Chapter 10
Probability

Section 10-5
Permutations and Combinations

Permutations

A **permutation** is an arrangement of objects in which order is important. For instance, the 6 possible permutations of the letters A, B, and C are shown.

ABC ACB BAC BCA CAB CBA

EXAMPLE 1 Counting Permutations

Consider the number of permutations of the letters in the word JULY. In how many ways can you arrange (a) all of the letters and (b) 2 of the letters?

SOLUTION

REMEMBER

Fundamental Counting Principle: If one event can occur in m ways and another event can occur in n ways, then the number of ways that both events can occur is $m \cdot n$. The Fundamental Counting Principle can be extended to three or more events.

a. Use the Fundamental Counting Principle to find the number of permutations of the letters in the word JULY.

$$\text{Number of permutations} = \left(\text{Choices for 1st letter}\right)\left(\text{Choices for 2nd letter}\right)\left(\text{Choices for 3rd letter}\right)\left(\text{Choices for 4th letter}\right)$$
$$= 4 \cdot 3 \cdot 2 \cdot 1$$
$$= 24$$

▶ There are 24 ways you can arrange all of the letters in the word JULY.

b. When arranging 2 letters of the word JULY, you have 4 choices for the first letter and 3 choices for the second letter.

$$\text{Number of permutations} = \left(\text{Choices for 1st letter}\right)\left(\text{Choices for 2nd letter}\right)$$
$$= 4 \cdot 3$$
$$= 12$$

▶ There are 12 ways you can arrange 2 of the letters in the word JULY.

1. In how many ways can you arrange the letters in the word HOUSE?

2. In how many ways can you arrange 3 of the letters in the word MARCH?

In Example 1(a), you evaluated the expression $4 \cdot 3 \cdot 2 \cdot 1$. This expression can be written as 4! and is read "4 *factorial*." For any positive integer n, the product of the integers from 1 to n is called ***n* factorial** and is written as

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1.$$

As a special case, the value of 0! is defined to be 1.

In Example 1(b), you found the permutations of 4 objects taken 2 at a time. You can find the number of permutations using the formulas on the next page.

Core Concept

Permutations

Formulas	**Examples**
The number of permutations of n objects is given by $${}_nP_n = n!.$$	The number of permutations of 4 objects is $${}_4P_4 = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24.$$
The number of permutations of n objects taken r at a time, where $r \le n$, is given by $${}_nP_r = \frac{n!}{(n-r)!}.$$	The number of permutations of 4 objects taken 2 at a time is $${}_4P_2 = \frac{4!}{(4-2)!} = \frac{4 \cdot 3 \cdot \cancel{2!}}{\cancel{2!}} = 12.$$

USING A GRAPHING CALCULATOR

Most graphing calculators can calculate permutations.

```
4  nPr  4
            24
4  nPr  2
            12
```

EXAMPLE 2 Using a Permutations Formula

Ten horses are running in a race. In how many different ways can the horses finish first, second, and third? (Assume there are no ties.)

SOLUTION

To find the number of permutations of 3 horses chosen from 10, find ${}_{10}P_3$.

$${}_{10}P_3 = \frac{10!}{(10-3)!} \qquad \text{Permutations formula}$$

$$= \frac{10!}{7!} \qquad \text{Subtract.}$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}} \qquad \text{Expand factorial. Divide out common factor, 7!.}$$

$$= 720 \qquad \text{Simplify.}$$

▶ There are 720 ways for the horses to finish first, second, and third.

STUDY TIP

When you divide out common factors, remember that 7! is a

EXAMPLE 3 Finding a Probability Using Permutations

For a town parade, you will ride on a float with your soccer team. There are 12 floats in the parade, and their order is chosen at random. Find the probability that your float is first and the float with the school chorus is second.

SOLUTION

Step 1 Write the number of possible outcomes as the number of permutations of the 12 floats in the parade. This is ${}_{12}P_{12} = 12!$.

Step 2 Write the number of favorable outcomes as the number of permutations of the other floats, given that the soccer team is first and the chorus is second. This is ${}_{10}P_{10} = 10!$.

Step 3 Find the probability.

$$P(\text{soccer team is 1st, chorus is 2nd}) = \frac{10!}{12!}$$ Form a ratio of favorable to possible outcomes.

$$= \frac{\cancel{10!}}{12 \cdot 11 \cdot \cancel{10!}}$$ Expand factorial. Divide out common factor, 10!.

$$= \frac{1}{132}$$ Simplify.

3. **WHAT IF?** In Example 2, suppose there are 8 horses in the race. In how many different ways can the horses finish first, second, and third? (Assume there are no ties.)

4. **WHAT IF?** In Example 3, suppose there are 14 floats in the parade. Find the probability that the soccer team is first and the chorus is second.

Combinations

A **combination** is a selection of objects in which order is *not* important. For instance, in a drawing for 3 identical prizes, you would use combinations, because the order of the winners would not matter. If the prizes were different, then you would use permutations, because the order would matter.

EXAMPLE 4 Counting Combinations

Count the possible combinations of 2 letters chosen from the list A, B, C, D.

SOLUTION

List all of the permutations of 2 letters from the list A, B, C, D. Because order is not important in a combination, cross out any duplicate pairs.

AB	AC	AD	~~BA~~	BC	BD
~~CA~~	~~CB~~	CD	~~DA~~	~~DB~~	~~DC~~

BD and DB are the same pair.

▶ There are 6 possible combinations of 2 letters from the list A, B, C, D.

Monitoring Progress

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5. Count the possible combinations of 3 letters chosen from the list A, B, C, D, E.

In Example 4, you found the number of combinations of objects by making an organized list. You can also find the number of combinations using the following formula.

USING A GRAPHING CALCULATOR

Most graphing calculators can calculate combinations.

```
4 nCr 2
              6
```

Core Concept

Combinations

Formula The number of combinations of n objects taken r at a time, where $r \leq n$, is given by

$$_nC_r = \frac{n!}{(n-r)! \cdot r!}.$$

Example The number of combinations of 4 objects taken 2 at a time is

$$_4C_2 = \frac{4!}{(4-2)! \cdot 2!} = \frac{4 \cdot 3 \cdot \cancel{2!}}{\cancel{2!} \cdot (2 \cdot 1)} = 6.$$

EXAMPLE 5 Using the Combinations Formula

You order a sandwich at a restaurant. You can choose 2 side dishes from a list of 8. How many combinations of side dishes are possible?

SOLUTION

The order in which you choose the side dishes is not important. So, to find the number of combinations of 8 side dishes taken 2 at a time, find ${}_8C_2$.

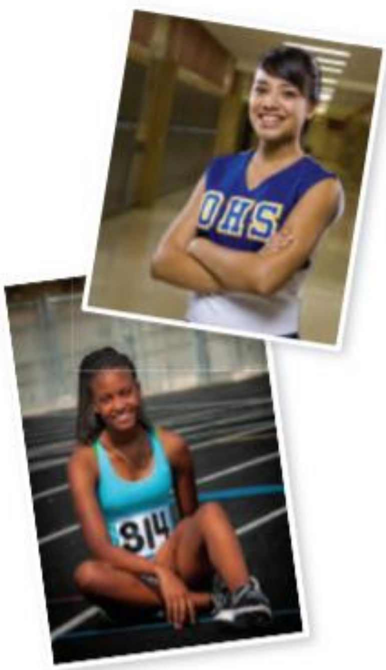
Check

```
8  nCr  2
            28
```

${}_8C_2 = \dfrac{8!}{(8-2)! \cdot 2!}$	Combinations formula
$= \dfrac{8!}{6! \cdot 2!}$	Subtract.
$= \dfrac{8 \cdot 7 \cdot \cancel{6!}}{\cancel{6!} \cdot (2 \cdot 1)}$	Expand factorials. Divide out common factor, 6!.
$= 28$	Multiply.

▶ There are 28 different combinations of side dishes you can order.

EXAMPLE 6 Finding a Probability Using Combinations



A yearbook editor has selected 14 photos, including one of you and one of your friend, to use in a collage for the yearbook. The photos are placed at random. There is room for 2 photos at the top of the page. What is the probability that your photo and your friend's photo are the 2 placed at the top of the page?

SOLUTION

Step 1 Write the number of possible outcomes as the number of combinations of 14 photos taken 2 at a time, or ${}_{14}C_2$, because the order in which the photos are chosen is not important.

${}_{14}C_2 = \dfrac{14!}{(14-2)! \cdot 2!}$	Combinations formula
$= \dfrac{14!}{12! \cdot 2!}$	Subtract.
$= \dfrac{14 \cdot 13 \cdot \cancel{12!}}{\cancel{12!} \cdot (2 \cdot 1)}$	Expand factorials. Divide out common factor, 12!.
$= 91$	Multiply.

Step 2 Find the number of favorable outcomes. Only one of the possible combinations includes your photo and your friend's photo.

Step 3 Find the probability.

$$P(\text{your photo and your friend's photos are chosen}) = \frac{1}{91}$$

6. **WHAT IF?** In Example 5, suppose you can choose 3 side dishes out of the list of 8 side dishes. How many combinations are possible?

7. **WHAT IF?** In Example 6, suppose there are 20 photos in the collage. Find the probability that your photo and your friend's photo are the 2 placed at the top of the page.

Binomial Expansions

In Section 4.2, you used Pascal's Triangle to find binomial expansions. The table shows that the coefficients in the expansion of $(a + b)^n$ correspond to combinations.

	n	Pascal's Triangle as Numbers	Pascal's Triangle as Combinations	Binomial Expansion
0th row	0	1	${}_0C_0$	$(a+b)^0 = 1$
1st row	1	1 1	${}_1C_0 \quad {}_1C_1$	$(a+b)^1 = 1a + 1b$
2nd row	2	1 2 1	${}_2C_0 \quad {}_2C_1 \quad {}_2C_2$	$(a+b)^2 = 1a^2 + 2ab + 1b^2$
3rd row	3	1 3 3 1	${}_3C_0 \quad {}_3C_1 \quad {}_3C_2 \quad {}_3C_3$	$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$

The results in the table are generalized in the **Binomial Theorem**.

Core Concept

The Binomial Theorem

For any positive integer n, the binomial expansion of $(a + b)^n$ is

$$(a + b)^n = {}_nC_0\, a^nb^0 + {}_nC_1\, a^{n-1}b^1 + {}_nC_2\, a^{n-2}b^2 + \cdots + {}_nC_n\, a^0b^n.$$

Notice that each term in the expansion of $(a + b)^n$ has the form ${}_nC_r\, a^{n-r}b^r$, where r is an integer from 0 to n.

Core Concept

The Binomial Theorem

For any positive integer n, the binomial expansion of $(a + b)^n$ is

$$(a + b)^n = {}_nC_0\, a^nb^0 + {}_nC_1\, a^{n-1}b^1 + {}_nC_2\, a^{n-2}b^2 + \cdots + {}_nC_n\, a^0b^n.$$

Notice that each term in the expansion of $(a + b)^n$ has the form ${}_nC_r\, a^{n-r}b^r$, where r is an integer from 0 to n.

EXAMPLE 7 Using the Binomial Theorem

a. Use the Binomial Theorem to write the expansion of $(x^2 + y)^3$.

b. Find the coefficient of x^4 in the expansion of $(3x + 2)^{10}$.

SOLUTION

a. $(x^2 + y)^3 = {}_3C_0(x^2)^3y^0 + {}_3C_1(x^2)^2y^1 + {}_3C_2(x^2)^1y^2 + {}_3C_3(x^2)^0y^3$

$= (1)(x^6)(1) + (3)(x^4)(y^1) + (3)(x^2)(y^2) + (1)(1)(y^3)$

$= x^6 + 3x^4y + 3x^2y^2 + y^3$

b. From the Binomial Theorem, you know

$$(3x + 2)^{10} = {}_{10}C_0(3x)^{10}(2)^0 + {}_{10}C_1(3x)^9(2)^1 + \cdots + {}_{10}C_{10}(3x)^0(2)^{10}.$$

Each term in the expansion has the form ${}_{10}C_r(3x)^{10-r}(2)^r$. The term containing x^4 occurs when $r = 6$.

$${}_{10}C_6(3x)^4(2)^6 = (210)(81x^4)(64) = 1{,}088{,}640x^4$$

▶ The coefficient of x^4 is 1,088,640.

8. Use the Binomial Theorem to write the expansion of (a) $(x + 3)^5$ and (b) $(2p - q)^4$.

9. Find the coefficient of x^5 in the expansion of $(x - 3)^7$.

10. Find the coefficient of x^3 in the expansion of $(2x + 5)^8$.

Section 10-5 Homework #3-39 odd and 51-63odd