

13

Limits and Derivatives

Short Answer Type Questions

Q. 1 Evaluate $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$.

Sol. Given,

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} &= \lim_{x \rightarrow 3} \frac{x^2 - (3)^2}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{(x + 3)(x - 3)}{(x - 3)} = \lim_{x \rightarrow 3} (x + 3) \\ &= 3 + 3 = 6\end{aligned}$$

Q. 2 Evaluate $\lim_{x \rightarrow 1/2} \frac{4x^2 - 1}{2x - 1}$.

Sol. Given,

$$\begin{aligned}\lim_{x \rightarrow 1/2} \frac{4x^2 - 1}{2x - 1} &= \lim_{x \rightarrow 1/2} \frac{(2x)^2 - (1)^2}{2x - 1} \\ &= \lim_{x \rightarrow 1/2} \frac{(2x + 1)(2x - 1)}{(2x - 1)} = \lim_{x \rightarrow 1/2} (2x + 1) \\ &= 2 \times \frac{1}{2} + 1 = 1 + 1 = 2\end{aligned}$$

Q. 3 Evaluate $\lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$.

Sol. Given,

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} &= \lim_{h \rightarrow 0} \frac{(x+h)^{1/2} - (x)^{1/2}}{x+h-x} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^{1/2} - (x)^{1/2}}{(x+h)-x} \\ &= \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-1/2} \\ &= \frac{1}{2\sqrt{x}}\end{aligned}$$

$\left[\because \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \right]$

$[\because h \rightarrow 0 \Rightarrow x + h \rightarrow x]$

Q. 4 Evaluate $\lim_{x \rightarrow 0} \frac{(x+2)^{1/3} - 2^{1/3}}{x}$.

Sol. Given, $\lim_{x \rightarrow 0} \frac{(x+2)^{1/3} - 2^{1/3}}{x} = \lim_{x \rightarrow 0} \frac{(x+2)^{1/3} - 2^{1/3}}{(x+2) - 2}$

$$= \frac{1}{3} \times 2^{\frac{1}{3}-1}$$

$$= \frac{1}{3} \times (2)^{-2/3}$$

$$= \frac{1}{3(2)^{2/3}}$$

$$\left[\because \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \right]$$

$$[\because x \rightarrow 0 \Rightarrow x+2 \rightarrow 2]$$

Q. 5 Evaluate $\lim_{x \rightarrow 0} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$.

Sol. Given, $\lim_{x \rightarrow 0} \frac{(1+x)^6 - 1}{(1+x)^2 - 1} = \lim_{x \rightarrow 0} \frac{\frac{(1+x)^6 - 1}{x}}{\frac{(1+x)^2 - 1}{x}}$

$$= \lim_{x \rightarrow 0} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$$

$$= \lim_{x \rightarrow 0} \frac{(1+x)^6 - (1)^6}{(1+x)^2 - (1)^2}$$

$$= \frac{6(1)^{6-1}}{2(1)^{2-1}}$$

$$= \frac{6 \times 1}{2 \times 1} = \frac{6}{2} = 3$$

[dividing numerator and denominator by x]

$$[\because x \rightarrow 0 \Rightarrow 1+x \rightarrow 1]$$

$$\left[\because \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \right]$$

$$\left[\because \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \right]$$

Q. 6 Evaluate $\lim_{x \rightarrow a} \frac{(2+x)^{5/2} - (a+2)^{5/2}}{x - a}$.

Sol. Given, $\lim_{x \rightarrow a} \frac{(2+x)^{5/2} - (a+2)^{5/2}}{x - a} = \lim_{x \rightarrow a} \frac{(2+x)^{5/2} - (a+2)^{5/2}}{(2+x) - (a+2)}$

$$= \frac{5}{2} (a+2)^{\frac{5}{2}-1}$$

$$= \frac{5}{2} (a+2)^{3/2}$$

$$\left[\because \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \right]$$

$$[\because x \rightarrow a \Rightarrow x+2 \rightarrow a+2]$$

Q. 7 Evaluate $\lim_{x \rightarrow 1} \frac{x^4 - \sqrt{x}}{\sqrt{x} - 1}$.

Sol. Given,
$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^4 - \sqrt{x}}{\sqrt{x} - 1} &= \lim_{x \rightarrow 1} \frac{\sqrt{x}[(x)^{7/2} - 1]}{\sqrt{x} - 1} \\ &= \lim_{x \rightarrow 1} \frac{(x)^{7/2} - 1}{\sqrt{x} - 1} \cdot \lim_{x \rightarrow 1} \sqrt{x} \quad \left[\because \lim_{x \rightarrow a} f(x) \cdot g(x) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) \right] \\ &= \lim_{x \rightarrow 1} \frac{\frac{x^{7/2} - 1}{x - 1}}{\frac{(x)^{1/2} - 1}{x - 1}} \cdot 1 \\ &= \frac{\lim_{x \rightarrow 1} \frac{x^{7/2} - 1}{x - 1}}{\lim_{x \rightarrow 1} \frac{(x)^{1/2} - 1}{x - 1}} \quad \left[\because \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \right] \\ &= \frac{\frac{7}{2}(1)^{\frac{7}{2}-1}}{\frac{1}{2}(1)^{\frac{1}{2}-1}} = \frac{\frac{7}{2}}{\frac{1}{2}} = 7 \end{aligned}$$

Q. 8 Evaluate $\lim_{x \rightarrow 2} \frac{x^2 - 4}{\sqrt{3x - 2} - \sqrt{x + 2}}$.

Sol. Given,
$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^2 - 4}{\sqrt{3x - 2} - \sqrt{x + 2}} &= \lim_{x \rightarrow 2} \frac{(x^2 - 4)\sqrt{3x - 2} + \sqrt{x + 2}}{(\sqrt{3x - 2} - \sqrt{x + 2})(\sqrt{3x - 2} + \sqrt{x + 2})} \\ &= \lim_{x \rightarrow 2} \frac{(x^2 - 4)(\sqrt{3x - 2} + \sqrt{x + 2})}{(\sqrt{3x - 2})^2 - (\sqrt{x + 2})^2} \\ &\quad [\because (a + b)(a - b) = a^2 - b^2] \\ &= \lim_{x \rightarrow 2} \frac{(x^2 - 4)(\sqrt{3x - 2} + \sqrt{x + 2})}{(3x - 2) - (x + 2)} \\ &= \lim_{x \rightarrow 2} \frac{(x^2 - 4)(\sqrt{3x - 2} + \sqrt{x + 2})}{3x - 2 - x - 2} \\ &= \lim_{x \rightarrow 2} \frac{(x^2 - 4)(\sqrt{3x - 2} + \sqrt{x + 2})}{2x - 4} \\ &= \lim_{x \rightarrow 2} \frac{(x + 2)(x - 2)(\sqrt{3x - 2} + \sqrt{x + 2})}{2(x - 2)} \\ &= \lim_{x \rightarrow 2} \frac{(x + 2)(\sqrt{3x - 2} + \sqrt{x + 2})}{2} \\ &= \frac{(2 + 2)(\sqrt{6 - 2} + \sqrt{2 + 2})}{2} \\ &= \frac{4(2 + 2)}{2} = 8 \end{aligned}$$

Q. 9 Evaluate $\lim_{x \rightarrow \sqrt{2}} \frac{x^4 - 4}{x^2 + 3\sqrt{2}x - 8}$.

Sol. Given,
$$\begin{aligned}\lim_{x \rightarrow \sqrt{2}} \frac{x^4 - 4}{x^2 + 3\sqrt{2}x - 8} &= \lim_{x \rightarrow \sqrt{2}} \frac{(x^2)^2 - (2)^2}{x^2 + 3\sqrt{2}x - 8} \\&= \lim_{x \rightarrow \sqrt{2}} \frac{(x^2 - 2)(x^2 + 2)}{x^2 + 4\sqrt{2}x - \sqrt{2}x - 8} \\&= \lim_{x \rightarrow \sqrt{2}} \frac{(x - \sqrt{2})(x + \sqrt{2})(x^2 + 2)}{x(x + 4\sqrt{2}) - \sqrt{2}(x + 4\sqrt{2})} \\&= \lim_{x \rightarrow \sqrt{2}} \frac{(x - \sqrt{2})(x + \sqrt{2})(x^2 + 2)}{(x - \sqrt{2})(x + 4\sqrt{2})} \\&= \lim_{x \rightarrow \sqrt{2}} \frac{(x + \sqrt{2})(x^2 + 2)}{(x + 4\sqrt{2})} \\&= \frac{(\sqrt{2} + \sqrt{2})[(\sqrt{2})^2 + 2]}{(\sqrt{2} + 4\sqrt{2})} \\&= \frac{2\sqrt{2}(2 + 2)}{5\sqrt{2}} = \frac{8}{5}\end{aligned}$$

Q. 10 Evaluate $\lim_{x \rightarrow 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$.

Sol. Given,
$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2} &\left[\frac{0}{0} \text{ form} \right] \\&= \lim_{x \rightarrow 1} \frac{x^7 - x^5 - x^5 + 1}{x^3 - x^2 - 2x^2 + 2} \\&= \lim_{x \rightarrow 1} \frac{x^5(x^2 - 1) - 1(x^5 - 1)}{x^2(x - 1) - 2(x^2 - 1)}\end{aligned}$$

On dividing numerator and denominator by $(x - 1)$, then

$$\begin{aligned}&\frac{x^5(x^2 - 1)}{(x - 1)} - \frac{1(x^2 - 1)}{(x - 1)} \\&= \lim_{x \rightarrow 1} \frac{\frac{x^5(x^2 - 1)}{(x - 1)} - \frac{2(x^2 - 1)}{(x - 1)}}{\frac{x^2(x - 1)}{(x - 1)} - \frac{2(x^2 - 1)}{(x - 1)}} \\&= \frac{\lim_{x \rightarrow 1} x^5(x + 1) - \lim_{x \rightarrow 1} \left(\frac{x^5 - 1}{x - 1} \right)}{\lim_{x \rightarrow 1} x^2 - \lim_{x \rightarrow 1} (x + 1)} \\&= \frac{1 \times 2 - 5 \times (1)^4}{1 - 2 \times 2} = \frac{2 - 5}{1 - 4} \\&= \frac{-3}{-3} = 1\end{aligned}$$

Q. 11 Evaluate $\lim_{x \rightarrow 0} \frac{\sqrt{1+x^3} - \sqrt{1-x^3}}{x^2}$.

Sol. Given, $\lim_{x \rightarrow 0} \frac{\sqrt{1+x^3} - \sqrt{1-x^3}}{x^2} = \lim_{x \rightarrow 0} \frac{\sqrt{1+x^3} - \sqrt{1-x^3}}{x^2} \cdot \frac{\sqrt{1+x^3} + \sqrt{1-x^3}}{\sqrt{1+x^3} + \sqrt{1-x^3}}$

$$= \lim_{x \rightarrow 0} \frac{(1+x^3) - (1-x^3)}{x^2(\sqrt{1+x^3} + \sqrt{1-x^3})}$$

$$= \lim_{x \rightarrow 0} \frac{1+x^3 - 1+x^3}{x^2(\sqrt{1+x^3} + \sqrt{1-x^3})}$$

$$= \lim_{x \rightarrow 0} \frac{2x^3}{x^2(\sqrt{1+x^3} + \sqrt{1-x^3})}$$

$$= \lim_{x \rightarrow 0} \frac{2x}{(\sqrt{1+x^3} + \sqrt{1-x^3})}$$

$$= 0$$

Q. 12 Evaluate $\lim_{x \rightarrow -3} \frac{x^3 + 27}{x^5 + 243}$.

Sol. Given, $\lim_{x \rightarrow -3} \frac{x^3 + 27}{x^5 + 243} = \lim_{x \rightarrow -3} \frac{\frac{x^3 + 27}{x + 3}}{\frac{x^5 + 243}{x + 3}}$

$$= \lim_{x \rightarrow -3} \frac{\frac{x^3 - (-3)^3}{x - (-3)}}{\frac{x^5 - (-3)^5}{x - (-3)}} = \frac{\lim_{x \rightarrow -3} \frac{x^3 - (-3)^3}{x - (-3)}}{\lim_{x \rightarrow -3} \frac{x^5 - (-3)^5}{x - (-3)}}$$

$$= \frac{3(-3)^{3-1}}{5(-3)^{5-1}} = \frac{3(-3)^2}{5(-3)^4}$$

$$= \frac{3}{5(-3)^2} = \frac{3}{45} = \frac{1}{15}$$

$$\left[\because \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \right]$$

$$\left[\because \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \right]$$

Q. 13 Evaluate $\lim_{x \rightarrow 1/2} \left(\frac{8x-3}{2x-1} - \frac{4x^2+1}{4x^2-1} \right)$.

Sol. Given, $\lim_{x \rightarrow 1/2} \left(\frac{8x-3}{2x-1} - \frac{4x^2+1}{4x^2-1} \right) = \lim_{x \rightarrow 1/2} \left[\frac{(8x-3)(2x-1) - (4x^2+1)}{(4x^2-1)} \right]$

$$= \lim_{x \rightarrow 1/2} \left[\frac{16x^2 + 8x - 6x - 3 - 4x^2 - 1}{4x^2 - 1} \right]$$

$$= \lim_{x \rightarrow 1/2} \left[\frac{12x^2 + 2x - 4}{4x^2 - 1} \right]$$

$$= \lim_{x \rightarrow 1/2} \frac{2(6x^2 + x - 2)}{4x^2 - 1}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 1/2} \frac{2(6x^2 + 4x - 3x - 2)}{4x^2 - 1} \\
&= \lim_{x \rightarrow 1/2} \frac{2[2x(3x + 2) - 1(3x + 2)]}{4x^2 - 1} \\
&= \lim_{x \rightarrow 1/2} \frac{2[(3x + 2)(2x - 1)]}{(2x)^2 - (1)^2} \\
&= \lim_{x \rightarrow 1/2} \frac{2(3x + 2)(2x - 1)}{(2x - 1)(2x + 1)} \\
&= \lim_{x \rightarrow 1/2} \frac{2(3x + 2)}{2x + 1} = \frac{2\left(3 \times \frac{1}{2} + 2\right)}{2 \times \frac{1}{2} + 1} \\
&= \frac{3}{2} + 2 = \frac{7}{2}
\end{aligned}$$

Q. 14 Find the value of n , if $\lim_{x \rightarrow 2} \frac{x^n - 2^n}{x - 2} = 80, n \in N$.

Sol. Given,

$$\lim_{x \rightarrow 2} \frac{x^n - 2^n}{x - 2} = 80$$

$$\Rightarrow n(2)^{n-1} = 80$$

$$\Rightarrow n(2)^{n-1} = 5 \times 16$$

$$\Rightarrow n \times 2^{n-1} = 5 \times (2)^4$$

$$\Rightarrow n \times 2^{n-1} = 5 \times (2)^{5-1}$$

$$\therefore n = 5$$

$$\left[\because \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \right]$$

Q. 15 Evaluate $\lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 7x}$.

Sol. Given,

$$\lim_{x \rightarrow 0} \frac{\frac{\sin 3x}{3x} \cdot 3x}{\frac{\sin 7x}{7x} \cdot 7x} = \frac{\lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \cdot 3x}{\lim_{x \rightarrow 0} \frac{\sin 7x}{7x} \cdot 7x}$$

$$= \frac{3 \cdot \lim_{x \rightarrow 0} \frac{\sin 3x}{3x}}{7 \cdot \lim_{x \rightarrow 0} \frac{\sin 7x}{7x}}$$

$$= \frac{3}{7}$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

$[\because x \rightarrow 0 \Rightarrow (kx \rightarrow 0), \text{ here } k \text{ is real number}]$

Q. 16 Evaluate $\lim_{x \rightarrow 0} \frac{\sin^2 2x}{\sin^2 4x}$.

Sol. Given,
$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin^2 2x}{\sin^2 4x} &= \lim_{x \rightarrow 0} \frac{\sin^2 2x}{[\sin 2(2x)]^2} \\ &= \lim_{x \rightarrow 0} \frac{\sin^2 2x}{(2\sin 2x \cos 2x)^2} \\ &= \lim_{x \rightarrow 0} \frac{\sin^2 2x}{4\sin^2 2x \cos^2 2x} && [\because \sin 2\theta = 2\sin\theta \cos\theta] \\ &= \lim_{x \rightarrow 0} \frac{1}{4\cos^2 2x} = \frac{1}{4} && [\because \cos 0 = 1]\end{aligned}$$

Q. 17 Evaluate $\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x^2}$.

Sol. Given,
$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x^2} &= \lim_{x \rightarrow 0} \frac{1 - 1 + 2\sin^2 x}{x^2} && [\because \cos 2x = 1 - 2\sin^2 x] \\ &= \lim_{x \rightarrow 0} \frac{2\sin^2 x}{x^2} = 2 \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} \\ &= 2 \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 && \left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right] \\ &= 2 \times 1 = 2\end{aligned}$$

Q. 18 Evaluate $\lim_{x \rightarrow 0} \frac{2\sin x - \sin 2x}{x^3}$.

Sol. Given,
$$\begin{aligned}\lim_{x \rightarrow 0} \frac{2\sin x - \sin 2x}{x^3} &= \lim_{x \rightarrow 0} \frac{2\sin x - 2\sin x \cos x}{x^3} && [\because \sin 2x = 2\sin x \cos x] \\ &= \lim_{x \rightarrow 0} \frac{2\sin x(1 - \cos x)}{x^3} \\ &= 2 \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{x^2} \right) \\ &= 2 \cdot 1 \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} && \left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right] \\ &= 2 \lim_{x \rightarrow 0} \frac{1 - 1 + 2\sin^2 \frac{x}{2}}{x^2} = 2 \lim_{x \rightarrow 0} \frac{2\sin^2 \frac{x}{2}}{4 \times \frac{x^2}{4}} \\ &= \frac{2 \cdot 2}{4} \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 = \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 = 1\end{aligned}$$

Q. 19 Evaluate $\lim_{x \rightarrow 0} \frac{1 - \cos mx}{1 - \cos nx}$.

Sol. Given, $\lim_{x \rightarrow 0} \frac{1 - \cos mx}{1 - \cos nx} = \lim_{x \rightarrow 0} \frac{1 - 1 + 2\sin^2 \frac{mx}{2}}{1 - 1 + 2\sin^2 \frac{nx}{2}} \quad \left[\begin{array}{l} \because \cos mx = 1 - 2\sin^2 \frac{mx}{2} \\ \text{and } \sin nx = 1 - 2\sin^2 \frac{nx}{2} \end{array} \right]$

$$= \lim_{x \rightarrow 0} \frac{\sin^2 \frac{mx}{2}}{\sin^2 \frac{nx}{2}} = \lim_{x \rightarrow 0} \frac{\frac{\sin^2 \frac{mx}{2}}{\left(\frac{mx}{2}\right)^2} \cdot \left(\frac{mx}{2}\right)^2}{\frac{\sin^2 \frac{nx}{2}}{\left(\frac{nx}{2}\right)^2} \cdot \left(\frac{nx}{2}\right)^2} = \frac{\lim_{x \rightarrow 0} \left(\frac{\sin \frac{mx}{2}}{\frac{mx}{2}}\right)^2 \cdot m^2 \frac{x^2}{4}}{\lim_{x \rightarrow 0} \left(\frac{\sin \frac{nx}{2}}{\frac{nx}{2}}\right)^2 \cdot n^2 \frac{x^2}{4}}$$

$$= \frac{m^2}{n^2} \cdot \frac{\lim_{x \rightarrow 0} \left(\frac{\sin \frac{mx}{2}}{\frac{mx}{2}}\right)^2}{\lim_{x \rightarrow 0} \left(\frac{\sin \frac{nx}{2}}{\frac{nx}{2}}\right)^2} = \frac{m^2}{n^2} \quad \left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

$$[\because x \rightarrow 0 \Rightarrow kx \rightarrow 0]$$

Q. 20 Evaluate $\lim_{x \rightarrow \pi/3} \frac{\sqrt{1 - \cos 6x}}{\sqrt{2}\left(\frac{\pi}{3} - x\right)}$.

Sol. Given, $\lim_{x \rightarrow \pi/3} \frac{\sqrt{1 - \cos 6x}}{\sqrt{2}\left(\frac{\pi}{3} - x\right)} = \lim_{x \rightarrow \pi/3} \frac{\sqrt{1 - 1 + 2\sin^2 3x}}{\sqrt{2}\left(\frac{\pi}{3} - x\right)} \quad [\because \cos 2x = 1 - 2\sin^2 x]$

$$= \lim_{x \rightarrow \pi/3} \frac{\sqrt{2} \sin 3x}{\sqrt{2}\left(\frac{\pi}{3} - x\right)} = \lim_{x \rightarrow \pi/3} \frac{\sin 3x}{\frac{\pi}{3} - x}$$

$$= \lim_{x \rightarrow \pi/3} \frac{\sin(\pi - 3x)}{\frac{\pi - 3x}{3}} \quad [\because \sin(\pi - \theta) = \sin \theta]$$

$$= 3 \lim_{x \rightarrow \pi/3} \frac{\sin(\pi - 3x)}{(\pi - 3x)} = 3 \times 1 \quad \left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

$$= 3 \quad \left[\because x \rightarrow \frac{\pi}{3} \Rightarrow \left(x - \frac{\pi}{3}\right) \rightarrow 0 \right]$$

Q. 21 Evaluate $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$.

Sol. Given, $\lim_{x \rightarrow \pi/4} \frac{\sqrt{2} \left(\sin x \cdot \frac{1}{\sqrt{2}} - \cos x \cdot \frac{1}{\sqrt{2}} \right)}{\left(x - \frac{\pi}{4} \right)} = \lim_{x \rightarrow \pi/4} \frac{\sqrt{2} \left(\sin x \cos \frac{\pi}{4} - \cos x \cdot \sin \frac{\pi}{4} \right)}{\left(x - \frac{\pi}{4} \right)}$

$$= \lim_{x \rightarrow \pi/4} \frac{\sqrt{2} \left\{ \sin \left(x - \frac{\pi}{4} \right) \right\}}{\left(x - \frac{\pi}{4} \right)}$$

$$= \sqrt{2} \lim_{x \rightarrow \pi/4} \frac{\sin \left(x - \frac{\pi}{4} \right)}{\left(x - \frac{\pi}{4} \right)} = \sqrt{2} \quad \left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

$$\left[\because x \rightarrow \frac{\pi}{4} \Rightarrow \left(x - \frac{\pi}{4} \right) \rightarrow 0 \right]$$

Q. 22 Evaluate $\lim_{x \rightarrow \pi/6} \frac{\sqrt{3} \sin x - \cos x}{x - \frac{\pi}{6}}$.

Sol. Given, $\lim_{x \rightarrow \pi/6} \frac{\sqrt{3} \sin x - \cos x}{x - \frac{\pi}{6}} = \lim_{x \rightarrow \pi/6} \frac{2 \left(\frac{\sqrt{3}}{2} \sin x - \frac{1}{2} \cos x \right)}{\left(x - \frac{\pi}{6} \right)}$

$$= \lim_{x \rightarrow \pi/6} \frac{2 \left(\sin x \cos \frac{\pi}{6} - \cos x \sin \frac{\pi}{6} \right)}{\left(x - \frac{\pi}{6} \right)} = 2 \lim_{x \rightarrow \pi/6} \frac{\sin \left(x - \frac{\pi}{6} \right)}{\left(x - \frac{\pi}{6} \right)}$$

$$[\because \sin A \cos B - \cos A \sin B = \sin (A - B)]$$

$$= 2 \quad \left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

$$\left[\because x \rightarrow \frac{\pi}{6} \Rightarrow \left(x - \frac{\pi}{6} \right) \rightarrow 0 \right]$$

Q. 23 Evaluate $\lim_{x \rightarrow 0} \frac{\sin 2x + 3x}{2x + \tan 3x}$.

Sol. Given, $\lim_{x \rightarrow 0} \frac{\sin 2x + 3x}{2x + \tan 3x} = \lim_{x \rightarrow 0} \frac{\frac{\sin 2x + 3x}{2x} \cdot 2x}{\frac{2x + \tan 3x}{3x} \cdot 3x}$

$$= \lim_{x \rightarrow 0} \frac{\frac{\left(\frac{\sin 2x}{2x} + \frac{3x}{2x} \right) 2x}{\left(\frac{2x}{3x} + \frac{\tan 3x}{3x} \right) 3x}} = \lim_{x \rightarrow 0} \frac{\frac{\sin 2x}{2x} + \frac{3}{2}}{\frac{2}{3} + \frac{\tan 3x}{3x}} \cdot \frac{2}{3}$$

$$\begin{aligned}
&= \frac{2}{3} \lim_{x \rightarrow 0} \frac{\frac{\sin 2x}{2x} + \frac{3}{2}}{\frac{2}{3} + \lim_{x \rightarrow 0} \frac{\tan 3x}{3x}} \\
&= \frac{2}{3} \left(\frac{1 + \frac{3}{2}}{\frac{2}{3} + 1} \right) \quad \left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ and } \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \right] \\
&= \frac{2}{3} \times \frac{\frac{5}{2}}{\frac{5}{3}} = \frac{2}{3} \times \frac{5}{2} \times \frac{3}{5} = 1
\end{aligned}$$

Q. 24 Evaluate $\lim_{x \rightarrow a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$.

Sol. Given, $\lim_{x \rightarrow a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}} = \lim_{x \rightarrow 0} \frac{2 \cos \left(\frac{x+a}{2} \right) \sin \left(\frac{x-a}{2} \right)}{\sqrt{x} - \sqrt{a}}$

$$\left[\because \sin C - \sin D = 2 \cos \frac{C+D}{2} \cdot \sin \frac{C-D}{2} \right]$$

$$\begin{aligned}
&= \lim_{x \rightarrow a} \frac{2 \cos \left(\frac{x+a}{2} \right) \sin \left(\frac{x-a}{2} \right) (\sqrt{x} + \sqrt{a})}{(\sqrt{x} - \sqrt{a}) (\sqrt{x} + \sqrt{a})} \\
&= \lim_{x \rightarrow 0} \frac{2 \cos \left(\frac{x+a}{2} \right) \sin \left(\frac{x-a}{2} \right) (\sqrt{x} + \sqrt{a})}{x - a} \\
&= 2 \lim_{x \rightarrow a} \cos \left(\frac{x+a}{2} \right) (\sqrt{x} + \sqrt{a}) \lim_{x \rightarrow 0} \frac{\sin \left(\frac{x-a}{2} \right)}{2 \left(\frac{x-a}{2} \right)} \\
&= 2 \lim_{x \rightarrow 0} \cos \left(\frac{x+a}{2} \right) (\sqrt{x} + \sqrt{a}) \cdot \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin \left(\frac{x-a}{2} \right)}{\left(\frac{x-a}{2} \right)} \\
&= 2 \cdot \cos \frac{a}{2} \cdot \sqrt{a} \cdot \frac{1}{2} \quad \left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right] \\
&= \sqrt{a} \cos \frac{a}{2}
\end{aligned}$$

Q. 25 Evaluate $\lim_{x \rightarrow \pi/6} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$.

Sol. Given $\lim_{x \rightarrow \pi/6} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2} = \lim_{x \rightarrow \pi/6} \frac{\operatorname{cosec}^2 x - 1 - 3}{\operatorname{cosec} x - 2}$ $[\because \operatorname{cosec}^2 x = 1 + \cot^2 x]$

$$\begin{aligned}
&= \lim_{x \rightarrow \pi/6} \frac{\operatorname{cosec}^2 x - 4}{\operatorname{cosec} x - 2} = \lim_{x \rightarrow \pi/6} \frac{(\operatorname{cosec} x)^2 - (2)^2}{\operatorname{cosec} x - 2} \\
&= \lim_{x \rightarrow \pi/6} \frac{(\operatorname{cosec} x + 2)(\operatorname{cosec} x - 2)}{(\operatorname{cosec} x - 2)} = \lim_{x \rightarrow \pi/6} (\operatorname{cosec} x + 2) \\
&= \operatorname{cosec} \frac{\pi}{6} + 2 = 2 + 2 = 4
\end{aligned}$$

Q. 26 Evaluate $\lim_{x \rightarrow 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x}$.

Sol. Given, $\lim_{x \rightarrow 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x} = \lim_{x \rightarrow 0} \frac{\sqrt{2} - \sqrt{1 + 2\cos^2 \frac{x}{2} - 1}}{\sin^2 x} \quad \left[\because \cos x = 2\cos^2 \frac{x}{2} - 1 \right]$

$$= \lim_{x \rightarrow 0} \frac{\sqrt{2} - \sqrt{2\cos^2 \frac{x}{2}}}{\sin^2 x} \quad \left[\because \sin x = 2\sin \frac{x}{2} \cos \frac{x}{2} \right]$$

$$= \lim_{x \rightarrow 0} \frac{\sqrt{2} \left(1 - \cos \frac{x}{2} \right)}{\sin^2 x} = \lim_{x \rightarrow 0} \frac{\sqrt{2} \left(1 - 1 + 2\sin^2 \frac{x}{4} \right)}{\sin^2 x}$$

$$= \lim_{x \rightarrow 0} \frac{\sqrt{2} \left(2\sin^2 \frac{x}{4} \right)}{\sin^2 x} = \lim_{x \rightarrow 0} 2\sqrt{2} \frac{\sin^2 \frac{x}{4}}{\left(\frac{x}{4} \right)^2} \cdot \frac{\left(\frac{x}{4} \right)^2}{\sin^2 x}$$

$$= 2\sqrt{2} \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{4}}{\frac{x}{4}} \right)^2 \cdot \lim_{x \rightarrow 0} \left(\frac{x}{\sin x} \right)^2 \cdot \frac{1}{16}$$

$$= 2\sqrt{2} \cdot 1 \cdot 1 \cdot \frac{1}{16} = \frac{1}{4\sqrt{2}}$$

Q. 27 Evaluate $\lim_{x \rightarrow 0} \frac{\sin x - 2\sin 3x + \sin 5x}{x}$.

Sol. Given,

$$\lim_{x \rightarrow 0} \frac{\sin x - 2\sin 3x + \sin 5x}{x} = \lim_{x \rightarrow 0} \frac{\sin 5x + \sin x - 2\sin 3x}{x}$$

$$= \lim_{x \rightarrow 0} \frac{2\sin 3x \cos 2x - 2\sin 3x}{x} = \lim_{x \rightarrow 0} \frac{2\sin 3x (\cos 2x - 1)}{x}$$

$$= \lim_{x \rightarrow 0} \frac{2\sin 3x}{\frac{1}{3} \times 3x} (\cos 2x - 1) = 6 \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} (\cos 2x - 1)$$

$$= 6 \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \cdot \lim_{x \rightarrow 0} (\cos 2x - 1) = 6 \times 1 \times 0 = 0$$

Q. 28 If $\lim_{x \rightarrow 1} \frac{x^4 - 1}{x - 1} = \lim_{x \rightarrow k} \frac{x^3 - k^3}{x^2 - k^2}$, then find the value of k .

Sol. Given,

$$\lim_{x \rightarrow 1} \frac{x^4 - 1}{x - 1} = \lim_{x \rightarrow k} \frac{x^3 - k^3}{x^2 - k^2}$$

$$\Rightarrow 4(1)^{4-1} = \lim_{x \rightarrow k} \frac{\frac{x^3 - k^3}{x - k}}{\frac{x^2 - k^2}{x - k}} \quad \left[\because \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \right]$$

$$\Rightarrow 4 = \frac{\lim_{x \rightarrow k} \frac{x^3 - k^3}{x - k}}{\lim_{x \rightarrow k} \frac{x^2 - k^2}{x - k}} \cdot 1$$

$$\Rightarrow 4 = \frac{3k^2}{2k} \Rightarrow 4 = \frac{3}{2}k$$

$$\therefore k = \frac{4 \times 2}{3} = \frac{8}{3}$$

$$\left[\because \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \right]$$

Differentiate each of the functions w.r.t. x in following questions

Q. 29 $\frac{x^4 + x^3 + x^2 + 1}{x}$

Sol.
$$\begin{aligned} \frac{d}{dx} \left(\frac{x^4 + x^3 + x^2 + 1}{x} \right) &= \frac{d}{dx} \left(x^3 + x^2 + x + \frac{1}{x} \right) \\ &= \frac{d}{dx} x^3 + \frac{d}{dx} x^2 + \frac{d}{dx} x + \frac{d}{dx} \left(\frac{1}{x} \right) \\ &= 3x^2 + 2x + 1 + \left(-\frac{1}{x^2} \right) \\ &= 3x^2 + 2x + 1 - \frac{1}{x^2} \\ &= \frac{3x^4 + 2x^3 + x^2 - 1}{x^2} \end{aligned}$$

Q. 30 $\left(x + \frac{1}{x} \right)^3$

Sol. Let $y = \left(x + \frac{1}{x} \right)^3$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} \left(x + \frac{1}{x} \right)^3 = 3 \left(x + \frac{1}{x} \right)^{3-1} \frac{d}{dx} \left(x + \frac{1}{x} \right) \quad \text{[by chain rule]}$$

$$= 3 \left(x + \frac{1}{x} \right)^2 \left(1 - \frac{1}{x^2} \right)$$

$$= 3x^2 - \frac{3}{x^2} - \frac{3}{x^4} + 3$$

Q. 31 $(3x + 5)(1 + \tan x)$

Sol. Let $y = (3x + 5)(1 + \tan x)$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} [(3x + 5)(1 + \tan x)]$$

$$= (3x + 5) \frac{d}{dx} (1 + \tan x) + (1 + \tan x) \frac{d}{dx} (3x + 5) \quad \text{[by product rule]}$$

$$= (3x + 5)(\sec^2 x) + (1 + \tan x) \cdot 3$$

$$= (3x + 5) \sec^2 x + 3(1 + \tan x)$$

$$= 3x \sec^2 x + 5 \sec^2 x + 3 \tan x + 3$$

Q. 32 $(\sec x - 1)(\sec x + 1)$ **Sol.** Let

$$y = (\sec x - 1)(\sec x + 1)$$

$$y = (\sec^2 x - 1) \quad [\because (a + b)(a - b) = a^2 - b^2]$$

$$= \tan^2 x$$

 $\therefore$

$$\frac{dy}{dx} = 2 \tan x \cdot \frac{d}{dx} \tan x$$

$$= 2 \tan x \cdot \sec^2 x \quad [\text{by chain rule}]$$

Q. 33 $\frac{3x + 4}{5x^2 - 7x + 9}$ **Sol.** Let

$$y = \frac{3x + 4}{5x^2 - 7x + 9}$$

$$\therefore \frac{dy}{dx} = \frac{(5x^2 - 7x + 9) \frac{d}{dx} (3x + 4) - (3x + 4) \frac{d}{dx} (5x^2 - 7x + 9)}{(5x^2 - 7x + 9)^2} \quad [\text{by quotient rule}]$$

$$= \frac{(5x^2 - 7x + 9) \cdot 3 - (3x + 4)(10x - 7)}{(5x^2 - 7x + 9)^2}$$

$$= \frac{15x^2 - 21x + 27 - 30x^2 + 21x - 40x + 28}{(5x^2 - 7x + 9)^2}$$

$$= \frac{-15x^2 - 40x + 55}{(5x^2 - 7x + 9)^2}$$

$$= \frac{55 - 15x^2 - 40x}{(5x^2 - 7x + 9)^2}$$

Q. 34 $\frac{x^5 - \cos x}{\sin x}$ **Sol.** Let

$$y = \frac{x^5 - \cos x}{\sin x}$$

 $\therefore$

$$\frac{dy}{dx} = \frac{\sin x \frac{d}{dx} (x^5 - \cos x) - (x^5 - \cos x) \frac{d}{dx} \sin x}{(\sin x)^2} \quad [\text{by quotient rule}]$$

$$= \frac{\sin x (5x^4 + \sin x) - (x^5 - \cos x) \cos x}{\sin^2 x}$$

$$= \frac{5x^4 \sin x + \sin^2 x - x^5 \cos x + \cos^2 x}{\sin^2 x}$$

$$= \frac{5x^4 \sin x - x^5 \cos x + \sin^2 x + \cos^2 x}{\sin^2 x}$$

$$= \frac{5x^4 \sin x - x^5 \cos x + 1}{\sin^2 x}$$

Q. 35 $\frac{x^2 \cos \frac{\pi}{4}}{\sin x}$

Sol. Let

$$y = \frac{x^2 \cos \frac{\pi}{4}}{\sin x} = \frac{x^2}{\sqrt{2} \sin x}$$

$$y = \frac{1}{\sqrt{2}} \cdot \frac{x^2}{\sin x}$$

$\therefore$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sqrt{2}} \left[\frac{\sin x \cdot \frac{d}{dx} x^2 - x^2 \frac{d}{dx} \sin x}{\sin^2 x} \right] && \text{[by quotient rule]} \\ &= \frac{1}{\sqrt{2}} \left[\frac{\sin x \cdot 2x - x^2 \cdot \cos x}{\sin^2 x} \right] \\ &= \frac{1}{\sqrt{2}} \cdot \frac{2x \sin x - x^2 \cos x}{\sin^2 x} \\ &= \frac{x}{\sqrt{2}} [2 \operatorname{cosec} x - x \cot x \operatorname{cosec} x] \\ &= \frac{x}{\sqrt{2}} \operatorname{cosec} [2 - x \cot x] \end{aligned}$$

Q. 36 $(ax^2 + \cot x)(p + q \cos x)$

Sol. Let $y = (ax^2 + \cot x)(p + q \cos x)$

$$\begin{aligned} \therefore \frac{dy}{dx} &= (ax^2 + \cot x) \frac{d}{dx} (p + q \cos x) + (p + q \cos x) \frac{d}{dx} (ax^2 + \cot x) && \text{[by product rule]} \\ &= (ax^2 + \cot x)(-q \sin x) + (p + q \cos x)(2ax - \operatorname{cosec}^2 x) \\ &= -q \sin x(ax^2 + \cot x) + (p + q \cos x)(2ax - \operatorname{cosec}^2 x) \end{aligned}$$

Q. 37 $\frac{a + b \sin x}{c + d \cos x}$

Sol. Let

$$y = \frac{a + b \sin x}{c + d \cos x}$$

$\therefore$

$$\begin{aligned} \frac{dy}{dx} &= \frac{(c + d \cos x) \frac{d}{dx} (a + b \sin x) - (a + b \sin x) \frac{d}{dx} (c + d \cos x)}{(c + d \cos x)^2} && \text{[by quotient rule]} \\ &= \frac{(c + d \cos x)(b \cos x) - (a + b \sin x)(-d \sin x)}{(c + d \cos x)^2} \\ &= \frac{bccosx + bd \cos^2 x + ad \sin x + bd \sin^2 x}{(c + d \cos x)^2} \\ &= \frac{bccosx + ad \sin x + bd(\cos^2 x + \sin^2 x)}{(c + d \cos x)^2} \\ &= \frac{bccosx + ad \sin x + bd}{(c + d \cos x)^2} \end{aligned}$$

Q. 38 $(\sin x + \cos x)^2$

Sol. Let $y = (\sin x + \cos x)^2$
 $\therefore \frac{dy}{dx} = 2(\sin x + \cos x)(\cos x - \sin x)$
 $= 2(\cos^2 x - \sin^2 x)$ [by chain rule]
 $= 2\cos 2x$ [$\because \cos 2x = \cos^2 x - \sin^2 x$]

Q. 39 $(2x - 7)^2(3x + 5)^3$

Sol. Let $y = (2x - 7)^2(3x + 5)^3$
 $\frac{dy}{dx} = (2x - 7)^2 \frac{d}{dx}(3x + 5)^3 + (3x + 5)^3 \frac{d}{dx}(2x - 7)^2$ [by product rule]
 $= (2x - 7)^2 (3)(3x + 5)^2 (3) + (3x + 5)^3 2(2x - 7)(2)$ [by chain rule]
 $= 9(2x - 7)^2(3x + 5)^2 + 4(3x + 5)^3(2x - 7)$
 $= (2x - 7)(3x + 5)^2 [9(2x - 7) + 4(3x + 5)]$
 $= (2x - 7)(3x + 5)^2(18x - 63 + 12x + 20)$
 $= (2x - 7)(3x + 5)^2(30x - 43)$

Q. 40 $x^2 \sin x + \cos 2x$

Sol. Let $y = x^2 \sin x + \cos 2x$
 $\frac{dy}{dx} = \frac{d}{dx}(x^2 \sin x) + \frac{d}{dx} \cos 2x$
 $= x^2 \cdot \cos x + \sin x \cdot 2x + (-\sin 2x) \cdot 2$ [by product rule]
 $= x^2 \cos x + 2x \sin x - 2 \sin 2x$ [by chain rule]

Q. 41 $\sin^3 x \cos^3 x$

Sol. Let $y = \sin^3 x \cos^3 x$
 $\therefore \frac{dy}{dx} = \sin^3 x \cdot \frac{d}{dx} \cos^3 x + \cos^3 x \cdot \frac{d}{dx} \sin^3 x$ [by product rule]
 $= \sin^3 x \cdot 3\cos^2 x(-\sin x) + \cos^3 x \cdot 3\sin^2 x \cos x$ [by chain rule]
 $= -3\cos^2 x \sin^4 x + 3\sin^2 x \cos^4 x$
 $= 3\sin^2 x \cos^2 x(\cos^2 x - \sin^2 x)$
 $= 3\sin^2 x \cos^2 x \cos 2x$
 $= \frac{3}{4}(2\sin x \cos x)^2 \cos 2x$
 $= \frac{3}{4} \sin^2 2x \cos 2x$

Q. 42 $\frac{1}{ax^2 + bx + c}$

Sol. Let

$$y = \frac{1}{ax^2 + bx + c} = (ax^2 + bx + c)^{-1}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= -(ax^2 + bx + c)^{-2}(2ax + b) && [\text{by chain rule}] \\ &= \frac{-(2ax + b)}{(ax^2 + bx + c)^2} \end{aligned}$$

Long Answer Type Questions

Differentiate each of the functions with respect to x in following questions using first principle.

Q. 43 $\cos(x^2 + 1)$

Sol. Let

$$f(x) = \cos(x^2 + 1) \text{ and } f(x + h) = \cos\{(x + h)^2 + 1\}$$

$$\begin{aligned} \therefore \frac{d}{dx} f(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos\{(x + h)^2 + 1\} - \cos(x^2 + 1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-2\sin\left\{\frac{(x + h)^2 + 1 + x^2 + 1}{2}\right\} \sin\left\{\frac{(x + h)^2 + 1 - x^2 - 1}{2}\right\}}{h} \\ &\quad \left[\because \cos C - \cos D = -2\sin\frac{C + D}{2} \cdot \sin\frac{C - D}{2} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[-2\sin\left\{\frac{(x + h)^2 + x^2 + 2}{2}\right\} \sin\left\{\frac{(x + h)^2 - x^2}{2}\right\} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[-2\sin\left\{\frac{(x + h)^2 + x^2 + 2}{2}\right\} \sin\left\{\frac{x^2 + h^2 + 2xh - x^2}{2}\right\} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[-2\sin\left\{\frac{(x + h)^2 + x^2 + 2}{2}\right\} \sin\left\{\frac{h^2 + 2hx}{2}\right\} \right] \\ &= -2 \lim_{h \rightarrow 0} \sin\left\{\frac{(x + h)^2 + x^2 + 2}{2}\right\} \lim_{h \rightarrow 0} \left\{ \frac{\sin h \left(\frac{h + 2x}{2}\right)}{h \left(\frac{h + 2x}{2}\right)} \right\} \times \left(\frac{h + 2x}{2}\right) \\ &= -2 \lim_{h \rightarrow 0} \sin\left\{\frac{(x + h)^2 + x^2 + 2}{2}\right\} \lim_{h \rightarrow 0} \left(\frac{h + 2x}{2}\right) \quad \left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right] \\ &= -2x \sin(x^2 + 1) && [\because x \rightarrow 0 \Rightarrow hx \rightarrow 0] \end{aligned}$$

Q. 44 $\frac{ax + b}{cx + d}$

Sol. Let

$$f(x) = \frac{ax + b}{cx + d}$$

$$f(x + h) = \frac{a(x + h) + b}{c(x + h) + d}$$

$\therefore$

$$\begin{aligned} \frac{d}{dx} f(x) &= \lim_{h \rightarrow 0} \frac{1}{h} [f(x + h) - f(x)] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{a(x + h) + b}{c(x + h) + d} - \frac{ax + b}{cx + d} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{ax + b + ah}{c(x + h) + d} - \frac{ax + b}{cx + d} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{(ax + ah + b)(cx + d) - (ax + b)\{c(x + h) + d\}}{\{c(x + h) + d\}(cx + d)} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{(ax + ah + b)(cx + d) - (ax + b)(cx + ch + d)}{\{c(x + h) + d\}(cx + d)} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{acx^2 + achx + bcx + adx + adh + bd - \{acx^2 + achx + adx + bcx + bch + bd\}}{\{c(x + h) + d\}(cx + d)} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{acx^2 + achx + bcx + adx + adh + bd - acx^2 - achx - adx - bcx - bch - bd}{\{c(x + h) + d\}(cx + d)} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{adh - bch}{\{c(x + h) + d\}(cx + d)} \right] \\ &= \lim_{h \rightarrow 0} \frac{ac - bd}{\{c(x + h) + d\}(cx + d)} \\ &= \frac{ac - bd}{(cx + d)^2} \end{aligned}$$

Q. 45 $x^{2/3}$

Sol. Let

$$f(x) = x^{2/3}$$

$$f(x + h) = (x + h)^{2/3}$$

Now,

$$\begin{aligned} \frac{d}{dx} f(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} [(x + h)^{2/3} - x^{2/3}] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[x^{2/3} \left(1 + \frac{h}{x} \right)^{2/3} - x^{2/3} \right] \end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[x^{2/3} \left(1 + \frac{h}{x} \cdot \frac{2}{3} + \frac{2}{3} \left(\frac{2}{3} - 1 \right) \frac{h^2}{x^2} + \dots \right) - 1 \right] \\
&\quad \left[\because (1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[x^{2/3} \left(\frac{2}{3} \cdot \frac{h}{x} - \frac{2}{9} \cdot \frac{h^2}{x^2} + \dots \right) \right] \\
&= \lim_{h \rightarrow 0} \frac{x^{2/3}}{h} \cdot \frac{2}{3} \frac{h}{x} \left(1 - \frac{1}{3} \cdot \frac{h}{x} + \dots \right) \\
&= \frac{2}{3} x^{2/3-1} = \frac{2}{3} x^{-1/3}
\end{aligned}$$

Alternate Method

Let $f(x) = x^{2/3}$

$$f(x+h) = (x+h)^{2/3}$$

$\therefore \frac{d}{dx} f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{(h \rightarrow 0)} \left[\frac{(x+h)^{2/3} - x^{2/3}}{h} \right] = \lim_{(x+h) \rightarrow x} \left[\frac{(x+h)^{2/3} - x^{2/3}}{(x+h) - x} \right]$$

$$= \frac{2}{3} (x)^{2/3-1} \quad \left[\because \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n a^{n-1} \right]$$

$$= \frac{2}{3} x^{-1/3}$$

Q. 46 $x \cos x$

Sol. Let $f(x) = x \cos x$

$\therefore f(x+h) = (x+h) \cos(x+h)$

$\therefore \frac{d}{dx} f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \rightarrow 0} \frac{1}{h} [(x+h) \cos(x+h) - x \cos x]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} [x \cos(x+h) + h \cos(x+h) - x \cos x]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} [x \{\cos(x+h) - \cos x\} + h \cos(x+h)]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[x \left\{ -2 \sin \left(\frac{2x+h}{2} \right) \sin \frac{h}{2} \right\} + h \cos(x+h) \right]$$

$$= \lim_{h \rightarrow 0} \left[-2x \sin \left(x + \frac{h}{2} \right) \frac{\sin \frac{h}{2}}{h} + \cos(x+h) \right]$$

$$\quad \left[\because \cos C - \cos D = -2 \sin \frac{C+D}{2} \cdot \sin \frac{C-D}{2} \right]$$

$$= -2 \lim_{h \rightarrow 0} x \sin \left(x + \frac{h}{2} \right) \lim_{h \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \cdot \frac{1}{2} + \lim_{h \rightarrow 0} \cos(x+h)$$

$$= -2 \cdot \frac{1}{2} x \sin x + \cos x$$

$$= \cos x - x \sin x$$

Evaluate each of the following limits in following questions

Q. 47 $\lim_{y \rightarrow 0} \frac{(x+y)\sec(x+y) - x\sec x}{y}$

Sol. Given, $\lim_{y \rightarrow 0} \frac{(x+y)\sec(x+y) - x\sec x}{y}$

$$\begin{aligned}
 &= \lim_{y \rightarrow 0} \frac{\frac{x+y}{\cos(x+y)} - \frac{x}{\cos x}}{y} \\
 &= \lim_{y \rightarrow 0} \frac{(x+y)\cos x - x\cos(x+y)}{y\cos x\cos(x+y)} \\
 &= \lim_{y \rightarrow 0} \left[\frac{x\cos x + y\cos x - x\cos(x+y)}{y\cos x\cos(x+y)} \right] \\
 &= \lim_{y \rightarrow 0} \left[\frac{x\cos x - x\cos(x+y) + y\cos x}{y\cos x\cos(x+y)} \right] \\
 &= \lim_{y \rightarrow 0} \frac{x\{\cos x - \cos(x+y)\} + y\cos x}{y\cos x\cos(x+y)} \\
 &= \lim_{y \rightarrow 0} \frac{x\left[-2\sin\left(x + \frac{y}{2}\right)\sin\left(\frac{-y}{2}\right)\right] + y\cos x}{y\cos x\cos(x+y)} \\
 &\quad \left[\because \cos C - \cos D = -2 \sin \frac{C+D}{2} \cdot \sin \frac{C-D}{2} \right] \\
 &= \lim_{y \rightarrow 0} \left[\frac{x\left\{2\sin\left(x + \frac{y}{2}\right)\sin\frac{y}{2}\right\} + y\cos x}{y\cos x\cos(x+y)} \right] \\
 &= \lim_{y \rightarrow 0} \frac{2x\sin\left(x + \frac{y}{2}\right)}{\cos x\cos(x+y)} \cdot \lim_{y \rightarrow 0} \frac{\sin\frac{y}{2}}{\frac{y}{2}} \cdot \frac{1}{2} + \lim_{y \rightarrow 0} \sec(x+y) \\
 &\quad \left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ and } x \rightarrow 0 \Rightarrow kx \rightarrow 0 \right] \\
 &= \lim_{y \rightarrow 0} \frac{2x\sin\left(x + \frac{y}{2}\right)}{\cos x\cos(x+y)} \cdot \frac{1}{2} + \lim_{y \rightarrow 0} \sec(x+y) \\
 &= \frac{2x\sin x}{\cos x\cos x} \cdot \frac{1}{2} + \sec x \\
 &= x\tan x\sec x + \sec x \\
 &= \sec x(x\tan x + 1)
 \end{aligned}$$

Q. 48 $\lim_{x \rightarrow 0} \frac{\sin(\alpha + \beta)x + \sin(\alpha - \beta)x + \sin 2\alpha x}{\cos 2\beta x - \cos 2\alpha x} \cdot x$

Sol. Given, $\lim_{x \rightarrow 0} \frac{[\sin(\alpha + \beta)x + \sin(\alpha - \beta)x + \sin 2\alpha x]}{\cos 2\beta x - \cos 2\alpha x} \cdot x$

$$= \lim_{x \rightarrow 0} \frac{[2\sin \alpha x \cdot \cos \beta x + \sin 2\alpha x]}{\cos 2\beta x - \cos 2\alpha x} \cdot x \quad \left[\because \sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2} \right]$$

$$= \lim_{x \rightarrow 0} \frac{[2\sin \alpha x \cos \beta x + \sin 2\alpha x]x}{2\sin(\alpha + \beta)x \sin(\alpha - \beta)x} \quad \left[\because \cos C - \cos D = 2 \sin \frac{C+D}{2} \cdot \sin \frac{D-C}{2} \right]$$

$$= \lim_{x \rightarrow 0} \frac{[2\sin \alpha x \cos \beta x + 2\sin \alpha x \cos \alpha x]x}{2\sin(\alpha + \beta)x \sin(\alpha - \beta)x}$$

$$= \lim_{x \rightarrow 0} \frac{2\sin \alpha x [\cos \beta x + \cos \alpha x]x}{2\sin(\alpha + \beta)x \sin(\alpha - \beta)x}$$

$$= \lim_{x \rightarrow 0} \frac{\sin \alpha x \left[2\cos\left(\frac{\alpha + \beta}{2}\right)x \cos\left(\frac{\alpha - \beta}{2}\right)x \right]x}{2\sin\left(\frac{\alpha + \beta}{2}\right)x \cos\left(\frac{\alpha + \beta}{2}\right)x \cdot 2\sin\left(\frac{\alpha - \beta}{2}\right)x \cos\left(\frac{\alpha - \beta}{2}\right)x}$$

$$\left[\because \cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2} \text{ and } \sin 2\theta = 2 \sin \theta \cos \theta \right]$$

$$= \lim_{x \rightarrow 0} \frac{\sin \alpha x \cdot x}{2\sin\left(\frac{\alpha + \beta}{2}\right)x \sin\left(\frac{\alpha - \beta}{2}\right)x}$$

$$= \frac{1}{2} \lim_{x \rightarrow 0} \frac{\frac{\sin \alpha x}{\alpha x} \cdot x \cdot (\alpha x)}{\sin\left(\frac{\alpha + \beta}{2}\right)x \cdot \sin\left(\frac{\alpha - \beta}{2}\right)x \cdot \left(\frac{\alpha + \beta}{2}\right)x \cdot \left(\frac{\alpha - \beta}{2}\right)x}$$

$$= \frac{\frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin \alpha x}{\alpha x} \cdot \alpha x^2}{\lim_{x \rightarrow 0} \sin\left(\frac{\alpha + \beta}{2}\right)x \lim_{x \rightarrow 0} \sin\left(\frac{\alpha - \beta}{2}\right)x \cdot \left(\frac{\alpha^2 - \beta^2}{4}\right)x^2}$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ and } x \rightarrow 0 \Rightarrow kx \rightarrow 0 \right]$$

$$= \frac{1}{2} \cdot \frac{\alpha \cdot 4}{\alpha^2 - \beta^2} \left[\frac{\lim_{x \rightarrow 0} \frac{\sin \alpha x}{\alpha x}}{\lim_{x \rightarrow 0} \sin\left(\frac{\alpha + \beta}{2}\right)x \lim_{x \rightarrow 0} \sin\left(\frac{\alpha - \beta}{2}\right)x} \right]$$

$$= \frac{1}{2} \cdot \frac{4\alpha}{\alpha^2 - \beta^2} = \frac{2\alpha}{\alpha^2 - \beta^2}$$

Q. 49 $\lim_{x \rightarrow \pi/4} \frac{\tan^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)}$

Sol. Given, $\lim_{x \rightarrow \pi/4} \frac{\tan^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)}$ $\left[\frac{0}{0} \text{ form}\right]$

$$\begin{aligned}
 &= \lim_{x \rightarrow \pi/4} \frac{\tan x (\tan^2 x - 1)}{\cos\left(x + \frac{\pi}{4}\right)} = \lim_{x \rightarrow \pi/4} \tan x \cdot \lim_{x \rightarrow \pi/4} \left(\frac{1 - \tan^2 x}{\cos\left(x + \frac{\pi}{4}\right)} \right) \\
 &= -1 \times \lim_{x \rightarrow \pi/4} \frac{(1 + \tan x)(1 - \tan x)}{\cos\left(x + \frac{\pi}{4}\right)} \quad [\because a^2 - b^2 = (a + b)(a - b)] \\
 &= - \lim_{x \rightarrow \pi/4} (1 + \tan x) \lim_{x \rightarrow \pi/4} \left[\frac{\cos x - \sin x}{\cos x \cdot \cos\left(x + \frac{\pi}{4}\right)} \right] \\
 &= -(1 + 1) \times \lim_{x \rightarrow \pi/4} \frac{\sqrt{2} \left[\frac{1}{\sqrt{2}} \cdot \cos x - \frac{1}{\sqrt{2}} \cdot \sin x \right]}{\cos x \cdot \cos\left(x + \frac{\pi}{4}\right)} = -2\sqrt{2} \lim_{x \rightarrow \pi/4} \left[\frac{\cos \frac{\pi}{4} \cdot \cos x - \sin \frac{\pi}{4} \cdot \sin x}{\cos x \cdot \cos\left(x + \frac{\pi}{4}\right)} \right] \\
 &\quad [\because \cos A \cdot \cos B - \sin A \sin B = \cos(A + B)] \\
 &= -2\sqrt{2} \lim_{x \rightarrow \pi/4} \frac{\cos\left(x + \frac{\pi}{4}\right)}{\cos x \cdot \cos\left(x + \frac{\pi}{4}\right)} = -2\sqrt{2} \times \frac{1}{\frac{1}{\sqrt{2}}} = -2\sqrt{2} \times \sqrt{2} = -4
 \end{aligned}$$

Q. 50 $\lim_{x \rightarrow \pi} \frac{1 - \sin \frac{x}{2}}{\cos \frac{x}{2} \left(\cos \frac{x}{4} - \sin \frac{x}{4} \right)}$

Sol. Given, $\lim_{x \rightarrow \pi} \frac{1 - \sin \frac{x}{2}}{\cos \frac{x}{2} \left(\cos \frac{x}{4} - \sin \frac{x}{4} \right)}$

$$\begin{aligned}
 &= \lim_{x \rightarrow \pi} \frac{\cos^2 \frac{x}{4} + \sin^2 \frac{x}{4} - 2 \cdot \sin \frac{x}{4} \cdot \cos \frac{x}{4}}{\cos \frac{x}{2} \cdot \left(\cos \frac{x}{4} - \sin \frac{x}{4} \right)} \quad [\because \sin^2 \theta + \cos^2 \theta = 1, \sin 2\theta = 2 \sin \theta \cos \theta] \\
 &= \lim_{x \rightarrow \pi} \frac{\left(\cos \frac{x}{4} - \sin \frac{x}{4} \right)^2}{\left(\cos^2 \frac{x}{4} - \sin^2 \frac{x}{4} \right) \left(\cos \frac{x}{4} - \sin \frac{x}{4} \right)} \quad [\because \cos^2 2\theta = \cos^2 \theta - \sin^2 \theta]
 \end{aligned}$$

$$= \lim_{x \rightarrow \pi} \frac{\left(\cos \frac{x}{4} - \sin \frac{x}{4} \right)}{\left(\cos \frac{x}{4} + \sin \frac{x}{4} \right) \left(\cos \frac{x}{4} - \sin \frac{x}{4} \right)}$$

$$\lim_{x \rightarrow \pi} \frac{1}{\cos \frac{x}{4} + \sin \frac{x}{4}} = \frac{1}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$[\because a^2 - b^2 = (a + b)(a - b)]$$

Q. 51 Show that $\lim_{x \rightarrow \pi/4} \frac{|x - 4|}{x - 4}$ does not exist,

Sol. Given,

$$\lim_{x \rightarrow \pi/4} \frac{|x - 4|}{x - 4}$$

$$\text{LHL} = \lim_{x \rightarrow \frac{\pi}{4}^-} \frac{-(x - 4)}{x - 4}$$

$$= -1$$

$$[\because |x - 4| = -(x - 4), x < 4]$$

$$\text{RHL} = \lim_{x \rightarrow \frac{\pi}{4}^+} \frac{(x - 4)}{x - 4} = 1$$

$$[\because |x - 4| = (x - 4), x > 4]$$

$\therefore$

$$\text{LHL} \neq \text{RHL}$$

So, limit does not exist.

Q. 52 If $f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{when } x \neq \frac{\pi}{2} \\ 3, & \text{when } x = \frac{\pi}{2} \end{cases}$ and $\lim_{x \rightarrow \pi/2} f(x) = f\left(\frac{\pi}{2}\right)$, then find the

value of k .

Sol. Given, $f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & x \neq \frac{\pi}{2} \\ 3, & x = \frac{\pi}{2} \end{cases}$

$$\therefore \text{LHL} = \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{k \cos x}{\pi - 2x} = \lim_{h \rightarrow 0} \frac{k \cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)}$$

$$= \lim_{h \rightarrow 0} \frac{k \sin h}{\pi - \pi + 2h} = \lim_{h \rightarrow 0} \frac{k \sin h}{2h}$$

$$= \frac{k}{2} \lim_{h \rightarrow 0} \frac{\sin h}{h} = \frac{k}{2} \cdot 1 = \frac{k}{2}$$

$$\left[\because \lim_{h \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

$$\text{RHL} = \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{k \cos x}{\pi - 2x} = \lim_{x \rightarrow \frac{\pi}{2}} + \frac{k \cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)}$$

$$= \lim_{h \rightarrow 0} \frac{-k \sin h}{\pi - \pi - 2h} = \lim_{h \rightarrow 0} \frac{k \sin h}{2h} = \frac{k}{2} \lim_{h \rightarrow 0} \frac{\sin h}{2h} = \frac{k}{2} \text{ and } f\left(\frac{\pi}{2}\right) = 3$$

It is given that,

$$\lim_{x \rightarrow \pi/2} f(x) = f\left(\frac{\pi}{2}\right) \Rightarrow \frac{k}{2} = 3$$

$\therefore$

$$k = 6$$

Q. 53 If $f(x) = \begin{cases} x+2, & x \leq -1 \\ cx^2, & x > -1 \end{cases}$, then find c when $\lim_{x \rightarrow -1} f(x)$ exists.

Sol. Given,

$$f(x) = \begin{cases} x+2, & x \leq -1 \\ cx^2, & x > -1 \end{cases}$$

$$\text{LHL} = \lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} (x+2)$$

$$= \lim_{h \rightarrow 0} (-1-h+2) = \lim_{h \rightarrow 0} (1-h) = 1$$

$$\text{RHL} = \lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} cx^2 = \lim_{h \rightarrow 0} c(-1+h)^2$$

$$\therefore \quad \quad \quad = C$$

If $\lim_{x \rightarrow -1} f(x)$ exist, then LHL = RHL

$$\therefore \quad \quad \quad C = 1$$

Objective Type Questions

Q. 54 $\lim_{x \rightarrow \pi} \frac{\sin x}{x - \pi}$ is equal to

(a) 1

(b) 2

(c) -1

(d) -2

Sol. (c) Given, $\lim_{x \rightarrow \pi} \frac{\sin x}{x - \pi} = \lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{\pi - (\pi - x)}$

$$[\because \sin \theta = \sin(\pi - \theta)]$$

$$= - \lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{(\pi - x)} = -1 \quad \left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ and } \pi - x \rightarrow 0 \Rightarrow x \rightarrow \pi \right]$$

Q. 55 $\lim_{x \rightarrow 0} \frac{x^2 \cos x}{1 - \cos x}$ is equal to

(a) 2

(b) $\frac{3}{2}$

(c) $\frac{-3}{2}$

(d) 1

Sol. (a) Given, $\lim_{x \rightarrow 0} \frac{x^2 \cos x}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{x^2 \cos x}{2 \sin^2 \frac{x}{2}}$

$$\left[\because 1 - \cos x = 2 \sin^2 \frac{x}{2} \right]$$

$$= 2 \lim_{x \rightarrow 0} \frac{\left(\frac{x}{2}\right)^2}{\sin^2 \frac{x}{2}} \cdot \lim_{x \rightarrow 0} \cos x = 2 \cdot 1 = 2$$

Q. 56 $\lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{x}$ is equal to

- (a) n (b) 1
(c) $-n$ (d) 0

Sol. (a) Given, $\lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{x} = \lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{(1+x) - 1} = \lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{(1+x) - 1}$

$$= \lim_{x \rightarrow 0} \frac{(1+x)^n - 1^n}{(1+x) - 1} = \lim_{(1+x) \rightarrow 1} \frac{(1+x)^n - 1^n}{(1+x) - 1}$$

$$= n \cdot (1)^{n-1} = n \quad \left[\because \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \right]$$

Q. 57 $\lim_{x \rightarrow 1} \frac{x^m - 1}{x^n - 1}$ is equal to

- (a) 1 (b) $\frac{m}{n}$
(c) $-\frac{m}{n}$ (d) $\frac{m^2}{n^2}$

Sol. (b) Given, $\lim_{x \rightarrow 1} \frac{x^m - 1}{x^n - 1} = \lim_{x \rightarrow 1} \frac{\frac{x^m - 1}{x - 1}}{\frac{x^n - 1}{x - 1}} = \frac{\lim_{x \rightarrow 1} \frac{x^m - 1^m}{x - 1}}{\lim_{x \rightarrow 1} \frac{x^n - 1^n}{x - 1}}$

$$= \frac{m(1)^{m-1}}{n(1)^{n-1}} = \frac{m}{n} \quad \left[\because \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \right]$$

Q. 58 $\lim_{\theta \rightarrow 0} \frac{1 - \cos 4\theta}{1 - \cos 6\theta}$ is equal to

- (a) $\frac{4}{9}$ (b) $\frac{1}{2}$
(c) $\frac{-1}{2}$ (d) -1

Sol. (a) Given, $\lim_{\theta \rightarrow 0} \frac{1 - \cos 4\theta}{1 - \cos 6\theta} = \lim_{\theta \rightarrow 0} \frac{2\sin^2 2\theta}{2\sin^2 3\theta}$ [$\because 1 - \cos 2\theta = 2\sin^2 \theta$]

$$= \frac{\lim_{\theta \rightarrow 0} \frac{\sin^2 2\theta}{(2\theta)^2} \cdot (2\theta)^2}{\lim_{\theta \rightarrow 0} \frac{\sin^2 3\theta}{(3\theta)^2} \cdot (3\theta)^2} = \frac{4}{9} \cdot \frac{\lim_{\theta \rightarrow 0} \left(\frac{\sin 2\theta}{2\theta} \right)^2}{\lim_{\theta \rightarrow 0} \left(\frac{\sin 3\theta}{3\theta} \right)^2}$$

$$= \frac{4}{9} \quad \left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ and } x \rightarrow 0 \Rightarrow kx \rightarrow 0 \right]$$

Q. 59 $\lim_{x \rightarrow 0} \frac{\operatorname{cosec} x - \cot x}{x}$ is equal to

(a) $-\frac{1}{2}$

(b) 1

(c) $\frac{1}{2}$

(d) 1

Sol. (c) Given,

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\operatorname{cosec} x - \cot x}{x} \\ &= \lim_{x \rightarrow 0} \frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{x} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x \cdot \sin x} \\ &= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{x \cdot 2 \sin \frac{x}{2} \cos \frac{x}{2}} = \lim_{x \rightarrow 0} \frac{\tan \frac{x}{2}}{x} \\ &= \lim_{x \rightarrow 0} \frac{\tan \frac{x}{2}}{\frac{x}{2}} \cdot \frac{1}{2} = \frac{1}{2} \end{aligned}$$

$$\left[\because \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1 \right]$$

Q. 60 $\lim_{x \rightarrow 0} \frac{\sin x}{\sqrt{x+1} - \sqrt{1-x}}$ is equal to

(a) 2

(b) 0

(c) 1

(d) -1

Sol. (c) Given,

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\sin x}{\sqrt{x+1} - \sqrt{1-x}} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{\sqrt{x+1} - \sqrt{1-x}} \cdot \frac{\sqrt{x+1} + \sqrt{1-x}}{\sqrt{x+1} + \sqrt{1-x}} \\ &= \lim_{x \rightarrow 0} \frac{\sin x (\sqrt{x+1} + \sqrt{1-x})}{(x+1) - (1-x)} \\ &= \lim_{x \rightarrow 0} \frac{\sin x (\sqrt{x+1} + \sqrt{1-x})}{x+1-1+x} \\ &= \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin x}{x} \lim_{x \rightarrow 0} (\sqrt{x+1} + \sqrt{1-x}) \\ &= \frac{1}{2} \cdot 1 \cdot 2 = 1 \end{aligned}$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

Q. 61 $\lim_{x \rightarrow \pi/4} \frac{\sec^2 x - 2}{\tan x - 1}$ is

(a) 3

(b) 1

(c) 0

(d) 2

Sol. (d) Given,

$$\begin{aligned} & \lim_{x \rightarrow \pi/4} \frac{\sec^2 x - 2}{\tan x - 1} \\ &= \lim_{x \rightarrow \pi/4} \frac{1 + \tan^2 x - 2}{\tan x - 1} = \lim_{x \rightarrow \pi/4} \frac{\tan^2 x - 1}{\tan x - 1} \\ &= \lim_{x \rightarrow \pi/4} \frac{(\tan x + 1)(\tan x - 1)}{(\tan x - 1)} = \lim_{x \rightarrow \pi/4} (\tan x + 1) \\ &= 2 \end{aligned}$$

Q. 62 $\lim_{x \rightarrow 1} \frac{(\sqrt{x} - 1)(2x - 3)}{2x^2 + x - 3}$ is equal to

(a) $\frac{1}{10}$

(b) $\frac{-1}{10}$

(c) 1

(d) None of these

Sol. (b) Given, $\lim_{x \rightarrow 1} \frac{(\sqrt{x} - 1)(2x - 3)}{2x^2 + x - 3} = \lim_{x \rightarrow 1} \frac{(\sqrt{x} - 1)(2x - 3)}{(2x + 3)(x - 1)}$

$$= \lim_{x \rightarrow 1} \frac{(\sqrt{x} - 1)(2x - 3)}{(2x + 3)(\sqrt{x} - 1)(\sqrt{x} + 1)}$$

$$= \lim_{x \rightarrow 1} \frac{2x - 3}{(2x + 3)(\sqrt{x} + 1)} = \frac{-1}{5 \times 2} = \frac{-1}{10}$$

Q. 63 If $f(x) = \begin{cases} \frac{\sin[x]}{[x]}, & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$, where $[.]$ denotes the greatest integer

function, then $\lim_{x \rightarrow 0} f(x)$ is equal to

(a) 1

(b) 0

(c) -1

(d) Does not exist

Sol. (d) Given, $f(x) = \begin{cases} \frac{\sin[x]}{[x]}, & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$

$\therefore$

$$\text{LHL} = \lim_{x \rightarrow 0^-} f(x)$$

$$= \lim_{x \rightarrow 0^-} \frac{\sin[x]}{[x]} = \lim_{h \rightarrow 0} \frac{\sin[0 - h]}{[0 - h]}$$

$$= \lim_{h \rightarrow 0} \frac{-\sin[-h]}{[-h]} = -1$$

$$\text{RHL} = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{\sin[x]}{[x]}$$

$$= \lim_{x \rightarrow 0^+} \frac{\sin[0 + h]}{[0 + h]} = \lim_{h \rightarrow 0} \frac{\sin[h]}{[h]} = 1$$

$\therefore$ LHL $\neq$ RHL

So, limit does not exist.

Q. 64 $\lim_{x \rightarrow 0} \frac{|\sin x|}{x}$ is equal to

(a) 1

(b) -1

(c) Does not exist

(d) None of these

Sol. (c) Given, $\text{limit} = \lim_{x \rightarrow 0} \frac{|\sin x|}{x}$

$\therefore$

$$\text{LHL} = \lim_{x \rightarrow 0^-} \left(\frac{-\sin x}{x} \right) = - \lim_{x \rightarrow 0^-} \frac{\sin x}{x} = -1$$

$$\text{RHL} = \lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$$

$\therefore$ LHL $\neq$ RHL

So, limit does not exist.

Q. 65 If $f(x) = \begin{cases} x^2 - 1, & 0 < x < 2 \\ 2x + 3, & 2 \leq x < 3 \end{cases}$, then the quadratic equation whose roots

are $\lim_{x \rightarrow 2^-} f(x)$ and $\lim_{x \rightarrow 2^+} f(x)$ is

(a) $x^2 - 6x + 9 = 0$

(b) $x^2 - 7x + 8 = 0$

(c) $x^2 - 14x + 49 = 0$

(d) $x^2 - 10x + 21 = 0$

Sol. (d) Given, $f(x) = \begin{cases} x^2 - 1, & 0 < x < 2 \\ 2x + 3, & 2 \leq x < 3 \end{cases}$

$$\begin{aligned} \therefore \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} (x^2 - 1) \\ &= \lim_{h \rightarrow 0} [(2 - h)^2 - 1] = \lim_{h \rightarrow 0} (4 + h^2 - 4h - 1) \\ &= \lim_{h \rightarrow 0} (h^2 - 4h + 3) = 3 \end{aligned}$$

$$\begin{aligned} \text{and } \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} (2x + 3) \\ &= \lim_{h \rightarrow 0} [2(2 + h) + 3] = \lim_{h \rightarrow 0} (4 + 2h + 3) = 7 \end{aligned}$$

So, the quadratic equation whose roots are 3 and 7 is $x^2 - (3 + 7)x + 3 \times 7 = 0$ i.e., $x^2 - 10x + 21 = 0$.

Q. 66 $\lim_{x \rightarrow 0} \frac{\tan 2x - x}{3x - \sin x}$ is equal to

(a) 2

(b) $\frac{1}{2}$

(c) $\frac{-1}{2}$

(d) $\frac{1}{4}$

Sol. (b) Given,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan 2x - x}{3x - \sin x} &= \lim_{x \rightarrow 0} \frac{x \left[\frac{\tan 2x}{x} - 1 \right]}{x \left[3 - \frac{\sin x}{x} \right]} \\ &= \frac{\lim_{x \rightarrow 0} 2 \times \frac{\tan 2x}{2x} - 1}{3 - \lim_{x \rightarrow 0} \frac{\sin x}{x}} = \frac{2 - 1}{3 - 1} = \frac{1}{2} \end{aligned}$$

Q. 67 If $f(x) = x - [x]$, $x \in R$, then $f'\left(\frac{1}{2}\right)$ is equal to

(a) $\frac{3}{2}$

(b) 1

(c) 0

(d) -1

Sol. (b) Given, $f(x) = x - [x]$

Now, first we have to check the differentiability of $f(x)$ at $x = \frac{1}{2}$.

$$\therefore Lf'\left(\frac{1}{2}\right) = \text{LHD} = \lim_{h \rightarrow 0} \frac{f\left(\frac{1}{2} - h\right) - f\left(\frac{1}{2}\right)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{\left(\frac{1}{2} - h\right) - \left(\frac{1}{2} - h\right) - \frac{1}{2} + \left(\frac{1}{2}\right)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{2} - h - 0 - \frac{1}{2} + 0}{h} = 1$$

$$\text{and } Rf'\left(\frac{1}{2}\right) = \text{RHD } \lim_{h \rightarrow 0} \frac{f\left(\frac{1}{2} + h\right) - f\left(\frac{1}{2}\right)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\left(\frac{1}{2} + h\right) - \left(\frac{1}{2} + h\right) - \frac{1}{2} + \left(\frac{1}{2}\right)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{2} + h - 0 - \frac{1}{2} + 0}{h} = 1$$

$$\therefore \text{LHL} = \text{RHD}$$

$$\therefore f'\left(\frac{1}{2}\right) = 1$$

Q. 68 If $y = \sqrt{x} + \frac{1}{\sqrt{x}}$, then $\frac{dy}{dx}$ at $x = 1$ is equal to

- (a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) 0

Sol. (d) Given, $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

$$\text{Now, } \frac{dy}{dx} = \frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}}$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=1} = \frac{1}{2} - \frac{1}{2} = 0$$

Q. 69 If $f(x) = \frac{x-4}{2\sqrt{x}}$, then $f'(1)$ is equal to

- (a) $\frac{5}{4}$ (b) $\frac{4}{5}$ (c) 1 (d) 0

Sol. (a) Given, $f(x) = \frac{x-4}{2\sqrt{x}}$

$$\text{Now, } f'(x) = \frac{2\sqrt{x} - (x-4) \cdot 2 \cdot \frac{1}{2\sqrt{x}}}{4x}$$

$$= \frac{2x - (x-4)}{4x^{3/2}} = \frac{2x - x + 4}{4x^{3/2}}$$

$$= \frac{x+4}{4x^{3/2}}$$

$$\therefore f'(1) = \frac{1+4}{4 \times (1)^{3/2}} = \frac{5}{4}$$

Q. 70 If $y = \frac{1 + \frac{1}{x^2}}{1 - \frac{1}{x^2}}$, then $\frac{dy}{dx}$ is equal to

- (a) $\frac{-4x}{(x^2-1)^2}$ (b) $\frac{-4x}{x^2-1}$
- (c) $\frac{1-x^2}{4x}$ (d) $\frac{4x}{x^2-1}$

Sol. (a) Given,

$$y = \frac{1 + \frac{1}{x^2}}{1 - \frac{1}{x^2}} \Rightarrow y = \frac{x^2 + 1}{x^2 - 1}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{(x^2 - 1)2x - (x^2 + 1)(2x)}{(x^2 - 1)^2} && \text{[by quotient rule]} \\ \frac{dy}{dx} &= \frac{2x(x^2 - 1 - x^2 - 1)}{(x^2 - 1)^2} \\ &= \frac{2x(-2)}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2} \end{aligned}$$

Q. 71 If $y = \frac{\sin x + \cos x}{\sin x - \cos x}$, then $\frac{dy}{dx}$ at $x = 0$ is equal to

- (a) -2 (b) 0 (c) $\frac{1}{2}$ (d) Does not exist

Sol. (a) Given, $y = \frac{\sin x + \cos x}{\sin x - \cos x}$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2} && \text{[by quotient rule]} \\ &= \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2} \\ &= \frac{-[(\sin x - \cos x)^2 + (\sin x + \cos x)^2]}{(\sin x - \cos x)^2} \\ &= \frac{-[\sin^2 x + \cos^2 x - 2\sin x \cos x + \sin^2 x + \cos^2 x + 2\sin x \cos x]}{(\sin x - \cos x)^2} \\ &= \frac{-2}{(\sin x - \cos x)^2} \\ \therefore \left(\frac{dy}{dx}\right)_{x=0} &= -2 \end{aligned}$$

Q. 72 If $y = \frac{\sin(x+9)}{\cos x}$, then $\frac{dy}{dx}$ at $x = 0$ is equal to

- (a) $\cos 9$ (b) $\sin 9$ (c) 0 (d) 1

Sol. (a) Given,

$$y = \frac{\sin(x+9)}{\cos x}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{\cos x \cos(x+9) - \sin(x+9)(-\sin x)}{(\cos x)^2} && \text{[by quotient rule]} \\ &= \frac{\cos x \cos(x+9) + \sin x \sin(x+9)}{\cos^2 x} \\ \therefore \left(\frac{dy}{dx}\right)_{x=0} &= \frac{\cos 9}{1} \\ &= \cos 9 \end{aligned}$$

Q. 73 If $f(x) = 1 + x + \frac{x^2}{2} + \dots + \frac{x^{100}}{100}$, then $f'(1)$ is equal to

(a) $\frac{1}{100}$

(b) 100

(c) 0

(d) Does not exist

Sol. (b) Given, $f(x) = 1 + x + \frac{x^2}{2} + \dots + \frac{x^{100}}{100}$

$\therefore f'(x) = 0 + 1 + 2 \times \frac{x}{2} + \dots + 100 \frac{x^{99}}{100}$

$f'(x) = 1 + x + x^2 + \dots + x^{99}$

Now, $f'(1) = 1 + 1 + 1 + \dots + 1$ (100 times)
 $= 100$

Q. 74 If $f(x) = \frac{x^n - a^n}{x - a}$ for some constant a , then $f'(a)$ is equal to

(a) 1

(b) 0

(c) $\frac{1}{2}$

(d) Does not exist

Sol. (d) Given, $f(x) = \frac{x^n - a^n}{x - a}$

$\therefore f'(x) = \frac{(x - a)nx^{n-1} - (x^n - a^n)(1)}{(x - a)^2}$ [by quotient rule]

$\Rightarrow f'(x) = \frac{nx^{n-1}(x - a) - x^n + a^n}{(x - a)^2}$

Now, $f'(a) = \frac{na^{n-1}(0) - a^n + a^n}{(x - a)^2}$

$\Rightarrow f'(a) = \frac{0}{0}$

So, $f'(a)$ does not exist,

Since, $f(x)$ is not defined at $x = a$.

Hence, $f'(x)$ at $x = a$ does not exist.

Q. 75 If $f(x) = x^{100} + x^{99} + \dots + x + 1$, then $f'(1)$ is equal to

(a) 5050

(b) 5049

(c) 5051

(d) 50051

Sol. (a) Given, $f(x) = x^{100} + x^{99} + \dots + x + 1$

$\therefore f'(x) = 100x^{99} + 99x^{98} + \dots + 1 + 0$
 $= 100x^{99} + 99x^{98} + \dots + 1$

Now, $f'(1) = 100 + 99 + \dots + 1$
 $= \frac{100}{2} [2 \times 100 + (100 - 1)(-1)]$

$= 50[200 - 99]$

$= 50 \times 101$

$= 5050$

$\left[\because S_n = \frac{n}{2} \{2a + (n - 1)d\} \right]$

Q. 76 If $f(x) = 1 - x + x^2 - x^3 + \dots - x^{99} + x^{100}$, then $f'(1)$ is equal to

- (a) 150 (b) -50 (c) -150 (d) 50

Sol. (d) Given, $f(x) = 1 - x + x^2 - x^3 + \dots - x^{99} + x^{100}$

$$f'(x) = 0 - 1 + 2x - 3x^2 + \dots - 99x^{98} + 100x^{99}$$

$$= -1 + 2x - 3x^2 + \dots - 99x^{98} + 100x^{99}$$

$$\therefore f'(1) = -1 + 2 - 3 + \dots - 99 + 100$$

$$= (-1 - 3 - 5 - \dots - 99) + (2 + 4 + \dots + 100) \quad \left[\because S_n = \frac{n}{2} \{2a + (n-1)d\} \right]$$

$$= -\frac{50}{2} [2 \times 1 + (50-1)2] + \frac{50}{2} [2 \times 2 + (50-1)2]$$

$$= -25 [2 + 49 \times 2] + 25 [4 + 49 \times 2]$$

$$= -25 (2 + 98) + 25 (4 + 98)$$

$$= -25 \times 100 + 25 \times 102$$

$$= -2500 + 2550$$

$$= 50$$

Fillers

Q. 77 If $f(x) = \frac{\tan x}{x - \pi}$, then $\lim_{x \rightarrow \pi} f(x) = \dots\dots\dots$

Sol. Given, $f(x) = \frac{\tan x}{x - \pi} = \lim_{x \rightarrow \pi} \frac{\tan x}{x - \pi} = \lim_{\pi - x \rightarrow 0} \frac{-\tan(\pi - x)}{-(\pi - x)} \quad \left[\because \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \right]$

$$= 1$$

Q. 78 $\lim_{x \rightarrow 0} \left(\sin mx \cot \frac{x}{\sqrt{3}} \right) = 2$, then $m = \dots\dots\dots$

Sol. Given, $\lim_{x \rightarrow 0} \left(\sin mx \cot \frac{x}{\sqrt{3}} \right) = 2$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sin mx}{mx} \cdot mx \cdot \frac{1}{\tan \frac{x}{\sqrt{3}}} = 2$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sin mx}{mx} \cdot mx \cdot \frac{\frac{x}{\sqrt{3}}}{\tan \frac{x}{\sqrt{3}}} \cdot \frac{1}{\frac{x}{\sqrt{3}}} = 2$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sin mx}{mx} \cdot \lim_{x \rightarrow 0} \frac{\frac{x}{\sqrt{3}}}{\tan \frac{x}{\sqrt{3}}} \cdot \lim_{x \rightarrow 0} \frac{mx}{\frac{x}{\sqrt{3}}} = 2$$

$$\Rightarrow \sqrt{3}x = 2$$

$$\therefore m = \frac{2\sqrt{3}}{3}$$

Q. 79 If $y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$, then $\frac{dy}{dx} = \dots\dots$

Sol. Given,

$\therefore$

$$\begin{aligned} y &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \\ \frac{dy}{dx} &= 0 + 1 + \frac{2x}{2} + \frac{3x^2}{6} + \frac{4x^3}{4!} \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots \\ &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ &= y \end{aligned}$$

Q. 80 $\lim_{x \rightarrow 3^+} \frac{x}{[x]} = \dots$

Sol. Given,

$$\begin{aligned} \lim_{x \rightarrow 3^+} \frac{x}{[x]} &= \lim_{h \rightarrow 0} \frac{(3+h)}{[3+h]} \\ &= \lim_{h \rightarrow 0} \frac{(3+h)}{3} = 1 \end{aligned}$$