# EXAM 1 C DIFFERENTIAL EQUATIONS Dr. Rapalje

1. Find a differential equation (without constants) for the following equations:

   a) $x^2 + y^2 - cx = 0$

   b) $y^2 = c_1 x + c_2$

Solve each of the following differential equations, subject to indicated conditions (if any!) Name the method of solution used in each case. Give all answers in simplest form.

2. $\dfrac{dy}{dx} = \dfrac{3x + xy^2}{y + x^2 y}$

3. $\dfrac{dy}{dx} = \dfrac{2xy}{x^2 - y^2}$ ; $y(2) = 1$

4. $y^2\,dx = (2xy + x^2)\,dy$

5. $y' = 3x + 2y$

6. $y\,y'' = y'$

7. $2xy\,dx + (x^2+1)\,dy = 0$ ; $y(1) = -3$

8. $\dfrac{dr}{d\theta} = \dfrac{r^2 \sin\theta}{2r\cos\theta - 1}$

9. $(x^2 - y^2)\,dx + 2xy\,dy = 0$

10. $y' \sin x = y \cos x + \sin^2 x$

11. $y' = (x+y)^2$ (Hint: let $v = x + y$)

Thought for today: "But God demonstrates His own love for us, in that while we were yet sinners, Christ died for us." Romans 5:8

Differential Equations Exam I C

Dr. Rapalje

1a) $x^2+y^2-cx=0 \qquad c=\frac{x^2+y^2}{x}$

$2x+2yy'-c=0$

$2x+2yy'-\frac{x^2+y^2}{x}=0$

$2x^2+2xyy'-x^2-y^2=0$

$2xyy'=y^2-x^2$

$y'=\frac{y^2-x^2}{2xy}$

b) $y^2=c_1x+c_2$

$2yy'=c_1$

$2yy''+y'(2y')=0$

$yy''+(y')^2=0$

2. Variables Separable:

$\frac{dy}{dx}=\frac{x(3+y^2)}{y(1+x^2)}$

$\int\frac{y\,dy}{3+y^2}=\int\frac{x\,dx}{1+x^2}$

$\frac{1}{2}\ln(3+y^2)=\frac{1}{2}\ln(1+x^2)+\frac{1}{2}\ln c$

$\ln(3+y^2)-\ln(1+x^2)=\ln c$

$\ln\frac{3+y^2}{1+x^2}=\ln c$

$\frac{3+y^2}{1+x^2}=c$

or $c(1+x^2)=3+y^2$

3. $\frac{dy}{dx}=\frac{2xy}{x^2-y^2} \qquad y(2)=1$ Integrating Factor

$2xy\,dx+(y^2-x^2)dy=0$

$\frac{\partial M}{\partial y}=2x \qquad \frac{\partial N}{\partial x}=-2x$

$IF=e^{\int-\frac{4x}{2xy}dy}=e^{-\int\frac{2}{y}dy}=\frac{1}{y^2}$

$2xy^{-1}dx+(1-x^2y^{-2})dy=0$

$$\left.\begin{aligned}u_1&=x^2y^{-1}+f(y)\\u_2&=y+x^2y^{-1}+f(x)\end{aligned}\right\}\; y+x^2y^{-1}=c$$

$y^2+x^2=cy$

$y(2)=1 \qquad 1+4=c\Rightarrow c=5.$

$y^2+x^2=5y$

4. $y^2dx=(2xy+x^2)dy$

$y^2dx-(2xy+x^2)dy=0$

$\frac{\partial M}{\partial y}=2y \qquad \frac{\partial N}{\partial x}=-2y-2x$

Integrating Factor $=e^{\int\frac{4y+2x}{-(2xy+x^2)}dx}$

$=e^{-\int\frac{2}{x}dx}=x^{-2}$ or $\frac{1}{x^2}$

$x^{-2}y^{+2}dx-(2x^{-1}y+1)dy=0$

$u_1=\int x^{-2}y\,dx=-x^{-1}y^2+f(y)$

$u_2=\int(-2x^{-1}y-1)dy=-x^{-1}y^2-y+f(x)$

$-x^{-1}y^2-y=-c$ or $y^2+xy=cx$

also Homogeneous.

5. $y'=3x+2y$ Linear

$y'-2y=3x$

$IF=e^{\int-2dx}=e^{-2x}$

$ye^{-2x}=\int3xe^{-2x}dx$

$u=3x \qquad dv=e^{-2x}$

$du=3 \qquad v=\frac{e^{-2x}}{-2}$

$ye^{-2x}=-\frac{3}{2}xe^{-2x}+\frac{3}{2}\int e^{-2x}dx$

$=-\frac{3}{2}xe^{-2x}-\frac{3}{4}e^{-2x}+\frac{c}{4}$

$4y=-6x-3+ce^{2x}$

$4y+6x+3=ce^{2x}$

6. $yy''=y'$

Variable Missing.

Let $v=y'$

$y''=v'=\frac{dv}{dy}\frac{dy}{dx}$

$yv'=v$

$y\left(\frac{dv}{dy}\cdot v\right)=v$

$dv=\frac{dy}{y}$

$v=\ln y+\ln c_1$

$\frac{dy}{dx}=\ln c_1y$

$\frac{dy}{\ln c_1y}=dx$

$x=\int\frac{dy}{\ln c_1y}+c_2$

(I knew it!)

7. $2xy\,dx + (x^2+1)\,dy = 0$

$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$ Exact.

$U_1 = \int 2xy\,dx = x^2y + f(y)$

$U_2 = \int (x^2+1)\,dy = x^2y + y + f(x)$

$x^2y + y = C$ or $y(x^2+1) = C$

8. $\frac{dr}{d\theta} = \frac{r^2\sin\theta}{2r\cos\theta - 1}$

$r^2\sin\theta\,d\theta + (1-2r\cos\theta)\,dr = 0$

$\frac{\partial M}{\partial r} = 2r\sin\theta = \frac{\partial N}{\partial \theta}$ Exact.

$U_1 = \int r^2\sin\theta\,d\theta = -r^2\cos\theta + f(r)$

$U_2 = \int (1-2r\cos\theta)\,dr = r - r^2\cos\theta + f(\theta)$

$r - r^2\cos\theta = C$

9. $(x^2-y^2)\,dx + 2xy\,dy = 0$

$\frac{\partial M}{\partial y} = -2y \quad \frac{\partial N}{\partial x} = 2y$

Integrating factor or Homogeneous.

$I.F. = e^{\int \frac{-4y}{2xy}dx} = \frac{1}{x^2}$

$(1 - x^{-2}y^2)\,dx + 2x^{-1}y\,dy = 0$

$U_1 = \int (1-x^{-2}y^2)\,dx = x + x^{-1}y^2 + f(y)$

$U_2 = \int 2x^{-1}y\,dy = x^{-1}y^2 + f(x)$

$x + x^{-1}y^2 = C$

$x^2 + y^2 = Cx$

— OR —

Homogeneous Let $y = vx$

$y' = v + v'x$

$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$

$v + v'x = \frac{1}{2}\left[\frac{y}{x} - \frac{x}{y}\right]$

$v + v'x = \frac{1}{2}\left[v - \frac{1}{v}\right]$

$v'x = \frac{1}{2}\left(\frac{v^2-1}{v}\right) - v$

$v'x = \frac{v^2 - 1 - 2v^2}{2v}$

$\frac{2v\,dv}{-1-v^2} = \frac{dx}{x}$ or $\frac{2v\,dv}{v^2+1} = -\frac{dx}{x}$

$\ln(v^2+1) = -\ln x + \ln C$

$(v^2+1) = \frac{C}{x}$

$\frac{y^2}{x^2} + 1 = \frac{C}{x}$ or $y^2 + x^2 = Cx$

10. $y'\sin x = y\cos x + \sin^2 x$

$y' - (\cot x)\,y = \sin x$ Linear.

$I.F. = e^{\int -\cot x\,dx} = e^{-\ln \sin x} = \frac{1}{\sin x}$

$y\frac{1}{\sin x} = \int (\sin x)\frac{1}{\sin x}\,dx$

$\frac{y}{\sin x} = x + C$

$y = x\sin x + C\sin x$

[If problem was read "$\sin^3 x$"

then $y' - (\cot x)\,y = \sin^2 x$

and $y\frac{1}{\sin x} = \int \sin^2 x \frac{1}{\sin x}\,dx$

$y\csc x = -\cos x + C$.]

11. $y' = (x+y)^2$ Let $v = x+y$

$v' = 1 + y'$

$v' - 1 = v^2$

$v' = v^2 + 1$

$\frac{dv}{v^2+1} = dx$

$\arctan v = x + C$

$v = \tan(x+C)$

$x + y = \tan(x+C)$