Chapter 4
Polynomial Functions

Section 4-5
Solving Polynomial Equations

Finding Solutions and Zeros

You have used the Zero-Product Property to solve factorable quadratic equations. You can extend this technique to solve some higher-degree polynomial equations.

EXAMPLE 1 Solving a Polynomial Equation by Factoring

Solve $2x^3 - 12x^2 + 18x = 0$.

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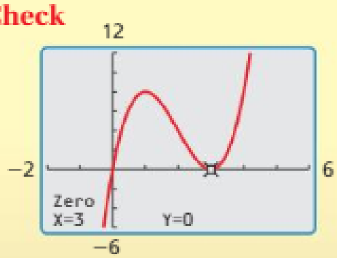


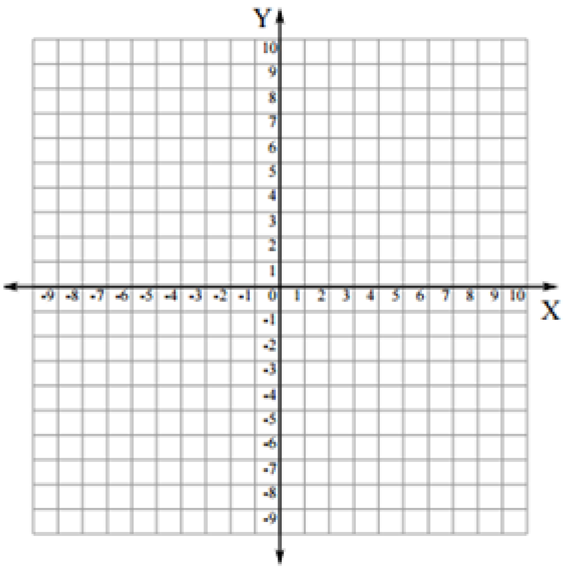
STUDY TIP

Because the factor $x - 3$ appears twice, the root $x = 3$ has a *multiplicity* of 2.

Tutorial

EXAMPLE 2 Finding Zeros of a Polynomial Function

Find the zeros of $f(x) = -2x^4 + 16x^2 - 32$. Then sketch a graph of the function.



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The Rational Root Theorem

The solutions of the equation $64x^3 + 152x^2 - 62x - 105 = 0$ are $-\frac{5}{2}$, $-\frac{3}{4}$, and $\frac{7}{8}$. Notice that the numerators (5, 3, and 7) of the zeros are factors of the constant term, -105. Also notice that the denominators (2, 4, and 8) are factors of the leading coefficient, 64. These observations are generalized by the *Rational Root Theorem*.

Core Concept

The Rational Root Theorem

If $f(x) = a_n x^n + \cdots + a_1 x + a_0$ has *integer* coefficients, then every rational solution of $f(x) = 0$ has the following form:

$$\frac{p}{q} = \frac{\text{factor of constant term } a_0}{\text{factor of leading coefficient } a_n}$$

STUDY TIP

Notice that you can use the Rational Root Theorem to list possible zeros of polynomial functions.

The Rational Root Theorem can be a starting point for finding solutions of polynomial equations. However, the theorem lists only *possible* solutions. In order to find the *actual* solutions, you must test values from the list of possible solutions.

Tutorial

EXAMPLE 3 Using the Rational Root Theorem

Find all real solutions of $x^3 - 8x^2 + 11x + 20 = 0$.

In Example 3, the leading coefficient of the polynomial is 1. When the leading coefficient is not 1, the list of possible rational solutions or zeros can increase dramatically. In such cases, the search can be shortened by using a graph.

EXAMPLE 4 Finding Zeros of a Polynomial Function

Find all real zeros of $f(x) = 10x^4 - 11x^3 - 42x^2 + 7x + 12$.

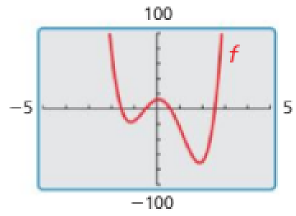
SOLUTION

Step 1 List the possible rational zeros of f: $\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{4}{1}, \pm\frac{6}{1}, \pm\frac{12}{1},$
$\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{4}{5}, \pm\frac{6}{5}, \pm\frac{12}{5}, \pm\frac{1}{10}, \pm\frac{3}{10}$

Step 2 Choose reasonable values from the list above to test using the graph of the function. For f, the values

$$x = -\frac{3}{2}, x = -\frac{1}{2}, x = \frac{3}{5}, \text{ and } x = \frac{12}{5}$$

are reasonable based on the graph shown at the right.



Step 3 Test the values using synthetic division until a zero is found.

$$\begin{array}{r|rrrrr} -\frac{3}{2} & 10 & -11 & -42 & 7 & 12 \\ & & -15 & 39 & \frac{9}{2} & -\frac{69}{4} \\ \hline & 10 & -26 & -3 & \frac{23}{2} & -\frac{21}{4} \end{array} \qquad \begin{array}{r|rrrrr} -\frac{1}{2} & 10 & -11 & -42 & 7 & 12 \\ & & -5 & 8 & 17 & -12 \\ \hline & 10 & -16 & -34 & 24 & 0 \end{array}$$

$-\frac{1}{2}$ is a zero.

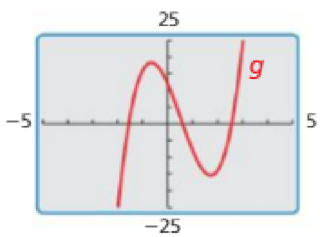
Step 4 Factor out a binomial using the result of the synthetic division.

$f(x) = \left(x + \frac{1}{2}\right)(10x^3 - 16x^2 - 34x + 24)$ Write as a product of factors.

$= \left(x + \frac{1}{2}\right)(2)(5x^3 - 8x^2 - 17x + 12)$ Factor 2 out of the second factor.

$= (2x + 1)(5x^3 - 8x^2 - 17x + 12)$ Multiply the first factor by 2.

Step 5 Repeat the steps above for $g(x) = 5x^3 - 8x^2 - 17x + 12$. Any zero of g will also be a zero of f. The possible rational zeros of g are:



$$x = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{4}{5}, \pm\frac{6}{5}, \pm\frac{12}{5}$$

The graph of g shows that $\frac{3}{5}$ may be a zero. Synthetic division shows that $\frac{3}{5}$ *is* a zero and $g(x) = \left(x - \frac{3}{5}\right)(5x^2 - 5x - 20) = (5x - 3)(x^2 - x - 4)$.
It follows that:

$$f(x) = (2x + 1) \cdot g(x) = (2x + 1)(5x - 3)(x^2 - x - 4)$$

Step 6 Find the remaining zeros of f by solving $x^2 - x - 4 = 0$.

$x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-4)}}{2(1)}$ Substitute 1 for *a*, −1 for *b*, and −4 for *c* in the Quadratic Formula.

$x = \dfrac{1 \pm \sqrt{17}}{2}$ Simplify.

▶ The real zeros of f are $-\frac{1}{2}, \frac{3}{5}, \frac{1 + \sqrt{17}}{2} \approx 2.56$, and $\frac{1 - \sqrt{17}}{2} \approx -1.56$.

6. Find all real zeros of $f(x) = 3x^4 - 2x^3 - 37x^2 + 24x + 12$.

The Irrational Conjugates Theorem

In Example 4, notice that the irrational zeros are *conjugates* of the form $a + \sqrt{b}$ and $a - \sqrt{b}$. This illustrates the theorem below.

Core Concept

The Irrational Conjugates Theorem

Let f be a polynomial function with rational coefficients, and let a and b be rational numbers such that $\sqrt{b}$ is irrational. If $a + \sqrt{b}$ is a zero of f, then $a - \sqrt{b}$ is also a zero of f.

EXAMPLE 5 Using Zeros to Write a Polynomial Function

Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 3 and $2 + \sqrt{5}$.