

$\frac{\partial z}{\partial x} = -\frac{1}{x^2} = -1$; $\frac{\partial z}{\partial y} = -\frac{1}{y^2} = -1$ at $(1,1,2)$

Direction numbers: $-1, -1, -1$

Line normal: $\frac{x-1}{-1} = \frac{y-1}{-1} = \frac{z-2}{-1}$

or $\boxed{x-1 = y-1 = z-2}$

8. $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & 2 \\ 2 & -3 & 0 \end{vmatrix}$

$= \vec{i}\begin{vmatrix} 3 & 2 \\ -3 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} 0 & 3 \\ 2 & -3 \end{vmatrix}$

$\boxed{= 6\vec{i} + 4\vec{j} - 6\vec{k}}$

7. $y^{1/2} + z^{1/2} = 7 \qquad (3, 16, 9)$

$\frac{\partial z}{\partial y}: \frac{1}{2}y^{-1/2} + \frac{1}{2}z^{-1/2}\frac{\partial z}{\partial y} = 0$

$\frac{\partial z}{\partial y} = \frac{z^{1/2}}{y^{1/2}} = \frac{-3}{4}$

$\frac{\partial z}{\partial x}: \frac{1}{2}z^{-1/2}\frac{\partial z}{\partial x} = 0$, so $\frac{\partial z}{\partial x} = 0$

$z - 9 = 0(x-3) + \frac{-3}{4}(y-16)$

$4z - 36 = -3y + 48$

$\boxed{3y + 4z - 84 = 0}$

MS 229 EXAM 4

1. Determine if the differentials are exact, and if they are find the functions of which each is the differential.

   a) $(x + y\cos x)\,dx + \sin x\,dy$

(7ea) b) $\dfrac{x\,dy - y\,dx}{x^2+y^2}$

   c) $(2xy + z^2)\,dx + (2yz + x^2)\,dy + (2xz + y^2)\,dz$

2. Evaluate $\int_c (x+2y)\,dx + (x^2-y^2)\,dy$ where c is the arc of $y = x^3$ from $(0,0)$ to $(1,1)$. Is this path independent? Evaluate where
(10) c is the path from $(0,0)$ to $(1,0)$ and then $(1,0)$ to $(1,1)$.

3. Evaluate $\int_C (3x^2+6xy)\,dx + (3x^2-3y^4)\,dy$ over any arc C from
(10) $(1,1)$ to $(3,2)$. (Note: the differential is exact.)

4. Change the order of integration and integrate: $\int_0^2 \int_{2y}^4 e^{x^2}\,dx\,dy$
(10)
5. Evaluate: $\iint_F 4y\,dA$ where F is the region bounded by $y = x^2$ and $x + y = 2$.
(10)
6. Set up the integral to find mass of the region F in two ways.
(10) Solve by either way: $\rho = 2x$; $F = \{(x,y):\ 0 \le x \le 1,\ x^3 \le y \le \sqrt{x}\}$

7. Evaluate by first changing to polar coordinates:

(10) $$\int_0^4 \int_{-\sqrt{4x-x^2}}^{\sqrt{4x-x^2}} \sqrt{x^2+y^2}\,dy\,dx$$

8. Set up a triple integral and solve to find the volume of the solid bounded by $x^2 + 4y^2 = 4$, $z = 0$, and $x + y + 4z = 4$.
(10)

9. Set up to find the area between $y = x^2$ and $x + y = 2$ (See #5)
(10) using $\iint dx\,dy$ and $\iint dy\,dx$. Solve by easier means.

Thought for today: Therefore if any man (or woman presumably!) be in Christ, he is a new creature: old things are passed away; behold, all things are become new. II Corinthians 5:11.

# MS 229 EXAM 4 Solutions

**1.a)** $(x+y\cos x)\,dx + \sin x\,dy$

$\frac{\partial P}{\partial y} = \cos x \qquad \frac{\partial Q}{\partial x} = \cos x$

$\boxed{\text{Exact}}$. $\boxed{f(x,y) = \frac{x^2}{2} + y\sin x + c}$

**b)** $\frac{x\,dy - y\,dx}{x^2+y^2} = \frac{-y}{x^2+y^2}\,dx + \frac{x\,dy}{x^2+y^2}$

$\frac{\partial P}{\partial y} = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$ $\boxed{\text{Exact}}$

$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$f(x,y) = \int \frac{-y}{x^2+y^2}\,\partial x = -\arctan\frac{x}{y} + c_y$

$f(x,y) = \int \frac{x}{x^2+y^2}\,\partial y = \arctan y/x + c$ (same)

**c)** $(2xy+z^2)\,dx + (2yz+x^2)\,dy + (2xz+y^2)\,dz$

$\frac{\partial P}{\partial y} = 2x \qquad \frac{\partial Q}{\partial x} = 2x$

$\frac{\partial P}{\partial z} = 2z \qquad \frac{\partial R}{\partial x} = 2z \qquad \frac{\partial Q}{\partial z} = 2y \qquad \frac{\partial R}{\partial y} = 2y$ $\boxed{\text{EXACT}}$

$\boxed{f(x,y,z) = x^2y + xz^2 + y^2z + c}$

**2.** $\int (x+2y)\,dx + (x^2-y^2)\,dy \qquad y = x^3,\ dy = 3x^2\,dx,\ (0,0) \text{ to } (1,1)$

$= \int_0^1 (x+2x^3)\,dx + (x^2-x^6)(3x^2\,dx)$

$= \frac{x^2}{2} + \frac{x^4}{2} + \frac{3x^5}{5} - \frac{x^9}{3}\Big|_0^1 = 1 + \frac{3}{5} - \frac{1}{3}$

$= 1 + \frac{4}{15} = \boxed{\frac{19}{15}}$

Not path independent.

$\frac{\partial P}{\partial y} = 2 \qquad \frac{\partial Q}{\partial x} = 2x.$

over $(0,0)$ to $(1,0)$ and $(1,0)$ to $(1,1)$

$y = 0,\ dy = 0 \qquad x = 1,\ dx = 0$

$\int_0^1 x\,dx + \int_0^1 1 - y^2\,dy = \frac{x^2}{2}\Big|_0^1 + \left(y - \frac{y^3}{3}\right)\Big|_0^1$

$= \frac{1}{2} + \frac{2}{3} = \boxed{\frac{7}{6}}$

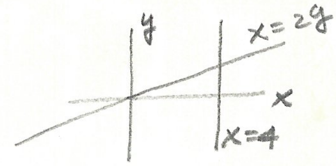
**3.** $\int (3x^2+6xy)\,dx + (3x^2-3y^2)\,dy \qquad (1,1) \text{ to } (3,2)$

$f(x,y) = x^3 + 3x^2y - y^3\Big|_{(1,1)}^{(3,2)}$

$= 27 + 54 - 8 - 3$

$= 81 - 11 = \boxed{70}$

**4.** $\int_0^2\int_{2y}^4 e^{x^2}\,dx\,dy$

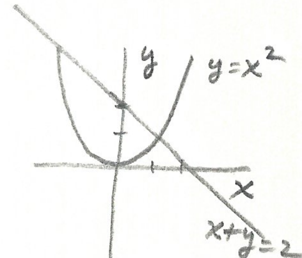



$\int_0^4\int_0^{x/2} e^{x^2}\,dy\,dx$

$= \int_0^4 e^{x^2}\frac{x}{2}\,dx \qquad u = x^2,\ du = 2x\,dx$

$= \frac{1}{4}e^{x^2}\Big|_0^4 = \boxed{\frac{1}{4}(e^{16}-1)}$

**5.** $\int_{-2}^1\int_{x^2}^{2-x} 4y\,dy\,dx$




$x + x^2 = 2$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = -2 \quad x = 1$

$y = 4 \quad y = 1$

$\int_{-2}^1 2y^2\Big|_{x^2}^{2-x}\,dx$

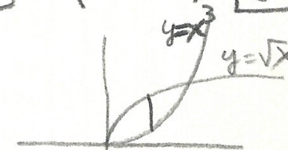
$= 2\int_{-2}^1 4 - 4x + x^2 - x^4\,dx$

$= 2\left[4x - 2x^2 + \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^1$

$= 2\left[4 - 2 + \frac{1}{3} - \frac{1}{5} - \left(-8 - 8 - \frac{8}{3} + \frac{32}{5}\right)\right]$

$= 2\left[2 + \frac{1}{3} - \frac{1}{5} + 16 + \frac{8}{3} - \frac{32}{5}\right] = 2\left(21 - \frac{33}{5}\right) = \boxed{\frac{154}{5}}$

**6.** $\int_0^1\int_{x^3}^{\sqrt{x}} 2x\,dy\,dx$ or $\int_0^1\int_{y^2}^{\sqrt[3]{y}} 2x\,dx\,dy$




$= \int_0^1 x^2\Big|_{y^2}^{\sqrt[3]{y}}\,dy = \int_0^1 y^{2/3} - y^4\,dy$

$= \frac{3y^{5/3}}{5} - \frac{y^5}{5}\Big|_0^1 = \boxed{\frac{2}{5}}$

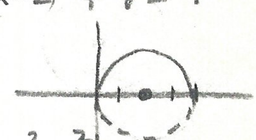
**7.** $\int_0^4\int_0^{\sqrt{4x-x^2}} \sqrt{x^2+y^2}\,dy\,dx$

$y = +\sqrt{4x-x^2}$

$y^2 = 4x - x^2$

$x^2 - 4x + 4 + y^2 = 4$

$(x-2)^2 + y^2 = 4$



$= \int_0^{\pi/2}\int_0^{4\cos\theta} r\ r\,dr\,d\theta$

$= \int_0^{\pi/2} \frac{r^3}{3}\Big|_0^{4\cos\theta}\,d\theta$

$= \frac{64}{3}\int_0^{\pi/2} \cos^3\theta\,d\theta$

$x^2 + y^2 = 4x$

$r^2 = 4r\cos\theta$

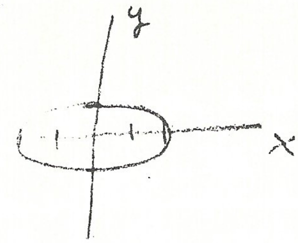
$r = 4\cos\theta$

$= \frac{64}{3}\int (1 - \sin^2\theta)\cos\theta\,d\theta$

$= \frac{64}{3}\int (1-u^2)\,du = \frac{64}{3}\left[u - \frac{u^3}{3}\right] \qquad u = \sin\theta,\ du = \cos\theta\,d\theta$

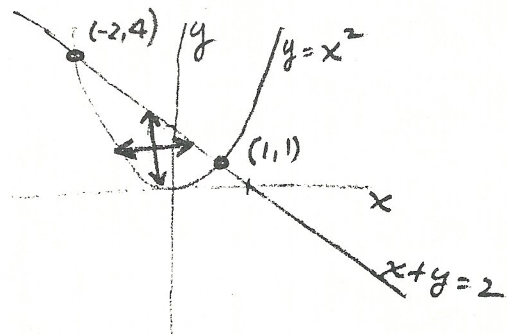
$= \frac{64}{3}\left[\sin\theta - \frac{\sin^3\theta}{3}\right]_0^{\pi/2} = \frac{64}{3}\cdot\frac{2}{3} = \boxed{\frac{128}{9}}$

8. $\displaystyle\int_{-1}^{1}\int_{-2\sqrt{1-y^2}}^{2\sqrt{1-y^2}}\int_{0}^{\frac{4-x-y}{4}} dz\,dx\,dy$ $\qquad x = \pm\sqrt{4-4y^2}$




or $\displaystyle\int_{-2}^{2}\int_{-\frac{\sqrt{4-x^2}}{2}}^{\frac{\sqrt{4-x^2}}{2}}\int_{0}^{\frac{4-x-y}{4}} dz\,dy\,dx$

9. $\displaystyle\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_{1}^{4}\int_{-\sqrt{y}}^{2-y} dx\,dy$




or $\displaystyle\int_{-2}^{1}\int_{x^2}^{2-x} dy\,dx = \int_{-2}^{1}(2-x-x^2)\,dx$

$\displaystyle = 2x - \frac{x^2}{2} - \frac{x^3}{3}\Big|_{-2}^{1}$

$\displaystyle = 2 - \frac{1}{2} - \frac{1}{3} - \left(-4 - 2 + \frac{8}{3}\right)$

$\displaystyle = 2 - \frac{1}{2} - \frac{1}{3} + 6 - \frac{8}{3}$

$\displaystyle = 5 - \frac{1}{2} = \boxed{9/2}$