

Find the general solution for the differential equations:

$$1. \quad y''' - y' = 0$$

$$2. \quad (D^3 - 2D^2 - D + 2)y = 0$$

$$3. \quad (4D^3 - 3D + 1)y = 0$$

$$4. \quad y'' - 6y' + 25y = 0$$

$$5. \quad y''' - y'' - 2y' = e^{-x}$$

$$6. \quad y'' + y = \cos x$$

$$7. \quad y'' + 4y = x^2 + \cos x$$

$$8. \quad (D^3 - 2D^2 - 3D)y = 3x^2 + \sin x$$

$$9. \quad x^2 y'' + x y' + 4y = x$$

(Let $x = e^z$)

$$E.C. (1) \quad D^6 y = 729y$$

$$(2) \quad (D-1)^2 y = \frac{e^x}{1-x^2}$$

Thought for today: This know also, that in the last days perilous times shall come. For men shall be lovers of their own selves, lovers of money, boasters, proud, blasphemers, disobedient to parents, unthankful, unholy, ... lovers of pleasure more than lovers of God; having a form of godliness, but denying the power thereof. II Tim 3:1-5.

1. $y''' - y' = 0$
 $m^3 - m = 0$
 $m(m^2 - 1) = 0$
 $m(m-1)(m+1) = 0$

$y = C_1 + C_2 e^x + C_3 e^{-x}$

2. $(D^3 - 2D^2 - D + 2)y = 0$
 $m^2(m-2) - (m-2) = 0$
 $(m^2-1)(m-2) = 0$
 $(m-1)(m+1)(m-2) = 0$

$y = C_1 e^x + C_2 e^{-x} + C_3 e^{2x}$

3. $(4D^3 - 3D + 1)y = 0$
 $4 \quad 0 \quad -3 \quad 1 \quad | \frac{1}{2}$
 $\quad \quad \quad 2 \quad 1 \quad -1$
 $\quad \quad \quad \hline$
 $4 \quad 2 \quad -2 \quad 0$

$4m^2 + 2m - 2 = 0$
 $2m^2 + m - 1 = 0$
 $(2m-1)(m+1) = 0$
 $m = \frac{1}{2} \quad m = -1$

$y = C_1 e^{-x} + C_2 e^{\frac{1}{2}x} + C_3 x e^{\frac{1}{2}x}$

4. $y'' - 6y' + 25y = 0$
 $m^2 - 6m + 25 = 0$
 $m^2 - 6m + 9 = -16$
 $(m-3) = \pm 4i$
 $m = 3 \pm 4i$

$y = e^{3x}(C_1 \sin 4x + C_2 \cos 4x)$

5. $y''' - y'' - 2y' = e^{-x}$
 $(m^3 - m^2 - 2m) = 0$
 $m(m^2 - m - 2) = 0$
 $m(m-2)(m+1) = 0$

$m=0 \quad m=2 \quad m=-1$
 $y_c = C_1 + C_2 e^{-x} + C_3 e^{2x}$

$y_p = Ax e^{-x}$
 $y_p' = -Ax e^{-x} + Ae^{-x}$
 $y_p'' = +Ax e^{-x} - Ae^{-x} - Ae^{-x}$
 $= Ax e^{-x} - 2Ae^{-x}$

$y_p''' = -Ax e^{-x} + Ae^{-x} + 2Ae^{-x}$
 $= -Ax e^{-x} + 3Ae^{-x}$

$-Ax e^{-x} + 3Ae^{-x} - Ax e^{-x} + 2Ae^{-x} + 2Ax e^{-x} - 2Ae^{-x} = e^{-x}$
 $A = \frac{1}{3}$

$y = C_1 + C_2 e^{-x} + C_3 e^{2x} + \frac{1}{3} x e^{-x}$

6. $y'' + y = \sec x$
 $y_c = C_1 \sin x + C_2 \cos x$

Let $y = A(x) \sin x + B(x) \cos x$

$y' = A(x) \cos x + A'(x) \sin x - B(x) \sin x + B'(x) \cos x$

$y'' = -A(x) \sin x + A'(x) \cos x - B(x) \cos x - B'(x) \sin x$
 $-A \cancel{\sin x} + A' \cos x - B \cancel{\cos x} - B' \sin x + A \cancel{\cos x} + B \cancel{\sin x} = \sec x$

$A'(x) \sin x + B'(x) \cos x = 0$ mult. by $\sin x$
 $A'(x) \cos x - B'(x) \sin x = \sec x$ mult. by $\cos x$

$A'(x) \sin^2 x + A'(x) \cos^2 x = 1$

$A'(x) = 1$ Then $\sin x + B'(x) \cos x = 0$

$A(x) = x + C_1$

$B'(x) = -\tan x$

$y = C_1 \sin x + C_2 \cos x + x \sin x + \cos x \ln |\cos x|$
 $3(x) = -\int \tan x dx = \ln |\cos x| + C_2$

7. $y'' + 4y = x^2 + \cos x$
 $y_c = C_1 \sin 2x + C_2 \cos 2x$

$y_p = ax^2 + bx + c + f \cos x + g \sin x$

$y_p' = 2ax + b + (-f \sin x) + g \cos x$

$y_p'' = 2a - f \cos x - g \sin x$

$(2a) - f \cos x - g \sin x + 4ax^2 + 4bx + (4c) + 4f \cos x + 4g \sin x = x^2 + \cos x$

$4a = 1 \quad 4b = 0 \quad 2a + 4c = 0 \quad 3f = 1 \quad 3g = 0$
 $a = \frac{1}{4} \quad b = 0 \quad \frac{1}{2} + 4c = 0 \quad f = \frac{1}{3} \quad g = 0$
 $c = -\frac{1}{8}$

$y = C_1 \sin 2x + C_2 \cos 2x + \frac{1}{4} x^2 - \frac{1}{8} + \frac{1}{3} \cos x$

8. $(D^3 - 2D - 3D)y = 3x + \sin x$

$m(m-3)(m+1) = 0$

$y_c = C_1 + C_2 e^{3x} + C_3 e^{-x}$

$y_p = ax^2 + bx + c + f \sin x + g \cos x$

Avoid repetition, so

$y_p = ax^3 + bx^2 + cx + f \sin x + g \cos x$

$y_p' = 3ax^2 + 2bx + c + f \cos x - g \sin x$

$y_p'' = 6ax + 2b - f \sin x - g \cos x$

$y_p''' = 6a - f \cos x + g \sin x$

$6a - f \cos x + g \sin x - 12ax - 4b + 2f \sin x + 2g \cos x - 9a x^2 - 6bx - 3c - 3f \cos x + 3g \sin x = 3x^2 + \sin x$

$-9a = 3 \quad -12a - 6b = 0 \quad 6a - 4b - 3c = 0$
 $a = -1/3 \quad 4 - 6b = 0 \quad -2 - 8/3 - 3c = 0$
 $b = 2/3 \quad -14/3 = 3c$
 $c = -14/9$

$-4f + 2g = 0 \quad 2f + g = 0$
 $2f + 4g = 1 \quad 2f + 4g = 1$
 $5g = 1 \quad g = 1/5$
 $2f = g \quad f = 1/10$

$y = C_1 + C_2 e^{3x} + C_3 e^{-x} - 1/3 x^3 + 2/3 x^2 - 14/9 x + 1/5 \cos x + 1/10 \sin x$

9. $x^2 y'' + x y' + 4y = x$

$x = e^z, \quad x y' = \frac{dy}{dz} \quad x^2 y'' = \frac{d^2 y}{dz^2} - \frac{dy}{dz}$
 $z = \ln x$

$\frac{d^2 y}{dz^2} - \frac{dy}{dz} + \frac{dy}{dz} + 4y = e^z$

$\frac{d^2 y}{dz^2} + 4y = e^z$

$y_c = C_1 \cos 2z + C_2 \sin 2z$

$y_p = A e^z$

$y_p' = A e^z$

$y_p'' = A e^z$

$A e^z + 4A e^z = e^z$

$5A e^z = e^z$

$A = 1/5$

$y = C_1 \cos(2 \ln x) + C_2 \sin(2 \ln x) + 1/5 x$

$(D^2 - 2D + 1)y = \frac{e^x}{1-x^2}$

$(2B + 2Bx + B' - 2A - 2Bx - 2B + A + Bx)e^x = \frac{e^x}{1-x^2}$
 $y'' = A e^x + (A' e^x) + B x e^x + B e^x + (B' e^x) + B e^x + B' e^x$
 $= (A + 2B) e^x + B x e^x + B' e^x$

$B' = \frac{1}{1-x^2} \quad A' = -B'x = \frac{x}{x^2-1}$
 $B = \frac{1}{2} \log \frac{1+x}{1-x} + C_2 \quad A = \frac{1}{2} \log |x^2-1| + C_1$

$y = C_1 e^x + C_2 x e^x + \frac{1}{2} \log |x^2-1| e^x + \frac{1}{2} x \log \frac{1+x}{1-x} e^x$

E.C. 1. $D^6 y - 729 y = 0$

$m^6 - 729 = 0$

$(m^3 - 27)(m^3 + 27) = 0$

$(m-3)(m^2 + 3m + 9)(m+3)(m^2 - 3m + 9) = 0$

$m = \pm 3, \quad m = \frac{-3 \pm \sqrt{9-36}}{2} \quad m = \frac{3 \pm \sqrt{9-36}}{2}$

$m = -\frac{3}{2} \pm \frac{3\sqrt{3}i}{2} \quad m = \frac{3}{2} \pm \frac{3\sqrt{3}i}{2}$

$y = C_1 e^{3x} + C_2 e^{-3x} + e^{-3x/2} (C_3 \cos \frac{3\sqrt{3}}{2} x + C_4 \sin \frac{3\sqrt{3}}{2} x)$
 $+ e^{3x/2} (C_5 \cos \frac{3\sqrt{3}}{2} x + C_6 \sin \frac{3\sqrt{3}}{2} x)$

2. $(D-1)^2 y = \frac{e^x}{1-x^2}$

$y_c = C_1 e^x + C_2 x e^x$

$A' + B'x = 0$

$y = A e^x + B x e^x$

$y' = A e^x + (A' e^x) + B x e^x + B e^x + (B' e^x)$

$y'' = A e^x + (A' e^x) + B x e^x + B e^x + (B' e^x) + B e^x + B' e^x$
 $= (A + 2B) e^x + B x e^x + B' e^x$

$y = C_1 e^x + C_2 x e^x + \frac{1}{2} \log |x^2-1| e^x + \frac{1}{2} x \log \frac{1+x}{1-x} e^x$