CALCULUS II EXAM 5C Dr. RAPALJE

Show all work on separate paper. Calculators, approved formula sheets are allowed. (For whatever it is worth, $\sqrt{1+\cos\theta} = \sqrt{2}\cos\frac{\theta}{2}$)

1. Eliminate the parameter and find the xy equation:
$$x = h + a\cos\theta, \quad y = k + b\sin\theta.$$

2. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $x = t^2+3t$, $y = t+1$.

3. Find the equation of the tangent line to $x = t^2 - t$, $y = t^3+3t$ at $t = -1$.

4. Find the arclength of $x = t^2$; $y = 4t^3 - 1$, $-1 \le t \le 1$.

5. Use parametric equations $x = a\cos t$, $y = b\cos t$, and the parametric area integral to show that the area enclosed by an ellipse is πab.

6. Find the polar equation for $x^2+y^2-2ax = 0$

7. Find the xy equation for $r = \dfrac{6}{2\cos\theta - 3\sin\theta}$.

8. If $r = 3\cos\theta$, show that $\frac{dy}{dx} = -\cot 2\theta$.

9. Find all tangents at the pole for $0 \le \theta \le 2\pi$ for $r = 2 + 4\cos\theta$ (i.e., for what values of θ does $r = 0$?)

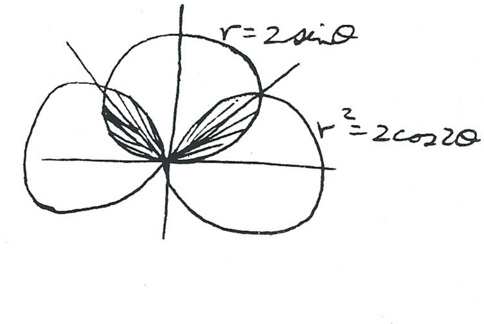
10. Sketch the graph of $r = \dfrac{3}{1-3\sin\theta}$.

11. Find the equation of the ellipse with focus at the pole, and with vertices at $(6,0)$ and $(2,\pi)$.

12. Find the $\tan\psi$ for $r = 4\sin 2\theta$ at $\theta = \pi/6$.

13. Set up only to find the area interior to the inner loop of $r = 1+2\cos\theta$.

14. Set up only to find the area (shaded) between the graphs $r = 2\sin\theta$ and $r^2 = 2\cos 2\theta$



15. Find the length of the curve $r = 5(1+\cos\theta)$ for $0 \le \theta \le 2\pi$.

CALCULUS II EXAM 5C Solutions Dr. RAPALJE

1. $x = h + a\cos\theta$, $y = k + b\sin\theta$

$\frac{(x-h)}{a} = \cos\theta \quad \frac{(y-k)}{b} = \sin\theta$

Since $\cos^2\theta + \sin^2\theta = 1$, $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

2. $x = t^2 + 3t$, $y = t + 1$

$\frac{dx}{dt} = 2t + 3 \quad \frac{dy}{dt} = 1$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{2t+3}$ → $\frac{dy}{dx} = (2t+3)^{-1}$

$\frac{d}{dt}\left(\frac{dy}{dx}\right) = -(2t+3)^{-2} \cdot 2$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{-2}{(2t+3)^2}}{2t+3} = \frac{-2}{(2t+3)^3}$

3. $x = t^2 - t$, $y = t^3 + 3t$ at $t = -1$

$x_0 = 1 - (-1) \quad y_0 = -1 - 3$

$x_0 = 2 \quad y_0 = -4$

$\frac{dx}{dt} = 2t - 1 \quad \frac{dy}{dt} = 3t^2 + 3 \quad \frac{dy}{dx} = \frac{3t^2+3}{2t-1}$

$y - y_0 = m(x - x_0) \quad m = \frac{3+3}{-3} = -2$

$y + 4 = -2(x - 2)$

$y = -2x$

4. $x = t^2 \quad y = 4t^3 - 1 \quad -1 \le t \le 1$

$\frac{dx}{dt} = 2t \quad \frac{dy}{dt} = 12t^2$

$s = \int_{-1}^{1} \sqrt{4t^2 + 144t^4}\, dt = 2\int_0^1 \sqrt{4t^2(1+36t^2)}\, dt$

$= 2\int_0^1 2t\sqrt{1+36t^2}\, dt \quad$ let $u = 1 + 36t^2$, $du = 72t\,dt$

$= 4\int_1^{37} u^{1/2} \frac{du}{72} = \frac{1}{9} \cdot \frac{2}{3} u^{3/2}\Big|_1^{37} = \frac{1}{27}\left(37^{3/2} - 1\right)$

5. $x = a\cos t \quad y = b\sin t$

$dx = -a\sin t\,dt$

QI $A = \int y\,dx = \int (b\sin t)(-a\sin t)\,dt$

$A = -ab\int_{\pi/2}^{0} \sin^2 t\, dt = -ab\int_{\pi/2}^{0} \frac{1-\cos 2t}{2}\, dt$

$= -\frac{ab}{2}\left[t - \frac{\sin 2t}{2}\right]_{\pi/2}^{0} = -\frac{ab}{2}\left[0 - \frac{\pi}{2}\right] = \frac{\pi ab}{4}$

Total Area $= 4\left[\frac{\pi ab}{4}\right] = \pi ab$

6. $x^2 + y^2 - 2ax = 0$

$r^2 - 2ar\cos\theta = 0$

$r(r - 2a\cos\theta) = 0$

$r = 0$ (pole) $\quad r = 2a\cos\theta$

7. $r = \frac{6}{2\cos\theta - 3\sin\theta}$

$2r\cos\theta - 3r\sin\theta = 6$

$2x - 3y = 6$

8. $r = 3\cos\theta$

$\frac{dr}{d\theta} = -3\sin\theta$

$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$

$= \frac{(3\cos\theta)\cos\theta + (-3\sin\theta)\sin\theta}{(-3\cos\theta)\sin\theta + (-3\sin\theta)\cos\theta}$

$= \frac{3(\cos^2\theta - \sin^2\theta)}{-3(2\sin\theta\cos\theta)} = -\frac{\cos 2\theta}{\sin 2\theta} = -\cot 2\theta$

9. $r = 2 + 4\cos\theta = 0$

$\cos\theta = -\frac{1}{2}$ QII, III.

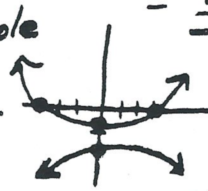
$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

10. $r = \frac{3}{1 - 3\sin\theta}$

$e = 3$ Dir. below pole

Hyperbola.

θ	0	$\pi/2$	π	$3\pi/2$
r	3	$-3/2$	3	$3/4$



11. $V(6,0)$ $V(2,\pi)$ Ellipse

Dir. left

$r = \frac{ep}{1 - e\cos\theta}$

$a = 4$

$c = 2$

$e = \frac{2}{4} = \frac{1}{2}$

$e = \frac{f \text{ to } V}{V \text{ to } d} = \frac{2}{u} = \frac{1}{2}$

$u = 4$

$p = 2 + 4 = 6$

$r = \frac{\frac{1}{2}\cdot 6}{1 - \frac{1}{2}\cos\theta}$

$r = \frac{6}{2 - \cos\theta}$

12. $r = 4\sin 2\theta$, $\theta = \pi/6$

$\frac{dr}{d\theta} = 4\cos 2\theta \cdot 2$

$\tan\psi = \frac{r}{\frac{dr}{d\theta}} = \frac{4\sin 2\theta}{8\cos 2\theta}$

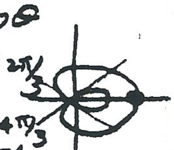
$= \frac{4 \cdot \frac{\sqrt{3}}{2}}{8 \cdot \frac{1}{2}} = \frac{\sqrt{3}}{2}$

13. $r = 1 + 2\cos\theta$

$1 + 2\cos\theta = 0$

$\cos\theta = -\frac{1}{2}$

$\theta = 2\pi/3, 4\pi/3$



$A = \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (1 + 2\cos\theta)^2\, d\theta$

$= 2 \cdot \frac{1}{2}\int_{2\pi/3}^{\pi} (1 + 2\cos\theta)^2\, d\theta$

Exam 5e Solutions (con't) Dr. RAPALJE

14. $r = 2\sin\theta \qquad r^2 = 2\cos 2\theta$

$r^2 = 4\sin^2\theta$

Intersect at $4\sin^2\theta = 2\cos 2\theta$

$= 2(1 - 2\sin^2\theta)$

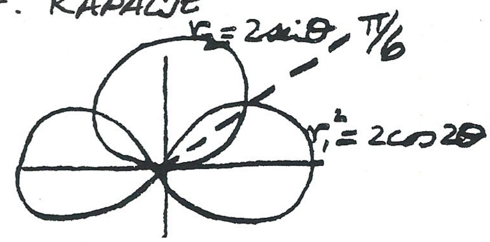
$4\sin^2\theta = 2 - 4\sin^2\theta$

$8\sin^2\theta = 2$

$\sin^2\theta = 1/4$

$\sin\theta = \pm\frac{1}{2}$

$\theta = \pi/6,\ 5\pi/6,\ 7\pi/6,\ \frac{11\pi}{6}$



$$A = 2\cdot\frac{1}{2}\int_0^{\pi/6} r_1^2\, d\theta + 2\cdot\frac{1}{2}\int_{\pi/6}^{\theta_1} r_2^2\, d\theta$$

θ_1 = First polar tangent of r_1

$2\cos 2\theta = 0$

$\cos 2\theta = 0$

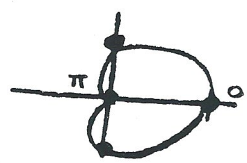
$2\theta = \pi/2,\ 3\pi/2$, etc.

$\theta = \pi/4$

$$A = 2\cdot\frac{1}{2}\left[\int_0^{\pi/6} 2\cos 2\theta\, d\theta + \int_{\pi/6}^{\pi/4} 4\sin^2\theta\, d\theta\right]$$

15. $r = 5(1+\cos\theta)$

$\frac{dr}{d\theta} = -5\sin\theta$



$s = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ ⟶ $r^2 = 25(1 + 2\cos\theta + \cos^2\theta)$

$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 25 + 50\cos\theta + 25\cos^2\theta + 25\sin^2\theta$

$= 25 + 50\cos\theta + 25(\cos^2\theta + \sin^2\theta)$

$= 50 + 50\cos\theta$

$= 50(1+\cos\theta)$

$= 2\int_0^{\pi} \sqrt{50(1+\cos\theta)}\, d\theta$

$= 2\int_0^{\pi} 5\sqrt{2}\left(\sqrt{2}\cos\frac{\theta}{2}\right) d\theta$

$= 20\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta$

$= 20\cdot\frac{2}{1}\sin\frac{\theta}{2}\Big|_0^{\pi} = 40$