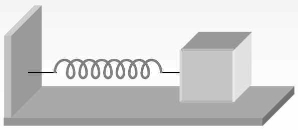
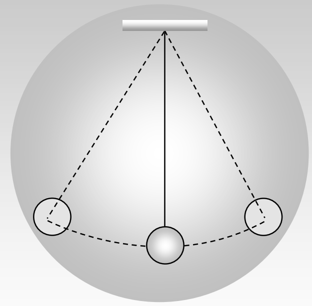


Chapter 15

Simple Harmonic Motion

Contents

On one occasion or another, most of us have watched a grandfather's clock or other timepiece based on a simple pendulum and marveled at this device as it swung back and forth majestically, often seeming hardly to move, yet maintaining remarkable accuracy day after day. The elegance and simplicity of this oscillator suggests that there is little to go wrong with it. After all, there is no hairspring to magnetize, rust, or tangle with regulator pins, no balance wheel to expand or contract with temperature or unbalance from one orientation to another. The oscillatory energy and the forces that shuttle it between its potential and kinetic forms are provided by gravity.



Simple Harmonic Motion

15.1 Periodic Motion

A motion, which repeat itself over and over again after a regular interval of time is called a periodic motion and the fixed interval of time after which the motion is repeated is called period of the motion.

Examples :

(i) Revolution of earth around the sun (period one year)

(ii) Rotation of earth about its polar axis (period one day)

(iii) Motion of hour's hand of a clock (period 12-hour)

(iv) Motion of minute's hand of a clock (period 1-hour)

(v) Motion of second's hand of a clock (period 1-minute)

(vi) Motion of moon around the earth (period 27.3 days)

15.2 Oscillatory or Vibratory Motion

Oscillatory or vibratory motion is that motion in which a body moves to and fro or back and forth repeatedly about a fixed point in a definite interval of time. In such a motion, the body is confined with in well-defined limits on either side of mean position.

Oscillatory motion is also called as harmonic motion.

Example :

(i) The motion of the pendulum of a wall clock.

(ii) The motion of a load attached to a spring, when it is pulled and then released.

(iii) The motion of liquid contained in U- tube when it is compressed once in one limb and left to itself.

(iv) A loaded piece of wood floating over the surface of a liquid when pressed down and then released executes oscillatory motion.

15.3 Harmonic and Non-harmonic Oscillation

Harmonic oscillation is that oscillation which can be expressed in terms of single harmonic function (*i.e.* sine or cosine function). *Example* : $y = a\sin\omega t$ or $y = a\cos\omega t$

Non-harmonic oscillation is that oscillation which can not be expressed in terms of single harmonic function. It is a combination of two or more than two harmonic oscillations. *Example* : $y = a\sin\omega t + b\sin 2\omega t$

15.4 Some Important Definitions

(1) **Time period :** It is the least interval of time after which the periodic motion of a body repeats itself.

S.I. units of time period is second.

(2) **Frequency :** It is defined as the number of periodic motions executed by body per second. S.I unit of frequency is hertz (*Hz*).

(3) **Angular Frequency :** Angular frequency of a body executing periodic motion is equal to product of frequency of the body with factor 2π. Angular frequency $\omega = 2\pi n$

S.I. units of ω is *Hz* [S.I.] ω also represents angular velocity. In that case unit will be *rad/sec.*

(4) **Displacement :** In general, the name displacement is given to a physical quantity which undergoes a change with time in a periodic motion.

Examples :

(i) In an oscillation of a loaded spring, displacement variable is its deviation from the mean position.

(ii) During the propagation of sound wave in air, the displacement variable is the local change in pressure

(iii) During the propagation of electromagnetic waves, the displacement variables are electric and magnetic fields, which vary periodically.

(5) **Phase :** phase of a vibrating particle at any instant is a physical quantity, which completely express the position and direction of motion, of the particle at that instant with respect to its mean position.

In oscillatory motion the phase of a vibrating particle is the argument of *sine* or *cosine* function involved to represent the generalised equation of motion of the vibrating particle.

$y = a\sin\theta = a\sin(\omega t + \phi_0)$ here, $\theta = \omega t + \phi_0$ = phase of vibrating particle.

(i) Initial phase or epoch : It is the phase of a vibrating particle at $t = 0$.

In $\theta = \omega t + \phi_0$, when $t = 0$; $\theta = \phi_0$ here, ϕ_0 is the angle of epoch.

(ii) Same phase : Two vibrating particle are said to be in same phase, if the phase difference between them is an even multiple of π or path difference is an even multiple of (λ / 2) or time interval is an even multiple of (T/ 2) because 1 time period is equivalent to 2π *rad* or 1 wave length (λ)

(iii) Opposite phase : When the two vibrating particles cross their respective mean positions at the same time moving in opposite directions, then the phase difference between the two vibrating particles is 180°

Opposite phase means the phase difference between the particle is an odd multiple of π (say π, 3π, 5π, 7π.....) or the path difference is an odd multiple of λ (say $\frac{\lambda}{2}, \frac{3\lambda}{2}$,.......) or the time interval is an odd multiple of (T / 2).

(iv) Phase difference : If two particles performs S.H.M and their equation are

$y_1 = a\sin(\omega t + \phi_1)$ and $y_2 = a\sin(\omega t + \phi_2)$

then phase difference $\Delta\phi = (\omega t + \phi_2) - (\omega t + \phi_1) = \phi_2 - \phi_1$

15.5 Simple Harmonic Motion

Simple harmonic motion is a special type of periodic motion, in which a particle moves to and fro repeatedly about a mean position under a restoring force which is always directed towards the mean position and whose magnitude at any instant is directly proportional to the displacement of the particle from the mean position at that instant.

Restoring force $\propto$ Displacement of the particle from mean position.

$F \propto -x$

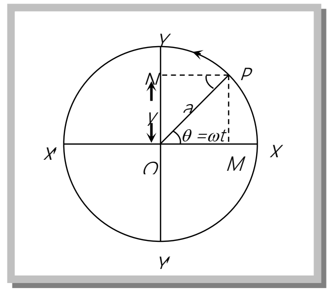
$F = -kx$

Where k is known as force constant. Its S.I. unit is Newton/meter and dimension is $[MT^{-2}]$.

15.6 Displacement in S.H.M.

The displacement of a particle executing S.H.M. at an instant is defined as the distance of particle from the mean position at that instant.

As we know that simple harmonic motion is defined as the projection of uniform circular motion on any diameter of circle of reference. If the projection is taken on *y*-axis.



then from the figure $y = a \sin \omega t$

$$y = a \sin \frac{2\pi}{T} t$$

$$y = a \sin 2\pi n t$$

$$y = a \sin(\omega t \pm \phi)$$

where a = Amplitude, ω = Angular frequency, t = Instantaneous time,

T = Time period, n = Frequency and ϕ = Initial phase of particle

If the projection of P is taken on X-axis then equations of S.H.M. can be given as

$$x = a \cos(\omega t \pm \phi)$$

$$x = a \cos\left(\frac{2\pi}{T} t \pm \phi\right)$$

$$x = a \cos(2\pi n t \pm \phi)$$

Important points

(i) $y = a \sin \omega t$ when the time is noted from the instant when the vibrating particle is at mean position.

(ii) $y = a \cos \omega t$ when the time is noted from the instant when the vibrating particle is at extreme position.

(iii) $y = a \sin(\omega t \pm \phi)$ when the vibrating particle is ϕ phase leading or lagging from the mean position.

(iv) Direction of displacement is always away from the equilibrium position, particle either is moving away from or is coming towards the equilibrium position.

(v) If t is given or phase (θ) is given, we can calculate the displacement of the particle.

If $t = \frac{T}{4}$ (or $\theta = \frac{\pi}{2}$) then from equation $y = a\sin\frac{2\pi}{T}t$, we get $y = a\sin\frac{2\pi}{T}\frac{T}{4} = a\sin\left(\frac{\pi}{2}\right) = a$

Similarly if $t = \frac{T}{2}$ (or $\theta = \pi$) then we get $y = 0$

Sample problems based on Displacement

Problem 1. A simple harmonic oscillator has an amplitude *A* and time period *T*. The time required by it to travel from $x = A$ to $x = A/2$ is [CBSE 1992; SCRA 1996]

(a) $T/6$ (b) $T/4$ (c) $T/3$ (d) $T/2$

Solution : (a) Because the S.H.M. starts from extreme position so $y = a\cos\omega t$ form of S.H.M. should be used.

$$\frac{A}{2} = A\cos\frac{2\pi}{T}t \Rightarrow \cos\frac{\pi}{3} = \cos\frac{2\pi}{T}t \Rightarrow t = T/6$$

Problem 2. A mass *m* = 100 *gms* is attached at the end of a light spring which oscillates on a friction less horizontal table with an amplitude equal to 0.16 meter and the time period equal to 2 *sec*. Initially the mass is released from rest at *t* = 0 and displacement *x* = – 0.16 *meter*. The expression for the displacement of the mass at any time (*t*) is [MP PMT 1995]

(a) $x = 0.16\cos(\pi t)$ (b) $x = -0.16\cos(\pi t)$ (c) $x = 0.16\cos(\pi t + \pi)$ (d) $x = -0.16\cos(\pi t + \pi)$

Solution : (b) Standard equation for given condition

$$x = a\cos\frac{2\pi}{T}t \Rightarrow x = -0.16\cos(\pi t)$$ [As *a* = – 0.16 *meter*, *T* = 2 *sec*]

Problem 3. The motion of a particle executing S.H.M. is given by $x = 0.01\sin 100\pi(t + .05)$. Where *x* is in meter and time *t* is in seconds. The time period is

(a) 0.01 *sec* (b) 0.02 *sec* (c) 0.1 *sec* (d) 0.2 *sec*

Solution : (b) By comparing the given equation with standard equation $y = a\sin(\omega t + \phi)$

$$\omega = 100\,\pi \quad \text{so} \quad T = \frac{2\pi}{\omega} = \frac{2\pi}{100\,\pi} = 0.02 \; sec$$

Problem 4. Two equations of two S.H.M. are $x = a\sin(\omega t - \alpha)$ and $y = b\cos(\omega t - \alpha)$. The phase difference between the two is [MP PMT 1985]

(a) 0° (b) α° (c) 90° (d) 180°

Solution : (c) $x = a\sin(\omega t - \alpha)$ and $y = b\cos(\omega t - \alpha) = b\sin(\omega t - \alpha + \pi/2)$

Now the phase difference = $(\omega t-\alpha+\frac{\pi}{2})-(\omega t-\alpha)=\pi/2=90^o$

15.7 Velocity in S.H.M.

Velocity of the particle executing S.H.M. at any instant, is defined as the time rate of change of its displacement at that instant.

In case of S.H.M. when motion is considered from the equilibrium position

$$y=a\sin\omega t$$

so $\quad v=\frac{dy}{dt}=a\omega\cos\omega t$

$\therefore \quad v=a\omega\cos\omega t$ ......(i)

or $\quad v=a\omega\sqrt{1-\sin^2\omega t}$ [As $\sin\omega t=y/a$]

or $\quad v=\omega\sqrt{a^2-y^2}$ .....(ii)

Important points

(i) In S.H.M. velocity is maximum at equilibrium position.

From equation (i) $\quad v_{max}=a\omega$ when $|\cos\omega t|=1$ *i.e.* $\theta=\omega t=0$

from equation (ii) $\quad v_{max}=a\omega$ when $y=0$

(ii) In S.H.M. velocity is minimum at extreme position.

From equation (i) $\quad v_{min}=0$ when $|\cos\omega t|=0$ *i.e* $\theta=\omega t=\frac{\pi}{2}$

From equation (ii) $\quad v_{min}=0$ when $y=a$

(iii) Direction of velocity is either towards or away from mean position depending on the position of particle.

Sample problems based on Velocity

Problem 5. A body is executing simple harmonic motion with an angular frequency 2 *rad/sec*. The velocity of the body at 20 *mm* displacement. When the amplitude of motion is 60 *mm* is [AFMC 1998]

(a) 40 *mm/sec* (b) 60 *mm/sec* (c) 113 *mm/sec* (d) 120 *mm/sec*

Solution : (c) $v=\omega\sqrt{a^2-y^2}=2\sqrt{(60)^2-(20)^2}$ = 113 *mm/sec*

Problem 6. A body executing S.H.M. has equation $y = 0.30\sin(220\,t + 0.64)$ in *meter*. Then the frequency and maximum velocity of the body is [AFMC 1998]

(a) $35\,Hz, 66\,m/s$ (b) $45\,Hz, 66\,m/s$ (c) $58\,Hz, 113\,m/s$ (d) $35\,Hz, 132\,m/s$

Solution : (a) By comparing with standard equation $y = a\sin(\omega t + \phi)$ we get $a = 0.30$; $\omega = 220$

$\therefore\ 2\pi n = 220 \Rightarrow n = 35\,Hz$ so $v_{max} = a\omega = 0.3 \times 220 = 66\,m/s$

Problem 7. A particle starts S.H.M. from the mean position. Its amplitude is *A* and time period is *T*. At the time when its speed is half of the maximum speed. Its displacement *y* is

(a) $A/2$ (b) $A/\sqrt{2}$ (c) $A\sqrt{3}/2$ (d) $2A/\sqrt{3}$

Solution : (c) $v = \omega\sqrt{a^2 - y^2} \Rightarrow \dfrac{a\omega}{2} = \omega\sqrt{a^2 - y^2} \Rightarrow \dfrac{a^2}{4} = a^2 - y^2 \Rightarrow y = \dfrac{\sqrt{3}A}{2}$ [As $v = \dfrac{v_{max}}{2} = \dfrac{a\omega}{2}$]

Problem 8. A particle perform simple harmonic motion. The equation of its motion is $x = 5\sin(4t - \frac{\pi}{6})$. Where *x* is its displacement. If the displacement of the particle is 3 units then its velocity is [MP PMT 1994]

(a) $2\pi/3$ (b) $5\pi/6$ (c) 20 (d) 16

Solution : (d) $v = \omega\sqrt{a^2 - y^2} = 4\sqrt{5^2 - 3^2} = 16$ [As $\omega = 4$, $a = 5$, $y = 3$]

Problem 9. A simple pendulum performs simple harmonic motion about $x = 0$ with an amplitude (A) and time period (T). The speed of the pendulum at $x = \dfrac{A}{2}$ will be [MP PMT 1987]

(a) $\dfrac{\pi A\sqrt{3}}{T}$ (b) $\dfrac{\pi A}{T}$ (c) $\dfrac{\pi A\sqrt{3}}{2T}$ (d) $\dfrac{3\pi^2 A}{T}$

Solution : (a) $v = \omega\sqrt{a^2 - y^2} \Rightarrow v = \dfrac{2\pi}{T}\sqrt{A^2 - \dfrac{A^2}{4}} = \dfrac{\pi A\sqrt{3}}{T}$ [As $y = A/2$]

Problem 10. A particle is executing S.H.M. if its amplitude is 2 *m* and periodic time 2 *seconds*. Then the maximum velocity of the particle will be

(a) 6π (b) 4π (c) 2π (d) π

Solution : (c) $v_{max} = a\omega = a\dfrac{2\pi}{T} = 2\dfrac{2\pi}{2} \Rightarrow v_{max} = 2\pi$

Problem 11. A S.H.M. has amplitude '*a*' and time period *T*. The maximum velocity will be [MP PMT 1985]

(a) $\dfrac{4a}{T}$ (b) $\dfrac{2a}{T}$ (c) $2\pi\sqrt{\dfrac{a}{T}}$ (d) $\dfrac{2\pi a}{T}$

Solution : (d) $v_{max} = a\omega = \dfrac{a2\pi}{T}$

Problem 12. A particle executes S.H.M. with a period of 6 second and amplitude of 3 *cm* its maximum speed in *cm/sec* is

[AIIMS 1982]

(a) $\pi/2$ (b) π (c) 2π (d) 3π

Solution : (b) $v_{\max} = a\omega = a\dfrac{2\pi}{T} = 3\dfrac{2\pi}{6} \Rightarrow v_{\max} = \pi$

Problem 13. A body of mass 5 *gm* is executing S.H.M. about a point with amplitude 10 *cm*. Its maximum velocity is 100 *cm/sec*. Its velocity will be 50 *cm/sec*, at a distance [CPMT 1976]

(a) 5 (b) $5\sqrt{2}$ (c) $5\sqrt{3}$ (d) $10\sqrt{2}$

Solution : (c) $v_{\max} = a\omega = 100\ cm/\sec$ and $a = 10\ cm$ so $\omega = 10\ rad/\sec$.

$\therefore\ v = \omega\sqrt{a^2 - y^2} \Rightarrow 50 = 10\sqrt{10^2 - y^2} \Rightarrow y = 5\sqrt{3}$

15.8 Acceleration in S.H.M.

The acceleration of the particle executing S.H.M. at any instant, is defined as the rate of change of its velocity at that instant. So acceleration $A = \dfrac{dv}{dt} = \dfrac{d}{dt}(a\omega\cos\omega t)$

$$A = -\omega^2 a\sin\omega t \qquad \text{......(i)}$$

$$A = -\omega^2 y \qquad \text{......(ii)} \qquad [\text{As } y = a\sin\omega t]$$

Important points

(i) In S.H.M. as $|\text{Acceleration}| = \omega^2 y$ is not constant. So equations of translatory motion can not be applied.

(ii) In S.H.M. acceleration is maximum at extreme position.

From equation (i) $|A_{\max}| = \omega^2 a$ when $|\sin\omega t| = \text{maximum} = 1$ *i.e.* at $t = \dfrac{T}{4}$ or $\omega t = \dfrac{\pi}{2}$

From equation (ii) $|A_{\max}| = \omega^2 a$ when $y = a$

(iii) In S.H.M. acceleration is minimum at mean position

From equation (i) $A_{\min} = 0$ when $\sin\omega t = 0$ *i.e.* at $t = 0$ or $t = \dfrac{T}{2}$ or $\omega t = \pi$

From equation (ii) $A_{\min} = 0$ when $y = 0$

(iv) Acceleration is always directed towards the mean position and so is always opposite to displacement

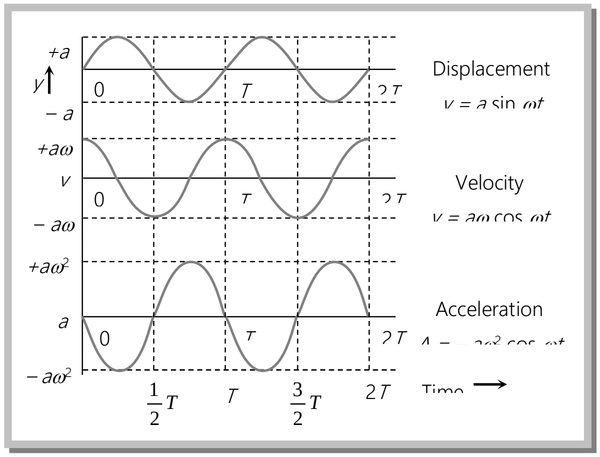
i.e., $A \propto -y$

15.9 Comparative Study of Displacement, Velocity and Acceleration

Displacement $y = a\sin\omega t$

Velocity $v = a\omega\cos\omega t = a\omega\sin(\omega t + \frac{\pi}{2})$

Acceleration $A = -a\omega^2 \sin\omega t = a\omega^2 \sin(\omega t + \pi)$



From the above equations and graphs we can conclude that.

(i) All the three quantities displacement, velocity and acceleration show harmonic variation with time having same period.

(ii) The velocity amplitude is ω times the displacement amplitude

(iii) The acceleration amplitude is ω^2 times the displacement amplitude

(iv) In S.H.M. the velocity is ahead of displacement by a phase angle $\pi/2$

(v) In S.H.M. the acceleration is ahead of velocity by a phase angle $\pi/2$

(vi) The acceleration is ahead of displacement by a phase angle of π

(vii) Various physical quantities in S.H.M. at different position :

Physical quantities	Equilibrium position (y = 0)	Extreme Position ($y = \pm a$)
Displacement $y = a\sin\omega t$	Minimum (Zero)	Maximum (a)
Velocity $v = \omega\sqrt{a^2 - y^2}$	Maximum ($a\omega$)	Minimum (Zero)
Acceleration $\lvert A \rvert = \omega^2 y$	Minimum (Zero)	Maximum ($\omega^2 a$)

15.10 Energy in S.H.M.

A particle executing S.H.M. possesses two types of energy : Potential energy and Kinetic energy

(1) **Potential energy :** This is an account of the displacement of the particle from its mean position.

The restoring force $F = -ky$ against which work has to be done

So $$U = -\int dw = -\int_0^x F dx = \int_0^y ky\,dy = \frac{1}{2}ky^2$$

$\therefore$ potential Energy $$U = \frac{1}{2}m\omega^2 y^2$$ [As $\omega^2 = k/m$]

$$U = \frac{1}{2}m\omega^2 a^2 \sin^2 \omega t$$ [As $y = a\sin\omega t$]

Important points

(i) Potential energy maximum and equal to total energy at extreme positions

$$U_{max} = \frac{1}{2}ka^2 = \frac{1}{2}m\omega^2 a^2$$ when $y = \pm a$; $\omega t = \pi/2$; $t = T/4$

(ii) Potential energy is minimum at mean position

$$U_{min} = 0$$ when $y = 0$; $\omega t = 0$; $t = 0$

(2) **Kinetic energy :** This is because of the velocity of the particle

Kinetic Energy $$K = \frac{1}{2}mv^2$$

$$K = \frac{1}{2}ma^2\omega^2 \cos^2 \omega t$$ [As $v = a\omega\cos\omega t$]

$$K = \frac{1}{2}m\omega^2(a^2 - y^2)$$ [As $v = \omega\sqrt{a^2 - y^2}$]

(i) Kinetic energy is maximum at mean position and equal to total energy at mean position.

$$K_{max} = \frac{1}{2}m\omega^2 a^2$$ when $y = 0$; $t = 0$; $\omega t = 0$

(ii) Kinetic energy is minimum at extreme position.

$$K_{min} = 0$$ when $y = a$; $t = T/4$, $\omega t = \pi/2$

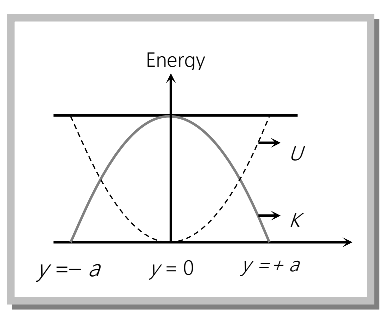
(3) **Total energy :** Total mechanical energy = Kinetic energy + Potential energy

$$E = \frac{1}{2}m\omega^2(a^2 - y^2) + \frac{1}{2}m\omega^2 y^2 = \frac{1}{2}m\omega^2 a^2$$

Total energy is not a position function *i.e.* it always remains constant.

(4) **Energy position graph :** Kinetic energy $(K) = \frac{1}{2}m\omega^2(a^2 - y^2)$

Potential Energy $(U) = \frac{1}{2}m\omega^2 y^2$



Total Energy (E) = $\frac{1}{2}m\omega^2 a^2$

It is clear from the graph that

(i) Kinetic energy is maximum at mean position and minimum at extreme position

(ii) Potential energy is maximum at extreme position and minimum at mean position

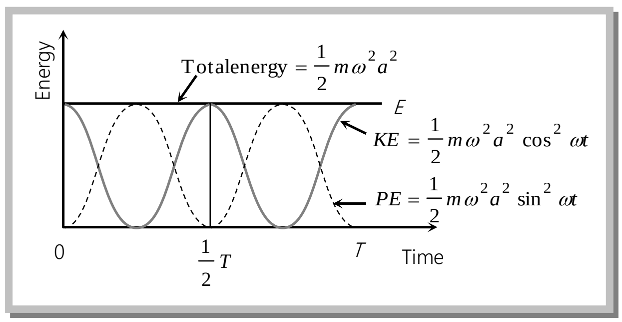
(iii) Total energy always remains constant.

(5) Kinetic Energy $K = \frac{1}{2}m\omega^2 a^2 \cos^2 \omega t = \frac{1}{4}m\omega^2 a^2 (1+\cos 2\omega t) = \frac{1}{2}E(1+\cos \omega' t)$

Potential Energy $U = \frac{1}{2}m\omega^2 a^2 \sin^2 \omega t = \frac{1}{4}m\omega^2 a^2 (1-\cos 2\omega t) = \frac{1}{2}E(1-\cos \omega' t)$

where $\omega' = 2\omega$ and $E = \frac{1}{2}m\omega^2 a^2$

i.e. in S.H.M., kinetic energy and potential energy vary periodically with double the frequency of S.H.M. (*i.e.* with time period $T' = T/2$)



From the graph we note that potential energy or kinetic energy completes two vibrations in a time during which S.H.M. completes one vibration. Thus the frequency of potential energy or kinetic energy double than that of S.H.M.

Sample problems based on Energy

Problem 14. A particle is executing simple harmonic motion with frequency *f*. The frequency at which its kinetic energy changes into potential energy is

(a) $f/2$ (b) f (c) $2f$ (d) $4f$

Solution : (c)

Problem 15. When the potential energy of a particle executing simple harmonic motion is one-fourth of the maximum value during the oscillation, its displacement from the equilibrium position in terms of amplitude '*a*' is

[CBSE 1993; MP PMT 1994; MP PET 1995, 96; MP PMT 2000]

(a) $a/4$ (b) $a/3$ (c) $a/2$ (d) $2a/3$

Solution : (c) According to problem potential energy $= \frac{1}{4}$ maximum Energy

$\Rightarrow \frac{1}{2} m\omega^2 y^2 = \frac{1}{4}\left(\frac{1}{2} m\omega^2 a^2\right) \Rightarrow y^2 = \frac{a^2}{4} \Rightarrow y = a/2$

Problem 16. A particle of mass 10 *grams* is executing S.H.M. with an amplitude of 0.5 *meter* and circular frequency of 10 *radian/sec*. The maximum value of the force acting on the particle during the course of oscillation is [MP PMT 2000]

(a) 25 *N* (b) 5 *N* (c) 2.5 *N* (d) 0.5 *N*

Solution : (d) Maximum force = mass × maximum acceleration $= m\omega^2 a = 10 \times 10^{-3}(10)^2(0.5)$ = 0.5 *N*

Problem 17. A body is moving in a room with a velocity of 20 *m/s* perpendicular to the two walls separated by 5 meters. There is no friction and the collision with the walls are elastic. The motion of the body is [MP PMT 1999]

(a) Not periodic (b) Periodic but not simple harmonic

(c) Periodic and simple harmonic (d) Periodic with variable time period

Solution : (b) Since there is no friction and collision is elastic therefore no loss of energy take place and the body strike again and again with two perpendicular walls. So the motion of the ball is periodic. But here, there is no restoring force. So the characteristics of S.H.M. will not satisfied.

Problem 18. Two particles executes S.H.M. of same amplitude and frequency along the same straight line. They pass one another when going in opposite directions. Each time their displacement is half of their amplitude. The phase difference between them is

(a) 30° (b) 60° (c) 90° (d) 120°

Solution : (d) Let two simple harmonic motions are $y = a\sin\omega t$ and $y = a\sin(\omega t + \phi)$

In the first case $\frac{a}{2} = a\sin\omega t \Rightarrow \sin\omega t = 1/2$ $\therefore$ $\cos\omega t = \frac{\sqrt{3}}{2}$

In the second case $\frac{a}{2} = a\sin(\omega t + \phi)$

$\Rightarrow \frac{1}{2} = [\sin\omega t.\cos\phi + \cos\omega t \sin\phi] \Rightarrow \frac{1}{2} = \left[\frac{1}{2}\cos\phi + \frac{\sqrt{3}}{2}\sin\phi\right]$

$\Rightarrow 1 - \cos\phi = \sqrt{3}\sin\phi \Rightarrow (1-\cos\phi)^2 = 3\sin^2\phi \Rightarrow (1-\cos\phi)^2 = 3(1-\cos^2\phi)$

By solving we get $\cos\phi = +1$ or $\cos\phi = -1/2$

i.e. $\phi = 0$ or $\phi = 120^o$

Problem 19. The acceleration of a particle performing S.H.M. is 12 *cm/sec*^{2} at a distance of 3 *cm* from the mean position. Its time period is [MP PET 1996; MP PMT 1997]

(a) 0.5 *sec* (b) 1.0 *sec* (c) 2.0 *sec* (d) 3.14 *sec*

Solution : (d) $A = \omega^2 y \Rightarrow \omega = \sqrt{\frac{A}{y}} = \sqrt{\frac{12}{3}} = 2$; but $T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi = 3.14$

Problem 20. A particle of mass 10 *gm* is describing S.H.M. along a straight line with period of 2 *sec* and amplitude of 10 *cm*. Its kinetic energy when it is at 5 cm. From its equilibrium position is

(a) $37.5\pi^2 erg$ (b) $3.75\pi^2 erg$ (c) $375\pi^2 erg$ (d) $0.375\pi^2 erg$

Solution : (c) Kinetic energy $= \frac{1}{2}m\omega^2(a^2 - y^2) = \frac{1}{2}10\frac{4\pi^2}{4}(10^2 - 5^2) = 375\pi^2 ergs$.

Problem 21. The total energy of the body executing S.H.M. is *E*. Then the kinetic energy when the displacement is half of the amplitude is [RPET 1996]

(a) $E/2$ (b) $E/4$ (c) $3E/4$ (d) $\sqrt{3}E/4$

Solution : (c) Kinetic energy $= \frac{1}{2}m\omega^2(a^2 - y^2) = \frac{1}{2}m\omega^2\left(a^2 - \frac{a^2}{4}\right) = \frac{3}{4}\left(\frac{1}{2}m\omega^2 a^2\right) = \frac{3E}{4}$ [As $y = \frac{a}{2}$]

Problem 22. A body executing simple harmonic motion has a maximum acceleration equal to 24 *m/sec*^{2} and maximum velocity equal to 16 *meter/sec*. The amplitude of simple harmonic motion is [MP PMT 1995]

(a) $\frac{32}{3} meters$ (b) $\frac{3}{32} meters$ (c) $\frac{1024}{9} meters$ (d) $\frac{64}{9} meters$

Solution : (a) Maximum acceleration $\omega^2 a = 24$ .....(i)

and maximum velocity $a\omega = 16$ ....(ii)

Dividing (i) by (ii) $\omega = \frac{3}{2}$

Substituting this value in equation (ii) we get $a = 32/3 meter$

Problem 23. The displacement of an oscillating particle varies with time (in seconds) according to the equation. $y(cm) = \sin\frac{\pi}{2}\left(\frac{t}{2} + \frac{1}{3}\right)$. The maximum acceleration of the particle approximately

(a) 5.21 *cm/sec*^{2} (b) 3.62 *cm/sec*^{2} (c) 1.81 *cm/sec*^{2} (d) 0.62 *cm/sec*^{2}

Solution : (d) By comparing the given equation with standard equation, $y = a\sin(\omega t + \phi)$

We find that $a = 1$ and $\omega = \pi/4$

Now maximum acceleration $= \omega^2 a = \left(\frac{\pi^2}{4}\right) = \left(\frac{3.14}{4}\right)^2 = 0.62 cm/\sec^2$

Problem 24. The potential energy of a particle executing S.H.M. at a distance *x* from the mean position is proportional to

[Roorkee 1992]

(a) $\sqrt{x}$ (b) x (c) x^2 (d) x^3

Solution : (c)

Problem 25. The kinetic energy and potential energy of a particle executing S.H.M. will be equal, when displacement is (amplitude = *a*) [MP PMT 1987; CPMT 1990]

(a) $a/2$ (b) $a\sqrt{2}$ (c) $a/\sqrt{2}$ (d) $\frac{a\sqrt{2}}{3}$

Solution : (c) According to problem Kinetic energy = Potential energy $\Rightarrow \frac{1}{2}m\omega^2(a^2-y^2)=\frac{1}{2}m\omega^2y^2$

$\Rightarrow a^2-y^2=y^2 \;\therefore\; y=a/\sqrt{2}$

Problem 26. The phase of a particle executing S.H.M. is $\frac{\pi}{2}$ when it has [MP PET 1985]

(a) Maximum velocity (b) Maximum acceleration (c) Maximum energy (d) Maximum displacement

Solution : (b, d) Phase $\pi/2$ means extreme position. At extreme position acceleration and displacement will be maximum.

Problem 27. The displacement of a particle moving in S.H.M. at any instant is given by $y = a\sin\omega t$. The acceleration after time $t=\frac{T}{4}$ is (where *T* is the time period) [MP PET 1984]

(a) $a\omega$ (b) $-a\omega$ (c) $a\omega^2$ (d) $-a\omega^2$

Solution : (d)

Problem 28. A particle of mass *m* is hanging vertically by an ideal spring of force constant *k*, if the mass is made to oscillate vertically, its total energy is

(a) Maximum at extreme position (b) Maximum at mean position

(c) Minimum at mean position (d) Same at all position

Solution : (d)

15.11 Time Period and Frequency of S.H.M.

For S.H.M. restoring force is proportional to the displacement

$F \propto y$ or $F=-ky$ ...(i) where *k* is a force constant.

For S.H.M. acceleration of the body $A=-\omega^2 y$ ...(ii)

$\therefore$ Restoring force on the body $F = mA = -m\omega^2 y$ ...(iii)

From (i) and (iii) $ky = m\omega^2 y \Rightarrow \omega = \sqrt{\frac{k}{m}}$

$\therefore$ Time period $(T) = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

or Frequency (n) $= \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

In different types of S.H.M. the quantities m and k will go on taking different forms and names.

In general m is called inertia factor and k is called spring factor.

Thus $T = 2\pi\sqrt{\frac{\text{Inertia factor}}{\text{Spring factor}}}$

or $n = \frac{1}{2\pi}\sqrt{\frac{\text{Spring factor}}{\text{Inertia factor}}}$

In linear S.H.M. the spring factor stands for force per unit displacement and inertia factor for mass of the body executing S.H.M. and in Angular S.H.M. k stands for restoring torque per unit angular displacement and inertial factor for moment of inertia of the body executing S.H.M.

For linear S.H.M. $T = 2\pi\sqrt{\frac{m}{k}} = \sqrt{\frac{m}{\text{Force/Displacement}}} = 2\pi\sqrt{\frac{m \times \text{Displacement}}{m \times \text{Acceleration}}} = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2\pi\sqrt{\frac{y}{A}}$

or $n = \frac{1}{2\pi}\sqrt{\frac{\text{Acceleration}}{\text{Dispalcement}}} = \frac{1}{2\pi}\sqrt{\frac{A}{y}}$

15.12 Differential Equation of S.H.M.

For S.H.M. (linear) Acceleration $\propto$ – (Displacement)

$$A \propto -y$$

or $$A = -\omega^2 y$$

or $$\frac{d^2y}{dt^2} = -\omega^2 y$$

or $$m\frac{d^2y}{dt^2} + ky = 0$$ [As $\omega = \sqrt{\frac{k}{m}}$]

For angular S.H.M. $\tau = -c\theta$ and $\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$

where $\omega^2 = \frac{c}{I}$ [As c = Restoring torque constant and I = Moment of inertia]

Sample problems based on Differential equation of S.H.M.

Problem **29.** A particle moves such that its acceleration a is given by $a = -bx$. Where x is the displacement from equilibrium position and b is a constant. The period of oscillation is

[NCERT 1984; CPMT 1991; MP PMT 1994; MNR 1995]

(a) $2\pi\sqrt{b}$ (b) $\frac{2\pi}{\sqrt{b}}$ (c) $\frac{2\pi}{b}$ (d) $2\sqrt{\frac{\pi}{b}}$

Solution : (b) We know that Acceleration = $-\omega^2$ (displacement) and $a = -bx$ (given in the problem)

Comparing above two equation $\omega^2 = b \Rightarrow \omega = \sqrt{b}$ $\therefore$ Time period $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{b}}$

Problem **30.** The equation of motion of a particle is $\frac{d^2y}{dt^2} + ky = 0$ where k is a positive constant. The time period of the motion is given by

(a) $\frac{2\pi}{k}$ (b) $2\pi k$ (c) $\frac{2\pi}{\sqrt{k}}$ (d) $2\pi\sqrt{k}$

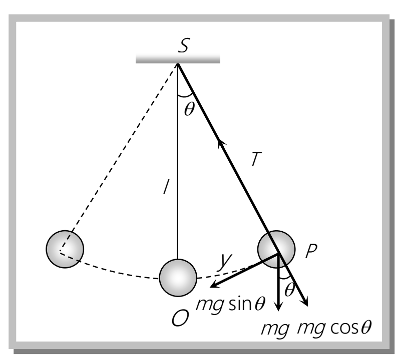
Solution : (c) Standard equation $m\frac{d^2y}{dt^2} + ky = 0$ and in a given equation m =1 and $k = k$

So, $T = 2\pi\sqrt{\frac{m}{k}} = \frac{2\pi}{\sqrt{k}}$

15.13 Simple Pendulum

An ideal simple pendulum consists of a heavy point mass body suspended by a weightless, inextensible and perfectly flexible string from a rigid support about which it is free to oscillate.

But in reality neither point mass nor weightless string exist, so we can never construct a simple pendulum strictly according to the definition.



Let mass of the bob = m

Length of simple pendulum = l

Displacement of mass from mean position (OP) = x

When the bob is displaced to position *P*, through a small angle θ from the vertical. Restoring force acting on the bob

$$F = -mg\sin\theta$$

or $$F = -mg\,\theta \qquad \left(\text{When } \theta \text{ is small } \sin\theta \simeq \theta = \frac{\text{Arc}}{\text{Length}} = \frac{OP}{l} = \frac{x}{l}\right)$$

or $$F = -mg\frac{x}{l}$$

$\therefore$ $$\frac{F}{x} = \frac{-mg}{l} = k \quad \text{(Spring factor)}$$

So time period $$T = 2\pi\sqrt{\frac{\text{Inertia factor}}{\text{Spring factor}}} = 2\pi\sqrt{\frac{m}{mg/l}} = 2\pi\sqrt{\frac{l}{g}}$$

Important points

(i) The period of simple pendulum is independent of amplitude as long as its motion is simple harmonic. But if θ is not small, $\sin\theta \neq \theta$ then motion will not remain simple harmonic but will become oscillatory. In this situation if θ_0 is the amplitude of motion. Time period

$$T = 2\pi\sqrt{\frac{l}{g}}\left[1 + \frac{1}{2^2}\sin^2\left(\frac{\theta_0}{2}\right) + \ldots\ldots\right] \approx T_0\left[1 + \frac{\theta_0^{\,2}}{16}\right]$$

(ii) Time period of simple pendulum is also independent of mass of the bob. This is why

(a) If the solid bob is replaced by a hollow sphere of same radius but different mass, time period remains unchanged.

(b) If a girl is swinging in a swing and another sits with her, the time period remains unchanged.

(iii) Time period $T \propto \sqrt{l}$ where l is the distance between point of suspension and center of mass of bob and is called effective length.

(a) When a sitting girl on a swinging swing stands up, her center of mass will go up and so l and hence T will decrease.

(b) If a hole is made at the bottom of a hollow sphere full of water and water comes out slowly through the hole and time period is recorded till the sphere is empty, initially and finally the center of mass will be at the center of the sphere. However, as water drains off the sphere, the center of mass of the system will first move down and then will come up. Due to this l and hence T first increase, reaches a maximum and then decreases till it becomes equal to its initial value.

(iv) If the length of the pendulum is comparable to the radius of earth then $T = 2\pi\sqrt{\dfrac{1}{g\left[\dfrac{1}{l}+\dfrac{1}{R}\right]}}$

(a) If $l << R$, then $\dfrac{1}{l} >> \dfrac{1}{R}$ so $T = 2\pi\sqrt{\dfrac{l}{g}}$

(b) If $l >> R(\rightarrow \infty) 1/l < 1/R$ so $T = 2\pi\sqrt{\dfrac{R}{g}} = 2\pi\sqrt{\dfrac{6.4\times 10^6}{10}} \cong 84.6$ *minutes*

and it is the maximum time period which an oscillating simple pendulum can have

(c) If $l = R$ so $T = 2\pi\sqrt{\dfrac{R}{2g}} \cong 1 hour$

(v) If the bob of simple pendulum is suspended by a wire then effective length of pendulum will increase with the rise of temperature due to which the time period will increase.

$l = l_0(1+\alpha\,\Delta\theta)$ (If $\Delta\theta$ is the rise in temperature, l_0 = initial length of wire, l = final length of wire)

$$\frac{T}{T_0} = \sqrt{\frac{l}{l_0}} = (1+\alpha\,\Delta\theta)^{1/2} \approx 1 + \frac{1}{2}\alpha\,\Delta\theta$$

So $\dfrac{T}{T_0} - 1 = \dfrac{1}{2}\alpha\,\Delta\theta$ *i.e.* $\dfrac{\Delta T}{T} \approx \dfrac{1}{2}\alpha\,\Delta\theta$

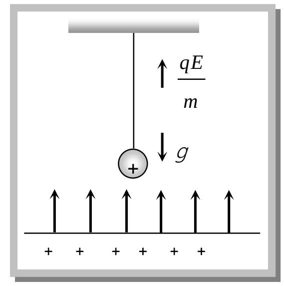
(vi) If bob a simple pendulum of density ρ is made to oscillate in some fluid of density σ (where $\sigma < \rho$) then time period of simple pendulum gets increased.

As thrust will oppose its weight therefore $mg' = mg$ – Thrust

or $g' = g - \dfrac{V\sigma g}{V\rho}$ *i.e.* $g' = g\left[1 - \dfrac{\sigma}{\rho}\right] \Rightarrow \dfrac{g'}{g} = \dfrac{\rho-\sigma}{\rho}$

$\therefore$ $\dfrac{T'}{T} = \sqrt{\dfrac{g}{g'}} = \sqrt{\dfrac{\rho}{\rho-\sigma}} > 1$

(vii) If a bob of mass *m* carries a positive charge *q* and pendulum is placed in a uniform electric field of strength *E* directed vertically upwards.



In given condition net down ward acceleration $g' = g - \frac{qE}{m}$

So $$T = 2\pi\sqrt{\frac{l}{g - \frac{qE}{m}}}$$

If the direction of field is vertically downward then time period $T = 2\pi\sqrt{\frac{l}{g + \frac{qE}{m}}}$

(viii) Pendulum in a lift : If the pendulum is suspended from the ceiling of the lift.

(a) If the lift is at rest or moving down ward /up ward with constant velocity.

$$T = 2\pi\sqrt{\frac{l}{g}} \quad \text{and} \quad n = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$$

(b) If the lift is moving up ward with constant acceleration *a*

$$T = 2\pi\sqrt{\frac{l}{g+a}} \quad \text{and} \quad n = \frac{1}{2\pi}\sqrt{\frac{g+a}{l}}$$

Time period decreases and frequency increases

(c) If the lift is moving down ward with constant acceleration *a*

$$T = 2\pi\sqrt{\frac{l}{g-a}} \quad \text{and} \quad n = \frac{1}{2\pi}\sqrt{\frac{g-a}{l}}$$

Time period increase and frequency decreases

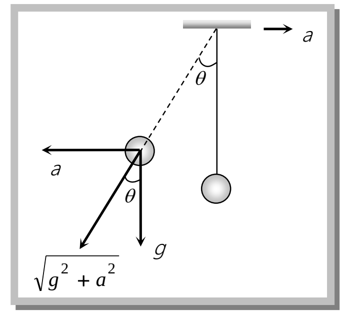
(d) If the lift is moving down ward with acceleration $a = g$

$$T = 2\pi\sqrt{\frac{l}{g-g}} = \infty \quad \text{and} \quad n = \frac{1}{2\pi}\sqrt{\frac{g-g}{l}} = 0$$

It means there will be no oscillation in a pendulum.

Similar is the case in a satellite and at the centre of earth where effective acceleration becomes zero and pendulum will stop.

(ix) The time period of simple pendulum whose point of suspension moving horizontally with acceleration *a*



$$T = 2\pi\sqrt{\frac{l}{(g^2 + a^2)^{1/2}}} \quad \text{and} \quad \theta = \tan^{-1}(a/g)$$

(x) If simple pendulum suspended in a car that is moving with constant speed v around a circle of radius r.

$$T = 2\pi\frac{\sqrt{l}}{\sqrt{g^2 + \left(\frac{v^2}{r}\right)^2}}$$

(xi) Second's Pendulum : It is that simple pendulum whose time period of vibrations is two seconds.

Putting $T = 2$ *sec* and $g = 9.8m/\sec^2$ in $T = 2\pi\sqrt{\frac{l}{g}}$ we get

$$l = \frac{4 \times 9.8}{4\pi^2} = 0.993\ m = 99.3\ cm$$

Hence length of second's pendulum is 99.3 *cm* or nearly 1 meter on earth surface.

For the moon the length of the second's pendulum will be 1/6 meter [As $g_{moon} = \frac{g_{\text{Earth}}}{6}$]

(xii) In the absence of resistive force the work done by a simple pendulum in one complete oscillation is zero.

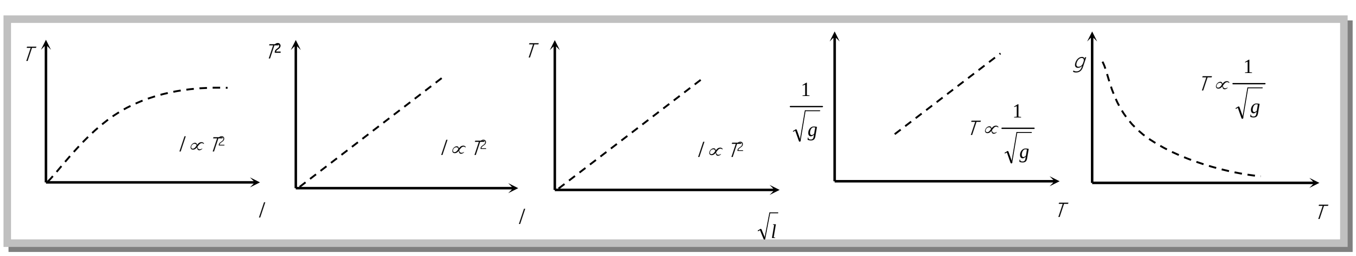
(xiii) Work done in giving an angular displacement θ to the pendulum from its mean position.

$$W = U = mgl(1 - \cos\theta)$$

(xiv) Kinetic energy of the bob at mean position = work done or potential energy at extreme

$$KE_{mean} = mgl(1 - \cos\theta)$$

(xv) Various graph for simple pendulum



Sample problems based on Simple pendulum

Problem 31. A clock which keeps correct time at 20°*C*, is subjected to 40°*C*. If coefficient of linear expansion of the pendulum is $12 \times 10^{-6} / °C$. How much will it gain or loose in time [BHU 1998]

(a) 10.3 *sec/day* (b) 20.6 *sec*/day (c) 5 *sec*/day (d) 20 *min*/day

Solution : (a) $\frac{\Delta T}{T} = \frac{1}{2}\alpha\Delta\theta = \frac{1}{2} \times 12 \times 10^{-6} \times (40 - 20)$; $\Delta T = 12 \times 10^{-5} \times 86400\ sec/\text{day}$ = 10.3 *sec/*day.

Problem 32. The metallic bob of simple pendulum has the relative density ρ. The time period of this pendulum is *T*. If the metallic bob is immersed in water, then the new time period is given by [SCRA 1998]

(a) $T\left(\frac{\rho - 1}{\rho}\right)$ (b) $T\left(\frac{\rho}{\rho - 1}\right)$ (c) $T\sqrt{\frac{\rho - 1}{\rho}}$ (d) $T\sqrt{\frac{\rho}{\rho - 1}}$

Solution : (d) Formula $\frac{T'}{T} = \sqrt{\frac{\rho}{\rho - \sigma}}$ Here $\sigma = 1$ for water so $T' = T\sqrt{\frac{\rho}{\rho - 1}}$.

Problem 33. The period of a simple pendulum is doubled when [CPMT 1974; MNR 1980; AFMC 1995]

(a) Its length is doubled

(b) The mass of the bob is doubled

(c) Its length is made four times

(d) The mass of the bob and the length of the pendulum are doubled

Solution : (c)

Problem 34. A simple pendulum is executing S.H.M. with a time period *T*. if the length of the pendulum is increased by 21% the percentage increase in the time period of the pendulum is [BHU 1994]

(a) 10% (b) 21% (c) 30% (d) 50%

Solution : (a) As $T \propto \sqrt{l}$ $\therefore$ $\frac{T_2}{T_1} = \sqrt{\frac{l_2}{l_1}} = \sqrt{1.21} \Rightarrow T_2 = 1.1\,T = T + 10\%T$.

Problem 35. The length of simple pendulum is increased by 1% its time period will [MP PET 1994]

(a) Increase by 1% (b) Increase by 0.5% (c) Decrease by 0.5% (d) Increase by 2%

Solution : (b) $T = 2\pi\sqrt{l/g}$ hence $T \propto \sqrt{l}$

Percentage increment in $T = \frac{1}{2}$(percentage increment in *l*) = 0.5%.

Problem 36. The bob of a simple pendulum of mass *m* and total energy *E* will have maximum linear momentum equal to

[MP PMT 1986]

(a) $\sqrt{\frac{2E}{m}}$ (b) $\sqrt{2mE}$ (c) 2*mE* (d) *mE*²

Solution : (b) $E = \frac{P^2}{2m}$ where *E* = Kinetic Energy, *P* = Momentum, *m* = Mass

So $P = \sqrt{2mE}$.

Problem 37. The mass and diameter of a planet are twice those of earth. The period of oscillation of pendulum on this planet will be (if it is a second's pendulum on earth) [IIT 1973]

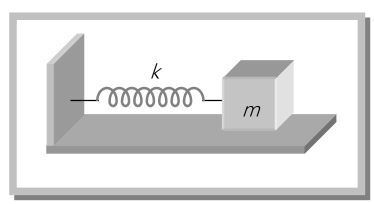
(a) $\frac{1}{\sqrt{2}}$ *sec* (b) $2\sqrt{2}$ *sec* (c) 2 *sec* (d) $\frac{1}{2}$ *sec*

Solution : (b) $g \propto \frac{M}{R^2}$; $g' = g/2$; $\frac{T'}{T} = \sqrt{\frac{g}{g'}}$ (*T* = 2 sec for second's pendulum)

$T' = 2\sqrt{2}$

15.14 Spring Pendulum

A point mass suspended from a mass less spring or placed on a frictionless horizontal plane attached with spring (fig.) constitutes a linear harmonic spring pendulum



Time period $T = 2\pi\sqrt{\frac{\text{inertia factor}}{\text{spring factor}}}$

$T = 2\pi\sqrt{\frac{m}{k}}$ and Frequency $n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

Important points

(i) Time period of a spring pendulum depends on the mass suspended

$$T \propto \sqrt{m} \quad \text{or} \quad n \propto \frac{1}{\sqrt{m}}$$

i.e. greater the mass greater will be the inertia and so lesser will be the frequency of oscillation and greater will be the time period.

(ii) The time period depends on the force constant *k* of the spring

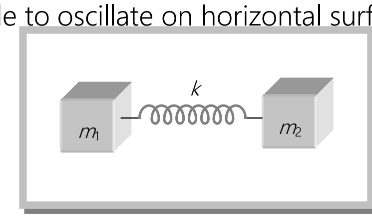
$$T \propto \frac{1}{\sqrt{k}} \qquad \text{or} \qquad n \propto \sqrt{k}$$

(iii) Time of a spring pendulum is independent of acceleration due to gravity. That is why a clock based on spring pendulum will keep proper time every where on a hill or moon or in a satellite and time period of a spring pendulum will not change inside a liquid if damping effects are neglected.

(iv) If the spring has a mass *M* and mass *m* is suspended from it, effective mass is given by $m_{eff} = m + \frac{M}{3}$

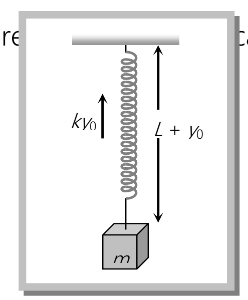
So that $$T = 2\pi\sqrt{\frac{m_{eff}}{k}}$$

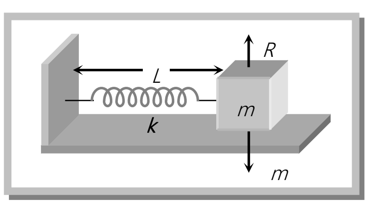
(v) If two masses of mass m_1 and m_2 are connected by a spring and made to oscillate on horizontal surface, the reduced mass m_r is given by $\frac{1}{m_r} = \frac{1}{m_1} + \frac{1}{m_2}$



So that $$T = 2\pi\sqrt{\frac{m_r}{k}}$$

(vi) If a spring pendulum, oscillating in a vertical plane is made to oscillate on a horizontal surface, (or on inclined plane) time period will remain unchanged. However, equilibrium position for a spring in a horizontal plain is the position of natural length of spring as weight is balanced by re case of vertical motion equilibrium position will be $L + y_0$ with $ky_0 = mg$



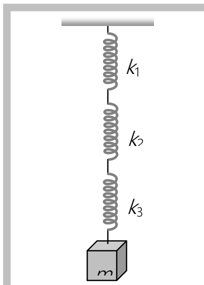


(vii) If the stretch in a vertically loaded spring is y_0 then for equilibrium of mass *m*, $ky_0 = mg$ *i.e.* $\frac{m}{k} = \frac{y_0}{g}$

So that $$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{y_0}{g}}$$

Time period does not depends on '*g*' because along with *g*, y_0 will also change in such a way that $\frac{y_0}{g} = \frac{m}{k}$ remains constant

(viii) Series combination : If *n* springs of different force constant are connected in series having force constant $k_1, k_2, k_3 ....$ respectively then



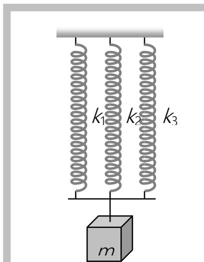
Series combination

$$\frac{1}{k_{eff}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + ...$$

If all spring have same spring constant then

$$k_{eff} = \frac{k}{n}$$

(ix) Parallel combination : If the springs are connected in parallel then



Parallel combination

$$k_{eff} = k_1 + k_2 + k_3 + ...$$

If all spring have same spring constant then

$$k_{eff} = nk$$

(x) If the spring of force constant *k* is divided in to *n* equal parts then spring constant of each part will become *nk* and if these *n* parts connected in parallel then

$$k_{eff} = n^2 k$$

(xi) The spring constant *k* is inversely proportional to the spring length.

As $\quad k \propto \frac{1}{\text{Extension}} \propto \frac{1}{\text{Length of spring}}$

That means if the length of spring is halved then its force constant becomes double.

(xii) When a spring of length *l* is cut in two pieces of length l_1 and l_2 such that $l_1 = nl_2$.

If the constant of a spring is *k* then $\quad$ Spring constant of first part $k_1 = \frac{k(n+1)}{n}$

Spring constant of second part $k_2 = (n+1)k$

and ratio of spring constant $\frac{k_1}{k_2} = \frac{1}{n}$

Sample problems based on Spring pendulum

Problem 38. A spring of force constant *k* is cut into two pieces such that one pieces is double the length of the other. Then the long piece will have a force constant of [IIT-JEE 1999]

(a) $2/3k$ (b) $3/2k$ (c) $3k$ (d) $6k$

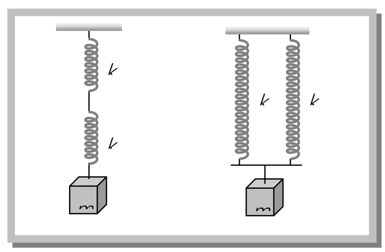
Solution : (b) If $l_1 = nl_2$ then $k_1 = \frac{(n+1)k}{n} = \frac{3}{2}k$ [As *n* = 2]

Problem 39. Two bodies *M* and *N* of equal masses are suspended from two separate mass less springs of force constants k_1 and k_2 respectively. If the two bodies oscillate vertically such that their maximum velocities are equal, the ratio of the amplitude of *M* to that of *N* is [MP PET/PMT 1997; IIT-JEE 1988; BHU 1998]

(a) k_1/k_2 (b) $\sqrt{k_1/k_2}$ (c) k_2/k_1 (d) $\sqrt{k_2/k_1}$

Solution : (d) Given that maximum velocities are equal $a_1\omega_1 = a_2\omega_2 \Rightarrow a_1\sqrt{\frac{k_1}{m}} = a_2\sqrt{\frac{k_2}{m}} \Rightarrow \frac{a_1}{a_2} = \sqrt{\frac{k_2}{k_1}}$.

Problem 40. Two identical springs of constant *k* are connected in series and parallel as shown in figure. A mass *m* is suspended from them. The ratio of their frequencies of vertical oscillation will be



(a) 2 : 1

(b) 1 : 1

(c) 1 : 2

(d) 4 : 1

Solution : (c) For series combination $n_1 \propto \sqrt{k/2}$

For parallel combination $n_2 \propto \sqrt{2k}$ so $\frac{n_1}{n_2} = \sqrt{\frac{k/2}{2k}} = \frac{1}{2}$.

Problem 41. A block of mass *m* attached to a spring of spring constant *k* oscillates on a smooth horizontal table. The other end of the spring is fixed to a wall. The block has a speed *v* when the spring is at its natural length. Before coming to an instantaneous rest, if the block moves a distance *x* from the Mean position, then

(a) $x = \sqrt{m/k}$ (b) $x = \frac{1}{v}\sqrt{\frac{m}{k}}$ (c) $x = v\sqrt{m/k}$ (d) $x = \sqrt{mv/k}$

Solution : (c) Kinetic energy of block $\left(\frac{1}{2}mv^2\right)$ = Elastic potential energy of spring $\left(\frac{1}{2}kx^2\right)$

By solving we get $x = v\sqrt{\frac{m}{k}}$.

Problem 42. A block is placed on a friction less horizontal table. The mass of the block is *m* and springs of force constant k_1, k_2 are attached on either side with if the block is displaced a little and left to oscillate, then the angular frequency of oscillation will be

(a) $\left(\frac{k_1+k_2}{m}\right)^{1/2}$ (b) $\left[\frac{k_1k_2}{m(k_1+k_2)}\right]^{1/2}$ (c) $\left[\frac{k_1k_2}{(k_1-k_2)m}\right]^{1/2}$ (d) $\left[\frac{k_1^2+k_2^2}{(k_1+k_2)m}\right]^{1/2}$

Solution : (a) Given condition match with parallel combination so $k_{eff}=k_1+k_2$ $\therefore\ \omega=\sqrt{\frac{k_{eff}}{m}}=\sqrt{\frac{k_1+k_2}{m}}$.

Problem 43. A particle of mass 200 *gm* executes S.H.M. The restoring force is provided by a spring of force constant 80 *N/m*. The time period of oscillations is [MP PET 1994]

(a) 0.31 sec (b) 0.15 sec (c) 0.05 sec (d) 0.02 sec

Solution : (a) $T=2\pi\sqrt{\frac{m}{k}}=2\pi\sqrt{\frac{0.2}{80}}=\frac{2\pi}{20}=0.31\text{ sec}$.

Problem 44. The length of a spring is *l* and its force constant is *k* when a weight *w* is suspended from it. Its length increases by *x*. if the spring is cut into two equal parts and put in parallel and the same weight *W* is suspended from them, then the extension will be

(a) $2x$ (b) x (c) $x/2$ (d) $x/4$

Solution : (d) As $F=kx$ so $x\propto\frac{1}{k}$ (if F = constant)

If the spring of constant *k* is divided in to two equal parts then each parts will have a force constant $2k$.

If these two parts are put in parallel then force constant of combination will becomes $4k$.

$x\propto\frac{1}{k}$ so, $\frac{x_2}{x_1}=\frac{k_1}{k_2}=\frac{k}{4k}\Rightarrow x_2=\frac{x}{4}$.

Problem 45. A mass *m* is suspended from a string of length *l* and force constant *k*. The frequency of vibration of the mass is f_1. The spring is cut in to two equal parts and the same mass is suspended from one of the parts. The new frequency of vibration of mass is f_2. Which of the following reaction between the frequencies is correct.

[NCERT 1983; CPMT 1986; MP PMT 1991]

(a) $f_1=\sqrt{2}f_2$ (b) $f_1=f_2$ (c) $f_1=2f_2$ (d) $f_2=\sqrt{2}f_1$

Solution : (d) $f\propto\sqrt{k}$

If the spring is divided in to equal parts then force constant of each part will becomes double

$\frac{f_2}{f_1}=\sqrt{\frac{k_2}{k_1}}=\sqrt{2}\Rightarrow f_2=\sqrt{2}f_1$

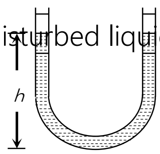
15.15 Various Formulae of S.H.M.

S.H.M. of a liquid in U tube

If a liquid of density ρ contained in a vertical U tube performs S.H.M. in its two limbs. Then time period $T = 2\pi\sqrt{\frac{L}{2g}} = 2\pi\sqrt{\frac{h}{g}}$

where L = Total length of liquid column,

h = Height of undisturbed liquid in each limb ($L=2h$)



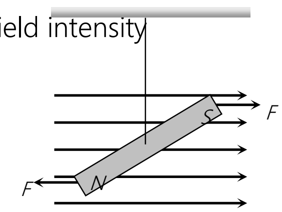
S.H.M. of a bar magnet in a magnetic field

$T = 2\pi\sqrt{\frac{I}{MB}}$

I = Moment of inertia of magnet

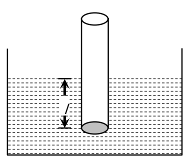
M = Magnetic moment of magnet

B = Magnetic field intensity



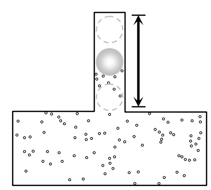
S.H.M. of a floating cylinder

If l is the length of cylinder dipping in liquid then time period $T = 2\pi\sqrt{\frac{l}{g}}$



S.H.M. of ball in the neck of an air chamber

$T = \frac{2\pi}{A}\sqrt{\frac{mV}{E}}$



m = mass of the ball

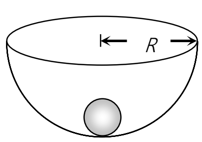
V = volume of air- chamber

A = area of cross section of neck

E = Bulk modulus for Air

S.H.M. of a small ball rolling down in hemi-spherical bowl

$T = 2\pi\sqrt{\frac{R-r}{g}}$

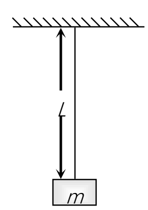


R = radius of the bowl

r =radius of the ball

S.H.M. of a body suspended from a wire

$T = 2\pi\sqrt{\frac{mL}{YA}}$

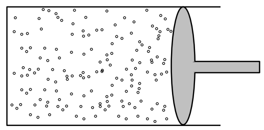
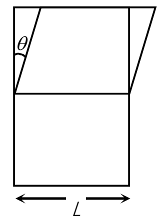
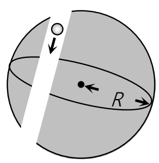
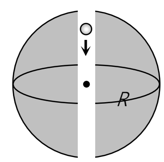
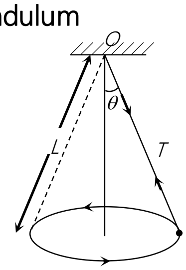


m = mass of the body

L = length of the wire

Y = young's modulus of wire

A = area of cross section of wire

S.H.M. of a piston in a cylinder	S.H.M of a cubical block
$T = 2\pi\sqrt{\dfrac{Mh}{PA}}$  M = mass of the piston A = area of cross section h = height of cylinder P = pressure in a cylinder	$T = 2\pi\sqrt{\dfrac{M}{\eta L}}$  M = mass of the block L = length of side of cube η = modulus of rigidity
S.H.M. of a body in a tunnel dug along any chord of earth	**S.H.M. of body in the tunnel dug along the diameter of earth**
$T = 2\pi\sqrt{\dfrac{R}{g}}$ = 84.6 minutes 	$T = 2\pi\sqrt{\dfrac{R}{g}}$  T = 84.6 minutes R = radius of the earth = 6400 km g = acceleration due to gravity = 9.8 m/s^2 at earth's surface
S.H.M. of a conical pendulum	**S.H.M. of L-C circuit**
$T = 2\pi\sqrt{\dfrac{L\cos\theta}{g}}$  L = length of string θ = angle of string from the vertical g = acceleration due to gravity	$T = 2\pi\sqrt{LC}$ L = coefficient of self inductance C = capacity of condenser

15.16 Important Facts and Formulae

(1) When a body is suspended from two light springs separately. The time period of vertical oscillations are T_1 and T_2 respectively.

$$T_1 = 2\pi\sqrt{\frac{m}{k_1}} \quad \therefore \quad k_1 = \frac{4\pi^2 m}{T_1^{\,2}} \qquad \text{and} \qquad T_2 = 2\pi\sqrt{\frac{m}{k_2}} \quad \therefore \ k_2 = \frac{4\pi^2 m}{T_2^{\,2}}$$

When these two springs are connected in series and the same mass m is attached at lower end and then for series combination $\dfrac{1}{k} = \dfrac{1}{k_1} + \dfrac{1}{k_2}$

By substituting the values of k_1, k_2 $\quad \frac{T^2}{4\pi^2 m} = \frac{T_1^2}{4\pi^2 m} + \frac{T_2^2}{4\pi^2 m}$

Time period of the system $T = \sqrt{T_1^2 + T_2^2}$

When these two springs are connected in parallel and the same mass m is attached at lower end and then for parallel combination $\quad k = k_1 + k_2$

By substituting the values of k_1, k_2 $\quad \frac{4\pi^2 m}{T^2} = \frac{4\pi^2 m}{T_1^2} + \frac{4\pi^2 m}{T_2^2}$

Time period of the system $T = \frac{T_1 T_2}{\sqrt{T_1^2 + T_2^2}}$

(2) The pendulum clock runs slow due to increase in its time period whereas it becomes fast due to decrease in time period.

(3) If infinite spring with force constant $k, 2k, 4k, 8k$.......... respectively are connected in series. The effective force constant of the spring will be $k/2$.

(4) If $y_1 = a\sin\omega t$ and $y_2 = b\cos\omega t$ are two S.H.M. then by the superimposition of these two S.H.M. we get

$$\vec{y} = \vec{y}_1 + \vec{y}_2$$

$$y = a\sin\omega t + b\cos\omega t$$

$$y = A\sin(\omega t + \phi)$$ this is also the equation of S.H.M.

where $A = \sqrt{a^2 + b^2}$ and $\phi = \tan^{-1}(b/a)$

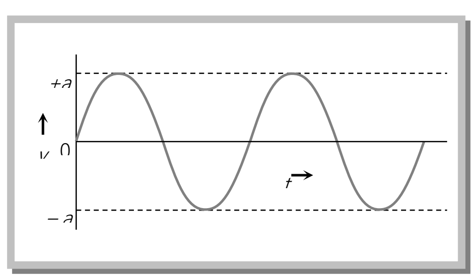
(5) If a particle performs S.H.M. whose velocity is v_1 at a x_1 distance from mean position and velocity v_2 at distance x_2

$$\omega = \sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}}; \quad T = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}} \quad a = \sqrt{\frac{v_1^2 x_2^2 - v_2^2 x_1^2}{v_1^2 - v_2^2}}; \quad v_{\max} = \sqrt{\frac{v_1^2 x_2^2 - v_2^2 x_1^2}{x_2^2 - x_1^2}}$$

15.17 Free, Damped, Forced and Maintained Oscillation

(1) **Free oscillation**

(i) The oscillation of a particle with fundamental frequency under the influence of restoring force are defined as free oscillations

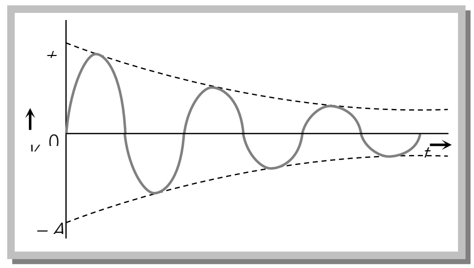


(ii) The amplitude, frequency and energy of oscillation remains constant

(iii) Frequency of free oscillation is called natural frequency because it depends upon the nature and structure of the body.

(2) **Damped oscillation**

(i) The oscillation of a body whose amplitude goes on decreasing with time are defined as damped oscillation



(ii) In these oscillation the amplitude of oscillation decreases exponentially due to damping forces like frictional force, viscous force, hysteresis *etc*.

(iii) Due to decrease in amplitude the energy of the oscillator also goes on decreasing exponentially

(3) **Forced oscillation**

(i) The oscillation in which a body oscillates under the influence of an external periodic force are known as forced oscillation

(ii) The amplitude of oscillator decrease due to damping forces but on account of the energy gained from the external source it remains constant.

(iii) Resonance : When the frequency of external force is equal to the natural frequency of the oscillator. Then this state is known as the state of resonance. And this frequency is known as resonant frequency.

(4) **Maintained oscillation**

The oscillation in which the loss of oscillator is compensated by the supplying energy from an external source are known as maintained oscillation.