MS 229 EXAM 5C Dr. Rapalje

1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$: $z^3 - 3x^2y + 6xyz = 0$

2. Show that if $z = \ln\left(\frac{y}{x}\right)$, then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$.

3. If $z = 2x^2 + xy - y^2 + 2x - 3y + 5$; $x = 2s - t$; $y = s + t$;
find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ in terms of s and t.

4. Find $\vec{\nabla} f$ and $D_a f$ where $f(x,y,z) = \sin xz + \cos xy$ at $P_1(0,-1,2)$ and in the direction from P_1 to $P_2(2,2,-4)$. (Note: $\vec{a}$ is a unit vector)

5. Find equations of the tangent plane and normal line to $z = 3x^2 + 2y^2 - 11$ at $(2,1,3)$.

6. Find a vector perpendicular to $\vec{u} = 3\vec{j} + 2\vec{k}$ and $\vec{v} = 2\vec{i} - 3\vec{j}$.

Use methods of partial differentiation to compute the derivatives.

7. Find $\frac{dy}{dx}$ where $\ln(x^2+y^2) - \arctan\left(\frac{y}{x}\right) = 0$. [Note: $d(\arctan u) = \frac{du}{1+u^2}$]

8. Find $\frac{\partial w}{\partial x}$ if $\sin(xyw) + x^2 + y^2 + w^2 = 3$.

9. If $x = u + v$ and $y = uv$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$

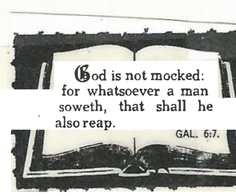
10. If $z = x\cos y - y\cos x$, find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$, $\frac{\partial^2 z}{\partial y^2}$,

11. Examine $f(x,y) = x^3 + y^3 + 3xy$ for maximum, minimum, saddle point

E.C. Prove Arf's Law. Given $F(x,y,z) = 0$ and $G(x,y,z) = 0$, find $\frac{dz}{dx}$.

Alt. problem: May be subtituted for any problem 1-11.

The volume of a rectangular box without a top is to be 500 ft³. Find the minimum surface area.



MS 229 EXAM Chapter 18

① $z^3 - 3x^2y + 6xyz = 0$

$3z^2\frac{\partial z}{\partial x} - 6xy + 6y\left(x\frac{\partial z}{\partial x} + z\right) = 0$

$3z^2\frac{\partial z}{\partial x} - 6xy + 6xy\frac{\partial z}{\partial x} + 6yz = 0$

$\frac{\partial z}{\partial x} = \frac{6xy - 6yz}{3z^2 + 6xy} = \boxed{\frac{2y(x-z)}{z^2+2xy}}$

$3z^2\frac{\partial z}{\partial y} - 3x^2 + 6x\left(y\frac{\partial z}{\partial y} + z\right) = 0$

$3z^2\frac{\partial z}{\partial y} - 3x^2 + 6xy\frac{\partial z}{\partial y} + 6xz = 0$

$\frac{\partial z}{\partial y} = \frac{3x^2 - 6xz}{3z^2 + 6xy} = \boxed{\frac{x(x-2z)}{z^2+2xy}}$

② $z = \ln(y/x)$

$z = \ln y - \ln x$

$\frac{\partial z}{\partial x} = -\frac{1}{x}$

$\frac{\partial z}{\partial y} = \frac{1}{y}$

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$

$x\cdot(-1/x) + y(1/y) = 0$

$-1 + 1 = 0$

$0 = 0$

③ $z = 2x^2 + xy - y^2 + 2x - 3y + 5$

$\frac{\partial z}{\partial x} = 4x + y + 2$

$\frac{\partial z}{\partial y} = x - 2y - 3$

$x = 2s - t \qquad y = s + t$

$\frac{\partial x}{\partial s} = 2 \qquad \frac{\partial y}{\partial s} = \frac{\partial y}{\partial t} = 1$

$\frac{\partial x}{\partial t} = -1$

$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$

$= (4x+y+2)2 + (x-2y-3)\cdot 1$

$= 9x + 1 = \boxed{18s - 9t + 1}$

$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$

$= (4x+y+2)(-1) + (x-2y-3)\cdot 1$

$= -3x - 3y - 5$

$= -3(2s-t) - 3(s+t) - 5$

$= \boxed{-9s - 5}$

④ $f(x,y,z) = \sin xz + \cos xy$ at $P(0,-1,2)$

$f_{,1} = z\cos xz - y\sin xy = 2$

$f_{,2} = -x\sin xy = 0$

$f_{,3} = x\cos xz = 0$

$\vec{\nabla} f = 2\vec{i} + 0\vec{j} + 0\vec{k}$

$\overrightarrow{P_1P_2} = 2\vec{i} + 3\vec{j} - 6\vec{k}$

$\vec{a} = \frac{2}{7}\vec{i} + \frac{3}{7}\vec{j} - \frac{6}{7}\vec{k}$

$D_a f = \vec{\nabla} f\cdot\vec{a} = \boxed{\frac{4}{7}}$

⑤ $z = 3x^2 + 2y^2 - 11$ at $(2,1,3)$

$\frac{\partial z}{\partial x} = 6x = 12$

$\frac{\partial z}{\partial y} = 4y = 4$

Plane: $z - z_0 = m_1(x-x_0) + m_2(y-y_0)$

$z - 3 = 12(x-2) + 4(y-1)$

$\boxed{12x + 4y - z - 25 = 0}$

Normal line: $\boxed{\frac{x-2}{12} = \frac{y-1}{4} = \frac{z-3}{-1}}$

⑥ $\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & 2 \\ 2 & -3 & 0 \end{vmatrix}$

$\boxed{\vec{w} = 6\vec{i} + 4\vec{j} - 6\vec{k}}$

⑦ $\frac{dy}{dx} = -\frac{f_x}{f_y}$

$\ln(x^2+y^2) - \arctan(y/x) = 0$ (yx^{-1})

$f_x = \frac{1}{x^2+y^2}\cdot 2x - \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)$

$= \frac{2x}{x^2+y^2} + \frac{y}{x^2+y^2}$

$f_y = \frac{1}{x^2+y^2}2y - \frac{1}{1+y^2/x^2}\left(\frac{1}{x}\right)$

$= \frac{2y}{x^2+y^2} - \frac{x}{x^2+y^2}$

$\frac{dy}{dx} = -\frac{2x+y}{2y-x}$

$= \boxed{\frac{2x+y}{x-2y}}$

⑧ $\sin(xyw) + x^2 + y^2 + w^2 = 3$

$F_x = yw\cos(xyw) + 2x$

$F_w = xy\cos(xyw) + 2w$

$\frac{\partial w}{\partial x} = -\frac{F_x}{F_w} = \boxed{\frac{-(2x + yw\cos xyw)}{2w + xy\cos xyw}}$

⑨ $x = u + v \qquad y = uv$

$F = u + v - x = 0 \qquad G = uv - y = 0$

$F_x = -1 \quad F_u = 1 \quad G_x = 0 \quad G_u = v$

$F_y = 0 \quad F_v = 1 \quad G_y = -1 \quad G_v = u$

$\frac{\partial u}{\partial x} = \frac{G_xF_v - G_vF_x}{F_uG_v - F_vG_u} = \frac{0 - u(-1)}{1\cdot u - 1\cdot v}$

$\frac{\partial v}{\partial y} = \frac{G_yF_u - G_uF_y}{F_vG_u - F_uG_v} = \frac{-1\cdot 1 - v\cdot 0}{1\cdot v - 1\cdot u}$

$\boxed{\frac{\partial u}{\partial x} = \frac{u}{u-v};\ \frac{\partial v}{\partial y} = \frac{1}{u-v}}$

⑩ $z = x\cos y - y\cos x$

$\frac{\partial z}{\partial x} = \cos y + y\sin x \qquad \frac{\partial z}{\partial y} = -x\sin y - \cos x$

$\boxed{\frac{\partial^2 z}{\partial x^2} = y\cos x} \qquad \boxed{\frac{\partial^2 z}{\partial y^2} = -x\cos y}$

$\boxed{\frac{\partial^2 z}{\partial x\partial y} = -\sin y + \sin x}$

11. $f(x,y) = x^3 + y^3 + 3xy$

$f_{,1} = 3x^2 + 3y = 0 \qquad -x^2 = y$

$f_{,2} = 3y^2 + 3x = 0 \qquad 3x^4 + 3x = 0$

$f_{11} = 6x = A \qquad 3x(x^3+1) = 0$

$f_{12} = 3 = B \qquad x = 0 \quad x = -1$

$\phantom{f_{12} = 3 = B} \qquad y = 0 \quad y = -1$

$f_{22} = 6y = C$

If $x=0,\ y=0,\ f(x,y) = 0,$

$AC - B^2 = -9 < 0$ Saddlepoint.

If $x = -1,\ y = -1,\ f(x,y) = 1$

$AC - B^2 = 36 - 9 > 0,\ A < 0$ local max.

Alt. problem: $V = 500$ cu ft. $= lwh \qquad wh = \frac{500}{l} \quad lw = \frac{500}{h}$

$A = 2wh + 2lh + lw$

$= \frac{1000}{l} + 2lh + \frac{500}{h}$

$\frac{\partial A}{\partial l} = -1000 l^{-2} + 2h = 0$

$\frac{\partial A}{\partial h} = 2l - 500 h^{-2} = 0$

$h = \frac{500}{l^2} \qquad 2l - 500 \cdot \frac{l^4}{500^2} = 0$

$1000 l - l^4 = 0$

$l(1000 - l^3) = 0$

$l = 10'$

$h = 5'$

$w = 10'$

Surface $= 100$
$+ 100$
$+ 100$
$\boxed{300 \text{ sq ft.}}$

Arf's law:

Given: $F(x,y,z) = 0 \qquad G(x,y,z) = 0$

$dF = F_x\,dx + F_y\,dy + F_z\,dz = 0$

$dG = G_x\,dx + G_y\,dy + G_z\,dz = 0$

Divide by dx: (1) $F_x + F_y \frac{dy}{dx} + F_z \frac{dz}{dx} = 0$

(2) $G_x + G_y \frac{dy}{dx} + G_z \frac{dz}{dx} = 0$

Eliminate $\frac{dy}{dx}$ by mult. eq. (1) by G_y and eq. (2) by $(-F_y)$.

$G_y F_x + \cancel{G_y F_y \frac{dy}{dx}} + G_y F_z \frac{dz}{dx} = 0$

$-F_y G_x - \cancel{F_y G_y \frac{dy}{dx}} - F_y G_z \frac{dz}{dx} = 0$

$G_y F_x - F_y G_x + (G_y F_z - F_y G_z)\frac{dz}{dx} = 0$

$(G_y F_z - F_y G_z)\frac{dz}{dx} = F_y G_x - G_y F_x$

$$\boxed{\frac{dz}{dx} = \frac{F_y G_x - G_y F_x}{G_y F_z - F_y G_z}}$$