Chapter 4
Polynomial Functions

Section 4-6
The Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra

The table shows several polynomial equations and their solutions, including repeated solutions. Notice that for the last equation, the repeated solution $x = -1$ is counted twice.

Equation	Degree	Solution(s)	Number of solutions
$2x - 1 = 0$	1	$\frac{1}{2}$	1
$x^2 - 2 = 0$	2	$\pm\sqrt{2}$	2
$x^3 - 8 = 0$	3	$2, -1 \pm i\sqrt{3}$	3
$x^3 + x^2 - x - 1 = 0$	3	$-1, -1, 1$	3

In the table, note the relationship between the degree of the polynomial $f(x)$ and the number of solutions of $f(x) = 0$. This relationship is generalized by the *Fundamental Theorem of Algebra*, first proven by German mathematician Carl Friedrich Gauss (1777−1855).

Core Concept

The Fundamental Theorem of Algebra

Theorem If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has at least one solution in the set of complex numbers.

Corollary If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has exactly n solutions provided each solution repeated twice is counted as two solutions, each solution repeated three times is counted as three solutions, and so on.

The corollary to the Fundamental Theorem of Algebra also means that an nth-degree polynomial function f has exactly n zeros.

EXAMPLE 1 Finding the Number of Solutions or Zeros

a. How many solutions does the equation $x^3 + 3x^2 + 16x + 48 = 0$ have?

b. How many zeros does the function $f(x) = x^4 + 6x^3 + 12x^2 + 8x$ have?

Remember that the number of solutions, number of zeros,and the number of x-intercepts all mean the same thing.

1. How many solutions does the equation $x^4 + 7x^2 - 144 = 0$ have?

2. How many zeros does the function $f(x) = x^3 - 5x^2 - 8x + 48$ have?

Complex Conjugates

Pairs of complex numbers of the forms $a + bi$ and $a - bi$, where $b \neq 0$, are called **complex conjugates**. In Example 2, notice that the zeros $2i$ and $-2i$ are complex conjugates. This illustrates the next theorem.

Core Concept

The Complex Conjugates Theorem

If f is a polynomial function with real coefficients, and $a + bi$ is an imaginary zero of f, then $a - bi$ is also a zero of f.

Tutorial

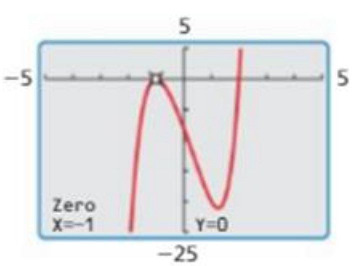
EXAMPLE 2 Finding the Zeros of a Polynomial Function

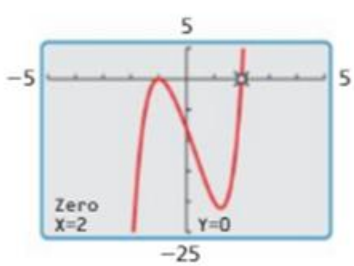
Find all zeros of $f(x) = x^5 + x^3 - 2x^2 - 12x - 8$.

STUDY TIP

Notice that you can use imaginary numbers to write $(x^2 + 4)$ as $(x + 2i)(x - 2i)$. In general, $(a^2 + b^2) = (a + bi)(a - bi)$.

The graph of f and the real zeros are shown. Notice that only the *real* zeros appear as x-intercepts. Also, the graph of f touches the x-axis at the repeated zero $x = -1$ and crosses the x-axis at $x = 2$.





EXAMPLE 3 Using Zeros to Write a Polynomial Function

Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 2 and $3 + i$.

EXAMPLE 5 Real-Life Application

A tachometer measures the speed (in revolutions per minute, or RPMs) at which an engine shaft rotates. For a certain boat, the speed x (in hundreds of RPMs) of the engine shaft and the speed s (in miles per hour) of the boat are modeled by

$$s(x) = 0.00547x^3 - 0.225x^2 + 3.62x - 11.0.$$

What is the tachometer reading when the boat travels 15 miles per hour?