Chapter 4
Polynomial Functions

Section 4-8
Analyzing Graphs of Polynomial Functions
And
Section 4-9
Modeling with Polynomial Functions

Graphing Polynomial Functions

In this chapter, you have learned that zeros, factors, solutions, and x-intercepts are closely related concepts. Here is a summary of these relationships.

Concept Summary

Zeros, Factors, Solutions, and Intercepts

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial function. The following statements are equivalent.

Zero:	k is a zero of the polynomial function f.
Factor:	$x - k$ is a factor of the polynomial $f(x)$.
Solution:	k is a solution (or root) of the polynomial equation $f(x) = 0$.
***x*-Intercept:**	If k is a real number, then k is an x-intercept of the graph of the polynomial function f. The graph of f passes through $(k, 0)$.

The Location Principle

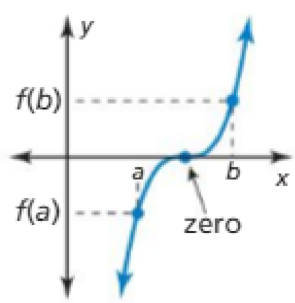
You can use the *Location Principle* to help you find real zeros of polynomial functions.

Core Concept

The Location Principle

If f is a polynomial function, and a and b are two real numbers such that $f(a) < 0$ and $f(b) > 0$, then f has at least one real zero between a and b.

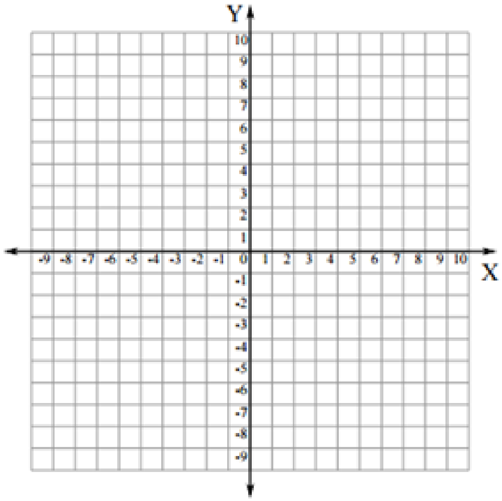
To use this principle to locate real zeros of a polynomial function, find a value a at which the polynomial function is negative and another value b at which the function is positive. You can conclude that the function has *at least* one real zero between a and b.



EXAMPLE 1 Using *x*-Intercepts to Graph a Polynomial Function

Graph the function

$f(x) = \frac{1}{6}(x + 3)(x - 2)^2$.



X	Y

EXAMPLE 2 Locating Real Zeros of a Polynomial Function

Find all real zeros of

$f(x) = 6x^3 + 5x^2 - 17x - 6$.

SOLUTION

Step 1 Use a graphing calculator to make a table.

X	Y1
0	−6
1	−12
2	28
3	150
4	390
5	784
6	1368

X=1

Step 2 Use the Location Principle. From the table shown, you can see that $f(1) < 0$ and $f(2) > 0$. So, by the Location Principle, f has a zero between 1 and 2. Because f is a polynomial function of degree 3, it has three zeros. The only possible *rational* zero between 1 and 2 is $\frac{3}{2}$. Using synthetic division, you can confirm that $\frac{3}{2}$ is a zero.

Step 3 Write $f(x)$ in factored form. Dividing $f(x)$ by its known factor $x - \frac{3}{2}$ gives a quotient of $6x^2 + 14x + 4$. So, you can factor $f(x)$ as

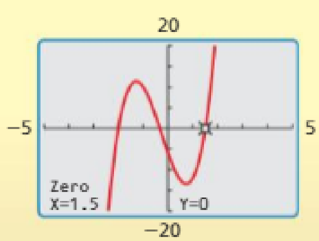
$$f(x) = \left(x - \tfrac{3}{2}\right)(6x^2 + 14x + 4)$$
$$= 2\left(x - \tfrac{3}{2}\right)(3x^2 + 7x + 2)$$
$$= 2\left(x - \tfrac{3}{2}\right)(3x + 1)(x + 2).$$

▶ From the factorization, there are three zeros. The zeros of f are

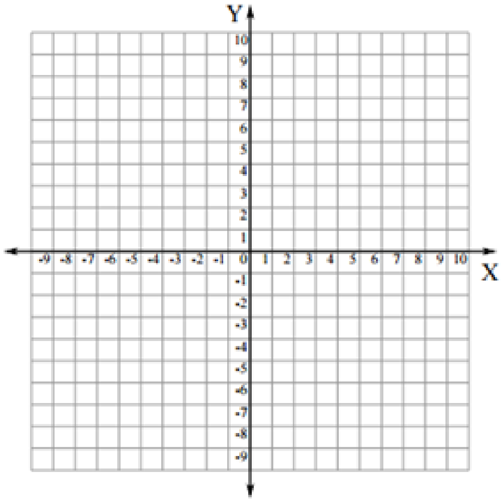
$\frac{3}{2}$, $-\frac{1}{3}$, and -2.

Check this by graphing f.

Check



3. Find all real zeros of $f(x) = 18x^3 + 21x^2 - 13x - 6$.



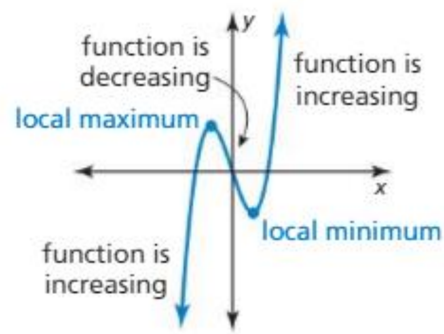
X	Y

Turning Points

Another important characteristic of graphs of polynomial functions is that they have *turning points* corresponding to local maximum and minimum values.

- The y-coordinate of a turning point is a **local maximum** of the function when the point is higher than all nearby points.
- The y-coordinate of a turning point is a **local minimum** of the function when the point is lower than all nearby points.

The turning points of a graph help determine the intervals for which a function is increasing or decreasing.



READING

Local maximum and local minimum are sometimes referred to as *relative maximum* and *relative minimum*.

Core Concept

Turning Points of Polynomial Functions

1. The graph of every polynomial function of degree n has *at most* $n - 1$ turning points.
2. If a polynomial function has n distinct real zeros, then its graph has *exactly* $n - 1$ turning points.

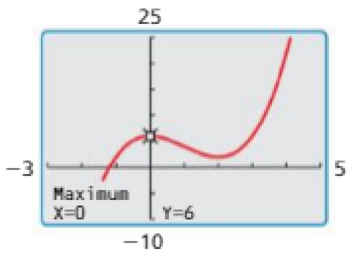
Tutorial

EXAMPLE 3 Finding Turning Points

Graph each function. Identify the x-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which each function is increasing or decreasing.

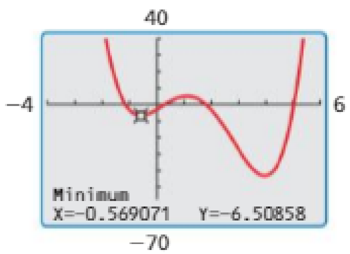
a. $f(x) = x^3 - 3x^2 + 6$ **b.** $g(x) = x^4 - 6x^3 + 3x^2 + 10x - 3$

SOLUTION



a. Use a graphing calculator to graph the function. The graph of f has one x-intercept and two turning points. Use the graphing calculator's *zero*, *maximum*, and *minimum* features to approximate the coordinates of the points.

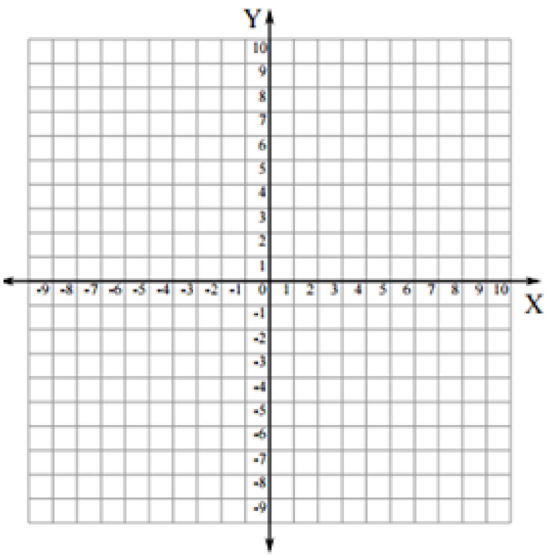
- The x-intercept of the graph is $x \approx -1.20$. The function has a local maximum at $(0, 6)$ and a local minimum at $(2, 2)$. The function is increasing when $x < 0$ and $x > 2$ and decreasing when $0 < x < 2$.



b. Use a graphing calculator to graph the function. The graph of g has four x-intercepts and three turning points. Use the graphing calculator's *zero*, *maximum*, and *minimum* features to approximate the coordinates of the points.

- The x-intercepts of the graph are $x \approx -1.14$, $x \approx 0.29$, $x \approx 1.82$, and $x \approx 5.03$. The function has a local maximum at $(1.11, 5.11)$ and local minimums at $(-0.57, -6.51)$ and $(3.96, -43.04)$. The function is increasing when $-0.57 < x < 1.11$ and $x > 3.96$ and decreasing when $x < -0.57$ and $1.11 < x < 3.96$.

4. Graph $f(x) = 0.5x^3 + x^2 - x + 2$. Identify the x-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which the function is increasing or decreasing.



X	Y

Even and Odd Functions

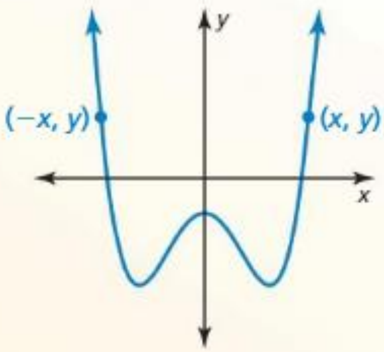
Core Concept

Even and Odd Functions

A function f is an **even function** when $f(-x) = f(x)$ for all x in its domain. The graph of an even function is *symmetric about the y-axis*.

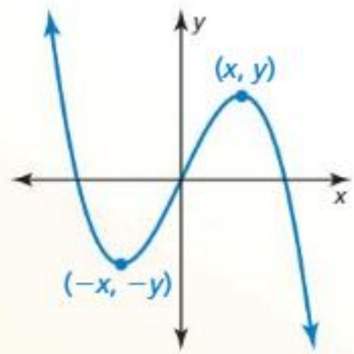
A function f is an **odd function** when $f(-x) = -f(x)$ for all x in its domain. The graph of an odd function is *symmetric about the origin*. One way to recognize a graph that is symmetric about the origin is that it looks the same after a 180° rotation about the origin.

Even Function



For an even function, if (x, y) is on the graph, then $(-x, y)$ is also on the graph.

Odd Function



For an odd function, if (x, y) is on the graph, then $(-x, -y)$ is also on the graph.

EXAMPLE 4 Identifying Even and Odd Functions

Tutorial

Determine whether each function is *even*, *odd*, or *neither*.

a. $f(x) = x^3 - 7x$ **b.** $g(x) = x^4 + x^2 - 1$ **c.** $h(x) = x^3 + 2$

SOLUTION

a. Replace x with $-x$ in the equation for f, and then simplify.

$$f(-x) = (-x)^3 - 7(-x) = -x^3 + 7x = -(x^3 - 7x) = -f(x)$$

▶ Because $f(-x) = -f(x)$, the function is odd.

b. Replace x with $-x$ in the equation for g, and then simplify.

$$g(-x) = (-x)^4 + (-x)^2 - 1 = x^4 + x^2 - 1 = g(x)$$

▶ Because $g(-x) = g(x)$, the function is even.

c. Replacing x with $-x$ in the equation for h produces

$$h(-x) = (-x)^3 + 2 = -x^3 + 2.$$

▶ Because $h(x) = x^3 + 2$ and $-h(x) = -x^3 - 2$, you can conclude that $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$. So, the function is neither even nor odd.

Determine whether the function is *even*, *odd*, or *neither*.

5. $f(x) = -x^2 + 5$

6. $f(x) = x^4 - 5x^3$

7. $f(x) = 2x^5$

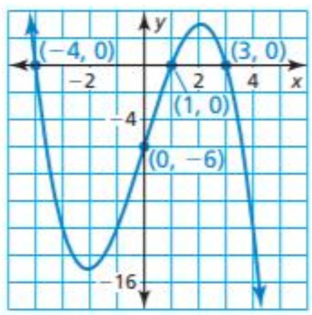
Section 4-9
Modeling with Polynomial Functions

Writing Polynomial Functions for a Set of Points

You know that two points determine a line and three points not on a line determine a parabola. In Example 1, you will see that four points not on a line or a parabola determine the graph of a cubic function.

EXAMPLE 1 **Writing a Cubic Function**

Write the cubic function whose graph is shown.



Finite Differences

When the x-values in a data set are equally spaced, the differences of consecutive y-values are called **finite differences**. Recall from Section 2.4 that the first and second differences of $y = x^2$ are:

equally-spaced x-values

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

first differences: -5 -3 -1 1 3 5

second differences: 2 2 2 2 2

Notice that $y = x^2$ has degree *two* and that the *second* differences are constant and nonzero. This illustrates the first of the two properties of finite differences shown on the next page.

Core Concept

Properties of Finite Differences

1. If a polynomial function $y = f(x)$ has degree n, then the nth differences of function values for equally-spaced x-values are nonzero and constant.
2. Conversely, if the nth differences of equally-spaced data are nonzero and constant, then the data can be represented by a polynomial function of degree n.

The second property of finite differences allows you to write a polynomial function that models a set of equally-spaced data.

EXAMPLE 2 Writing a Function Using Finite Differences

Use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function.

x	1	2	3	4	5	6	7
$f(x)$	1	4	10	20	35	56	84