Categorical syllogisms – a set theory do-over

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Abstract

One gives a set interpretation to categorical syllogisms. The standard S,P,M terms become sets, and the standard O,I, (resp. A,E), particular, (resp. universal,) quantifiers, lay set elements in, (resp. empty), subsets of a universal set U which contains S,P,M and their complementary sets S',P',M'. One can see that any pair of categorical premises (PCP), if it entails a logical conclusion (LC) at all, (and thus generates a valid categorical argument (VCA)), its LC may have only one of the following three formats: (#1) Three subsets of M, the middle term, (or of M'), turn out to be empty. (U is partitioned into 8 subsets: MSP, MS'P, MSP', MS'P', M'SP, M'S'P, M'SP', M'S'P'; each of the other sets but U is partitioned into 4 subsets.) The type #1 LC is entailed by any PCP made of two universal premises both containing, or "acting on", either M or M'. Note that each and every such pair entails the above type of LC (not only the pairs A(M,P)A(M,S) and E(M,P)A(M,S) which generate valid syllogisms (VS) via existential import (ei) on M, and whose LCs are I(S,P), Darapti, and, resp., O(S,P), Felapton/Fesapo). (#2) One of the 8 subsets of U turns out to be non-empty. Any PCP made of one universal premise and one particular premise, both acting either on M or on M', entails such an LC. (#3) The sets P, (or P'), and S, (or S'), have each three empty subsets as a result of any PCP made of two universal premises, one acting on M and the other one acting on M'. Note that the PCPs that generate valid syllogisms (VS), are those whose LC, by definition, consists of one of the A,O,E,I quantifiers applied to the ordered pair (S,P). The VS set is a subset of the VCA set. Since sets are not "distributed", do not appear in "figures", do not have "moods", etc., such concepts are (blissfully) ignored by the "set treatment" of the categorical syllogisms. Using set relabelings of S,P,M,S',P',M', one can show that the VCAs inside each of the three VCA classes are transformed into each other, i.e., are equivalent to one another up to a "renaming transformation". The rest of the PCPs do not entail any LC.

Keywords. *categorical premises • valid syllogisms • valid categorical arguments • cylindrical Venn diagram • Karnaugh map • categorical operators*

1. Sets and categorical premises

By definition a categorical syllogism is made of a PCP to which one tacks a 3rd statement, an (S,P)-conclusion, i.e., one of the categorical operators A,O,E,I applied to the ordered pair (S,P). If the conclusion is truly entailed by the PCP one has a valid syllogism (VS), otherwise the syllogism is invalid. There are 36 distinct pairs of categorical premises (PCP) expressed via only the S,P,M terms and the categorical operators (or quantifiers) A,O,E,I applied to pairs of these 3 terms. Modeling such premises on a cylindrical Venn diagram or Karnaugh map with n=3, where the middle term M occupies one row and M', its complementary set, occupies a 2nd row, one sees that the PCPs do not act symmetrically on the M and M' rows: four P-premises (resp. S-premises) out of six act on M and only two act on M'. Only by adding premises explicitly containing the M' term, namely A(M',*) and O(M',*) where the * stands for either S or P, one can get to 8 P-premises and 8 S-premises, with 4 P-premises, (resp. 4 S-premises) acting on the M set and 4 P-premises, (resp. 4 S-premises) acting in similar ways on the M' set. Only with the "M' premises addition" one gets to a set of 64 distinct PCPs which remains unchanged under the relabelings M↔M', S↔S' and P↔P'. The 36 distinct PCPs which are worded using only the S,P,M terms, as well as the "extended" 64 PCPs, are naturally partitioned into 5 subsets: two subsets whose PCPs do not entail any logical conclusion (LC), and three subsets whose PCPs do each entail at least one LC and thus generate a VCA. This way one obtains three classes of VCAs, each class containing some of the VS. The LCs of the VCA\VS set have one of the formats A(P,S), O(P,S) or I(S',P'). Using a P↔S relabeling one transforms a VCA\VS PCP which entails an A(P,S) or O(P,S) LC, into a PCP which has an A(S,P) or O(S,P) LC; if the latter PCP is a VS PCP, then one may argue that such VCA\VS are of no interest since they are, up to a P↔S relabeling, identical to usual VSs. One may also argue that LCs of the format I(S',P') are of no practical interest, (maybe because an I(S',P') LC introduces the S', P' terms which do not appear in the (English) wording of the 64 PCPs). Such arguments, (outside of set theory), would play an equivalent role to the rules of valid syllogisms – whose main role is to eliminate, (under the "guise" of distribution, of "two negative premises are not allowed", etc.), the VCA\VS set as containing only invalid syllogisms, even if all of them are valid categorical arguments.

The five PCP subsets are characterized as: **#1PCPs).** Two universal premises both acting on either M or M'. There are 8 such PCPs, with each LC being that either M or M' is reduced to just one subset out of its 4 subsets. Then one needs existential import (ei) on either M or M' in order to express the LC without any reference to M (or M'). (M or M' appearing in the LC would mean repeating the premises' content instead of stating the "new knowledge" the VCA or VS "should" bring (Striker 2009: 20).) **#2PCPs).** One universal premise and one particular premise both acting on either M or M'. There are 16 such PCPs; each of their LCs is of the type: one subset out of the 8 subsets of U is ≠Ø. Since U has only 8 subsets, each one being ≠Ø appears as an LC twice: for example, SP'M≠Ø, meaning O(S,P), is the LC of E(M,P)I(M,S), Ferio/Festino/Ferison/Fresison, and also the LC of I(M,P')E(M,S'), Bocardo. **#3PCPs).** Two universal premises, acting one on M and one on M'. The result of such a PCP is the emptying of 4 subsets of U, two subsets being located on the same column, and the other two being located on each side of that column but on different rows (one on the M row, one on the M' row). Consequently, the two LCs of such a PCP are that both sets in one of the pairs of sets, (S,P), (S',P'), (S,P'), (S',P), whose intersection is a column of U, are each reduced to one subset out of 4. For example, Barbara's PCP empties the column SP' and leaves S=SPM and P'=S'P'M', which leads to the following LCs: A(S,P), A(P',S'), and via ei on S and P' to I(S,P) and I(S',P'). Note that the two universal LCs are not independent since A(S,P),=A(P',S') cfm. contraposition or set definitions, but I(S,P) and I(S',P') are independent ei LCs. **#4PCPs).** Both premises are particular. Obviously there are 16 such PCPs and they entail no LC. **#5PCPs).** One premise is universal and one particular, and they act one on M, and the other one on M'. These are 16 more PCPs not entailing any LC.

Each VCA (and VS) LC is easily found either via a “tree like method” (which eliminates, (i.e., closes), any subset (i.e., branch), emptied by a universal premise), or, by looking at the cylindrical Venn diagram below. The VCA are partitioned into three equivalence classes, each class being generated by the PCPs from the subsets #3 to #5 above. Inside each VCA equivalence class, via a relabeling transformation of the sets S,P,M, S',P',M', any of the VCA can be recast (or reformulated) as any other VCA (or VS) from the same class (Radulescu 2017). Out of the 64 distinct PCP (or just pairs), only 32 pairs entail at least one LC and thus generate VCAs, out of which 8 are distinct VS, and 6 are distinct existential import (ei) VS. (If syllogistic figures are used, then one counts 15 VS and 9 ei VS, but this means that, e.g., the same content VS, Ferio/Festino/Ferison/Fresison, receives four different names and counts as 4 distinct VS, when in reality one deals with one PCP, E(M,P)I(M,S), and one LC: O(S,P). The VCA\VS subset contains 16 non ei VCAs and 18 ei VCAs. The “tree like method” is easier to apply if one first writes any premise using only the E or I statements. For any PCP in subset #2 one starts the (very short) tree with the non-empty intersection of the two sets appearing in the particular premise: in Ferio/Festino/Ferison/Fresison's case, Ø≠MS:=M∩S= MSP+MSP' =MSP' since the premise E(M,P) says MP= Ø. Thus the LC is MSP'≠Ø or O(S,P). (One denotes by + the union of disjoint subsets.) For any PCP in subset #1, the “LC subset” is found by “starting a tree” with either M or M' – the set which appears in both universal premises. It will result that M (or M') equals its intersection with the complements of the other two sets appearing in the two **#1PCPs** universal premises. For any PCP in subset #3 each of the two “LC subsets” can be found via two short trees, each starting with one of the “letter sets” other than M and M' and continuing by eliminating its subsets emptied by the two universal premises. For example, in the case of Barbara's PCP, A(M,P)A(S,M)=E(M,P')E(M',S) start with S=SM'+SM =SM=SPM+SP'M=SPM and P'=MP'+M'P'=M'P' =S'P'M' +SP'M'=S'P'M'. The LCs are A(S,P)=A(P',S') (Barbara), and via ei on S, I(S,P) (Barbari), plus, via ei on P', I(S',P'). (This explains why there are 48 distinct VCAs, generated by only 32 PCPs.)

2. The Cylindrical Venn diagram (the Karnaugh map for n=3)

S'P'M	SP'M	SPM	S'PM
S'P'M'	SP'M'	SPM'	S'PM'

Fig. 1

The universal set U is graphed as a rectangle – but the left and right borders of the rectangle are glued together to generate a cylinder, so that S'PM and S'P'M are adjacent, and S'PM' and S'P'M' are adjacent, too – as in the usual 3-circle Venn diagram. On this "cylindrical Venn diagram" - or Karnaugh map with n=3 all the syllogistic conclusions are graphically obvious. It took me about a year to "invent" the cylindrical Venn diagram, only to find out - after I sent to the publisher the first version of a previous paper (Radulescu, 2017) – that Alan Marquand invented it in 1881, and then, in 1952 and 1953, Edward Veitch and Maurice Karnaugh used Karnaugh (-Veitch) maps for n=3, n=4, etc., to find the optimal design of digital circuits. After more than 130 years, these maps, (which have chapters dedicated to them in engineering books), apparently, never made it into logic textbooks – which are still using the 3-circle Venn diagrams.

3. New Notations

The universal operators A and E are "emptying operators", and the particular operators I and O are "element laying" operators. A(M,P) means "All M is P", i.e., the set P'M :=P' ∩M is empty. Thus, in Fig. 1, A(M,P) acts on the M row, by emptying (two "horizontally adjacent" subsets) P'M = SP'M + S'P'M. One always list PCPs with the P-premise written first and the S-premise written after it, and inside each categorical operator the M or M' term will be listed first. Moreover, since A(P,M) acts on M' as E acts on M, (E(M,P) means PM=Ø, A(P,M) means PM'=Ø), we'll denote E':=E(M',*) =A(*,M), I'=I(M',*)=O(*,M) where * stands for either S or P. The above three conventions allow for a shorthand notation for the PCPs: for example EI clearly means E(M,P)I(M,S), AE' means A(M,P)E(M',S)=A(M,P)A(S,M), i.e., Barbara's premises, and AA=A(M,P)A(M,S) are Darapti's premises. We'll also introduce two more operators acting on M': A':=A(M',*) and O':=O(M',*). Now one can order all the 8 possible P-premises, (resp. all 8 possible S-premises), as vector components: $\mathbb{P}_i$ ={E(M,P), E(M,P'), E(M',P), E(M',P'), I(M,P), I(M,P'), I(M',P), I(M',P')}=(E,A,E',A', I,O,I',O')(P), resp., $\mathbb{S}_i$ ={E(M,S), E(M,S'), E(M',S), E(M',S'), I(M,S), I(M,S'), I(M',S), I(M',S')}=(E,A,E',A', I,O,I',O')(S). All the possible PCP are the components of the direct product of these two vectors $L_{ij} = \mathbb{P}_i \otimes \mathbb{S}_j$, i,j = 1,..,8. Without the "M' premises addition", there are 36 distinct PCPs partitioned in the same 5 subsets characterized in the introduction. Note how easy it is now to decide in which one of the 5 classes a PCP falls: AA – two universal premises acting on the M row, EE' (Celarent/Cesare) or AE' – two universal premises acting on two different rows, AO' – no LC, etc.

4. The matrix L_{ij} , i,j = 1,..,8 and its symmetries

The "PCP matrix" L:= $L_{ij} = \mathbb{P}_i \otimes \mathbb{S}_j$, i,j = 1,8 contains 32 PCPs which do not entail any LC, and 32 PCPs which entail at least one LC. The latter ones appear as eight 2 by 2 sub matrices of L. The **#1PCPs** appear in $L^{(1)}:=L_{ij}$, i,j = 1,2, (EE,EA,AE,AA), (whose each LC is M=one of the its 4 subsets), and in $L^{(2)}:=L_{ij}$, i,j=3,4, (E'E',E'A',A'E',A'A') - whose each LC is M'=one of the its 4 subsets. (See Fig. 1.) The following four 2 by 2 sub matrices all contain one particular and one universal premises, (the **#2PCPs** described in Section 2), both acting on either M or M' and whose each LC is that one subset of U is ≠Ø. Since U has only 8 subsets and one has 16 such PCPs, it follows that the LC: some particular subset of U is ≠Ø, appears twice for each subset of U. The four sub matrices containing only **#2PCPs** are: $L^{(3)}:=L_{ij}$, i=1,2, j=5,6, (EI,EO,AI,AO); $L^{(4)}:=L_{ij}$, i=3,4, j=7,8, (E'I',E'O',A'I',A'O'); $L^{(5)}:=L_{ij}$, i= 5,6, j= 1,2, (IE, IA, OE,OA); $L^{(6)}:=L_{ij}$, i=7,8, j=3,4 (I'E',I'A',O'E',O'A'). Finally, the **#3PCPs** appear in $L^{(7)}:=L_{ij}$, i= 1,2, j=3,4, (EE',EA',AE',AA') and $L^{(8)}:=L_{ij}$, i=3,4, j=1,2, (E'E,E'A,A'E,A'A). [The **#4PCPs**, (two particular premises, no LC), appear in the sub-matrix L_{ij}, i,j =5,6,7,8; the **#5PCPs**, (no LC), "span" another four 2 by 2 sub matrices of L.]

An M↔M' relabeling transforms the un-primed premises, operators and LCs into the respective primed ones and vice-versa. An S↔P relabeling transforms, for any pair (i,j), the L_{ij} PCP and its LC into the L_{ji} PCP and its respective LC. One may group the 32 PCPs which entail at least one LC and thus generate VCAs, into 8 subsets of 4 PCPs (generating 6 VCAs per subset). The first PCP in such a group of 4 PCPs belongs to the **#1PCPs**, the following two belong to **#2PCPs** and the last one to the **#3PCPs** class of PCPs. We'll say that each of the 4 PCPs subsets is "bound to" a subset of U: the one **#1PCP**, the following two **#2PCPs** and the one **#3PCP** do not act on the subset of U on which they are "bound", but act on some of its "neighbours" in the cylindrical Venn diagram. To each of the 8 subsets of U one "attaches" a group of four PCPs "bound" to it:

1. VCAs "bound to" the subset S'P'M:

EE=E(M,P)E(M,S) M=S'P'M. If M≠Ø: I(S',P'), No name
IE=I(M,P)E(M,S) S'PM≠Ø or O(P,S), No name
EI=E(M,P)I(M,S) SP'M≠Ø or O(S,P), Ferio/Festino/Ferison/Fresison
EE'=E(M,P)E(M',S) S=SP'M, P=S'PM', E(S,P), Celarent/Cesare
O(S,P) if S≠Ø, Celaront/Cesaro; O(P,S) if P≠Ø, No name

2. VCAs bound to the subset SP'M:

EA=E(M,P)E(M,S') M=SP'M. If M≠Ø: O(S,P), Felapton/Fesapo
IA=I(M,P)E(M,S') SPM≠Ø or I(S,P), Disamis/Dimaris
EO=E(M,P)I(M,S') S'P'M≠Ø or I(S',P'), No name
EA'=E(M,P)E(M',S') S'=S'P'M, P=SPM', A(P,S)=A(S',P'),
I(S,P) if P≠Ø, Bramantip'; I(S',P') if S'≠Ø, No
name

3. VCAs bound to the subset S'PM:

AE=E(M,P')E(M,S)   M=S'PM. If M≠Ø: O(P,S), No name
OE=I(M,P')E(M,S)   S'P'M≠Ø or I(S',P'), No name
AI=E(M,P')I(M,S)   SPM≠Ø or I(S,P), Darii/Datisi
AE'=E(M,P')E(M',S)   S=SPM, P'=S'P'M', A(S,P), Barbara
I(S,P) if S≠Ø, Barbari; I(S',P') if P'≠Ø, No name

4. VCAs bound to the subset SPM:
AA=E(M,P')E(M,S')   M=SPM. If M≠Ø: I(S,P), Darapti
OA=I(M,P')E(M,S')   SP'M ≠Ø or O(S,P), Bocardo
AO=E(M,P')I(M,S')   S'PM≠Ø or O(P,S), No name
AA'=E(M,P')E(M',S')   S'=S'PM, P'=SP'M', E(S',P'), No name
O(P,S) if S'≠Ø, No name; O(S,P) if P'≠Ø, No name

M' row VCAs:
5. VCAs bound to the subset S'P'M':
E'E'=E(M',P)E(M',S)   M'=S'P'M'. If M'≠Ø: I(S',P'), No name
I'E'=I(M',P)E(M',S)   S'PM'≠Ø or O(P,S), No name
E'I'=E(M',P)I(M',S)   SP'M'≠Ø or O(S,P), Baroco
E'E=E(M',P)E(M,S)   S=SP'M', P=S'PM, E(S,P), Camestres/Camenes
O(S,P) if S≠Ø, Camestros/Camenos; O(P,S) if P≠Ø, No name

6. VCAs bound to the subset SP'M':
E'A'=E(M',P)E(M',S')   M'=SP'M'. If M'≠Ø: O(S,P), Felapton'/Fesapo'
I'A'=I(M',P)E(M',S')   SPM'≠Ø or I(S,P), Disamis'/Dimaris'
E'O'=E(M',P)I(M',S')   S'P'M≠Ø or I(S',P'), No name
E'A=E(M',P)E(M,S')   S'=S'P'M', P=SPM, E(S',P)=A(P,S), No name
I(S,P) if P≠Ø, Bramantip, I(S',P') if S'≠Ø, No name

7. VCAs bound to the subset S'PM':
A'E'=E(M',P')E(M',S)   M'=S'PM'. If M'≠Ø: O(P,S), No name
O'E'=I(M',P')E(M',S)   S'P'M'≠Ø or I(S',P'), No name
A' I'=E(M',P')I(M',S)   SPM'≠Ø or I(S,P), Darii'/Datisi'
A'E=E(M',P')E(M,S)   S=SPM', P'=S'P'M, A(S,P)=A(P',S'), Barbara'
I(S,P) if S≠Ø, Barbari'; I(S',P') if P'≠Ø, No name

8. VCAs bound to the subset SPM':
A'A'=E(M',P')E(M',S')   M'=SPM'. If M'≠Ø: I(S,P), Darapti'
O'A'=I(M',P')E(M',S')   SP'M' ≠Ø or O(S,P), Bocardo'
A'O'=E(M',P')I(M',S')   S'PM≠Ø or O(P,S), No name
A'A=E(M',P')E(M,S')   S'=S'PM', P'=SP'M, E(S',P'), No name
O(P,S) if S'≠Ø, No name; O(S,P) if P'≠Ø, No name

One can now define a "relabeling group" acting on the above VCAs subsets 1,2,...,8.

Let p:= P↔P', s:=S↔S', m:=M↔M'. One can see that compositions of s,p,m generate a commutative group G with eight distinct elements:1,s,p,m,sp,sm,pm,spm. Obviously $1=s^2=p^2=m^2=(spm)^2=(ms)^2=(ps)^2=(pm)^2$. This group acts on the above VCAs subsets 1,2,...,8, as follows:

p(1)=3, p(2)=4, p(5)=7, p(6)=8; s(1)=2, s(3)=4, s(5)=6, s(7)=8; m(1)=5, m(2)=6, m(3)=7, m(4)=8.

One can check that {G(1)}= {G(2)}= …={G(8)}= {1,2,3,...,8}. This shows that any VCA from any of the three VCA classes can be recast as any other VCA in the same class. For example, spm(E'E')=AA=A(M,P) A(M,S)=E(M,P')E(M,S') which are Darapti's premises. This means that E'E'=E(M',P)E(M',S)=A(P,M)A(S,M) become Darapti's premises after an spm relabeling. E'E' generates a VCA, and because M is not distributed in either premise, one "knows" that it can not be a VS. In fact one can just apply the definition of a VS as a VCA whose LC is one of the A,O,E,I operators applied to the ordered pair (S,P), and one can see that E'E' is not a VS; the "distribution talk" is unnecessary. (See also Radulescu, 2017.) The attached table shows for each PCP to which one of the eight PCP groups, (equivalent to each other via a relabeling), that PCP belongs. Each such "bound" or equivalency group is made of four PCPs – the 1st group contains the EE, EE', EI, IE PCPs, etc.

References

Karnaugh, Maurice (1953), The map method for synthesis of combinational logic circuits, Transactions of the American Institute of Electrical Engineers, Part 1, 72, 593-599.

Marquand, Allan (1881), On logical diagrams for n terms, Philosophical Magazine 12, 266-270.

Radulescu, Dan C., A cylindrical Venn diagram model for categorical syllogisms. Submitted, 2017.

Striker, Gisela (Translation, Introduction and Commentary, 2009) Aristotle's Prior Analytics Book I. Oxford University Press (Clarendon Aristotle Series), Oxford, p. 20

Veitch, Edward, W., A chart method for simplifying truth functions, Proceedings of the Association for Computing Machinery, pp. 127–133, 1952.

Wikipedia.org, https://en.wikipedia.org/wiki/Karnaugh_map

Wikipedia.org, https://en.wikipedia.org/wiki/Quine%E2%80%93McCluskey_algorithm

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<table>
<tr><th></th><th>E</th><th>A</th><th>E'</th><th>A'</th><th>I</th><th>O</th><th>I'</th><th>O'</th></tr>
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1</td><td>2</td><td>EE'
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1</td><td>2</td><td colspan="2" rowspan="2">One universal + one particular premise, one acting on M, the other on M'. No LC.</td></tr>
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<tr><td>E'</td><td>5</td><td>6</td><td>5</td><td>6</td><td colspan="2" rowspan="2">One universal + one particular premise, one acting on M, the other on M'. No LC.</td><td>5</td><td>6</td></tr>
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<tr><td>I</td><td>IE
1</td><td>2</td><td colspan="2" rowspan="2">One universal + one particular premise, one acting on M, the other on M'. No LC.</td><td colspan="4" rowspan="4">Two particular premises PCPs. No LC.</td></tr>
<tr><td>O</td><td>3</td><td>4</td></tr>
<tr><td>I'</td><td colspan="2" rowspan="2">One universal + one particular premise, one acting on M, the other on M'. No LC.</td><td>5</td><td>6</td></tr>
<tr><td>O'</td><td>7</td><td>8</td></tr>
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The eight PCP groups, equivalent to each other via a relabeling, each made of four PCPs.