CALCULUS III EXAM 5 A 15.6–15.7, 16.1–16.5

10

Show all work. Graph all regions!

1. Set up a triple integral for the volume of the solid enclosed by $y^2 = x$, $z = 0$, and $x + z = 1$.

2. Evaluate $\int_{-5}^{5}\int_{0}^{\sqrt{25-x^2}}\int_{0}^{\frac{1}{x^2+y^2}} \sqrt{x^2+y^2}\, dz\, dy\, dx$ by first changing to cylindrical coordinates.

3. Determine which vector fields are conservative. For those that are conservative, find the potential function.

a) $\vec{F}(x,y) = \left(4x^3y^3 + \frac{1}{x}\right)\hat{\imath} + \left(3x^4y^2 - \frac{1}{y}\right)\hat{\jmath}$

b) $\vec{F}(x,y) = \dfrac{2x\hat{\imath} + 2y\hat{\jmath}}{(x^2+y^2)^2}$

c) $\vec{F}(x,y,z) = \frac{1}{y}\hat{\imath} - \frac{x}{y^2}\hat{\jmath} + (2z-1)\hat{k}$

4. Evaluate $\int_C xy\,dx + y\,dy$ from $(0,0)$ to $(2,4)$

a) where C is the straight line from $(0,0)$ to $(2,4)$

b) where C is the parabola $y = x^2$

c) Is it path independent? Justify!

5. Find value of $\int_C (x^2+y^2)\,dx + 2xy\,dy$ THINK!

a) where C is the line from $(3,0)$ to $(0,3)$

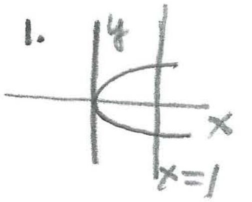
b) where C is the quarter circle from $(3,0)$ to $(0,3)$

c) where C is the path around the circle $x^2+y^2=9$

d) Is it path independent?

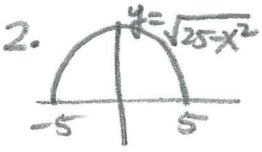
6. Use Green's Theorem $\left[\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA\right]$ to find the $\int_C [\sin(x^2) + y]\,dx + [3x - \arctan(e^y)]\,dy$ where C is the square with vertices $(0,0)$, $(2,0)$, $(2,2)$, and $(0,2)$

1. $\displaystyle\int_0^1\int_{-\sqrt{x}}^{\sqrt{x}}\int_0^{1-x} dz\,dy\,dx$

1.


Roof: $z = 1-x$ plane.
Floor: $z=0$

or $\displaystyle\int_{-1}^1\int_{y^2}^1\int_0^{1-x} dz\,dx\,dy$

2.


2. $\displaystyle\int_{-5}^5\int_0^{\sqrt{25-x^2}}\int_0^{\frac{1}{x^2+y^2}} \sqrt{x^2+y^2}\,dz\,dy\,dx$

$\displaystyle = \int_0^\pi\int_0^5\int_0^{\frac{1}{r^2}} r\,r\,dz\,dr\,d\theta.$

$\displaystyle = \int_0^\pi\int_0^5 r^2 z\Big|_0^{\frac{1}{r^2}} dr\,d\theta$

$\displaystyle = \int_0^\pi\int_0^5 r^2\frac{1}{r^2}\,dr\,d\theta$

$\displaystyle = \int_0^\pi r\Big|_0^5 d\theta = 5\pi$

3a) $\vec{F}(x,y) = \left(4x^3y^3+\frac{1}{x}\right)\hat{\imath} + \left(3x^4y^2-\frac{1}{y}\right)\hat{\jmath}$

$\displaystyle\frac{\partial M}{\partial y} = 12x^3y^2 = \frac{\partial N}{\partial x}$ Yes

$\displaystyle f(x,y) = \int\left(4x^3y^3+\frac{1}{x}\right)\partial x = x^4y^3 + \ln x + C(y)$

$\displaystyle f(x,y) = \int\left(3x^4y^2-\frac{1}{y}\right)\partial y = x^4y^3 - \ln y + C(x)$

$f(x,y) = x^4y^3 + \ln x - \ln y + C$

3b) $\displaystyle \vec{F}(x,y) = \frac{2x\,\hat{\imath}}{(x^2+y^2)^2} + \frac{2y\,\hat{\jmath}}{(x^2+y^2)^2}$

$\displaystyle\frac{\partial M}{\partial y} = -4x(x^2+y^2)^{-3}\cdot 2y$

$\displaystyle = -\frac{8xy}{(x^2+y^2)^3}$ ← Yes

$\displaystyle\frac{\partial N}{\partial x} = -4y(x^2+y^2)^{-3}\cdot 2x$

$\displaystyle f(x,y) = \int\frac{2x\,dx}{(x^2+y^2)^2}$ $\quad u = x^2+y^2$, $du = 2x\,dx$

$\displaystyle = \int u^{-2}du = -u^{-1}+C$

$\displaystyle = -\frac{1}{x^2+y^2}+C$

$\displaystyle\int\frac{2y\,\partial y}{(x^2+y^2)^2}$ = Same

3c) $\displaystyle F(x,y,z) = \underset{M}{\frac{1}{y}}\hat{\imath} - \underset{N}{\frac{x}{y^2}}\hat{\jmath} + \underset{P}{(2z-1)}\hat{k}$

$\displaystyle\frac{\partial M}{\partial y} = -\frac{1}{y^2} = \frac{\partial N}{\partial x}$ $\quad\displaystyle\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$

$\displaystyle\frac{\partial N}{\partial z} = 0 = \frac{\partial P}{\partial y}$

$\displaystyle f(x,y,z) = \int\frac{1}{y}\partial x = \frac{x}{y} + C(y,z)$

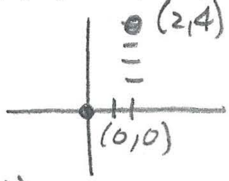
$\displaystyle f(x,y,z) = \int -\frac{x}{y^2}\partial y = \frac{x}{y} + C(x,z)$

$\displaystyle f(x,y,z) = \int(2z-1)\partial z = z^2 - z + C(x,y)$

$\displaystyle f(x,y,z) = \frac{x}{y} + z^2 - z + C$

4. $\int xy\,dx + y\,dy$ $\quad(0,0)$ to $(2,4)$

a) $(0,0)$ to $(2,4)$



$y = 2x$
$dy = 2dx$

$\displaystyle\int_0^2 x(2x)\,dx + 2x(2dx)$

$\displaystyle = \int_0^2(2x^2+4x)\,dx = \frac{2x^3}{3}+2x^2\Big|_0^2$

$\displaystyle = \frac{16}{3}+8 = \frac{40}{3}$

b) $y = x^2$
$dy = 2x\,dx$

$\displaystyle\int_0^2 x\,x^2dx + x^2(2x\,dx)$

$\displaystyle = \int_0^2 3x^3dx = \frac{3x^4}{4}\Big|_0^2 = 12$

c) $\displaystyle\frac{\partial M}{\partial y} = x \neq \frac{\partial N}{\partial x} = 0$ NOT INDEP.

5d) $\int(x^2+y^2)\,dx + 2xy\,dy$

$\displaystyle\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x}$ Path Independent.

$\displaystyle f(x,y) = \frac{x^3}{3} + xy^2\Big|_{(3,0)}^{(0,3)} = 0 - 9$

Any path $\quad = -9$ 5a) 5b)

c) Closed path, conservative $= 0$

6. $\int[\sin(x^2)+y]\,dx + [3x - \arctan(e^y)]\,dy$

$\displaystyle = \iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial m}\right)dA = \iint(3-1)\,dA$

$= 2\iint dA = 2(\text{Area}) = 2\cdot 4 = 8$