Chapter 3 - MATRICES

Applications

The below are the uses of matrices:

- Compact and simple methods of solving system of linear equations.
- Used as a representation of the coefficients in system of linear equations
- Matrix notation and operations are used in electronic spreadsheet programs for personal computer, which in turn is used in different areas of business and science like budgeting, sales projection, cost estimation, analysing the results of an experiment etc.
- Used in physical operations such as magnification, rotation and reflection through a plane
- Used in cryptography.
- Also used in various branches of sciences like genetics, economics, sociology, modern psychology and industrial management.
- Used in 3D math, where they are primarily used to describe the relationship between two coordinate spaces.

Matrix

- **Definition:** A matrix is an ordered rectangular array of numbers (may be real or complex) or functions.
- **Elements or Entries of the matrix:** The numbers or functions in the array
- **Rows of Matrix:** The horizontal lines of elements
- **Columns of Matrix:** The vertical lines of elements
- **General Format:**
  - Simple Matrix:
    - Has one Row and Column
    - Represented as (Row, Column)
  - Higher Level Matrix
    - Has many Rows and Columns
    - Represented as below:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \dots\dots\dots & a_{1n} \\ a_{21} & a_{22} & a_{23} \dots \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} \dots \dots & a_{3n} \end{bmatrix}$$

(or)

$$\begin{array}{cc} A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \dots\dots\dots & a_{1n} \\ a_{21} & a_{22} & a_{23} \dots \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} \dots \dots & a_{3n} \end{pmatrix} & \textbf{Rows} \\ \textbf{Columns} & \end{array}$$

    - Furthermore, Generalized format is as below:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

(Or)

A= $[a_{ij}]_{m \times n}$

Where, $1 \le i \le m$, $1 \le j \le n$ and i, j € N

- i^{th} Row Elements are $a_{i1}, a_{i2}, a_{i3}, \ldots\ldots\ldots\ a_{in}$
- j^{th} Row Elements are $a_{1j}, a_{2j}, a_{3j} \ldots\ldots\ldots\ldots\ldots\ldots, a_{nj}$

Order of a Matrix

- If a matrix has **"m"** rows and **"n"** columns, then order of the matrix is m x n
- The total numbers of elements in the matrix of m x n will be *mn*
- Eg: Order of the below matrix is 3 x 2

  $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 5 & 9 \end{bmatrix}$

  With total number of elements = product of 3 and 2 = 6
- Any point (x,y) in a plane can be represented in the matrix format as below:

  P =$[\,x, y\,]$ or $\begin{bmatrix} x \\ y \end{bmatrix}$
- Suppose if the vertices of a Quadrilateral ABCD are given then it can be represented as matrix format as below

  Points: A(1,0) , B(3,2), C(1,3) and D(-1,-2)

$$X = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 3 & 1 & -1 \\ 0 & 2 & 3 & 2 \end{bmatrix}_{2\times 4} \end{array} \quad \text{or} \quad Y = \begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 1 & 3 \\ -1 & 2 \end{bmatrix}_{4\times 2}$$

- Suppose if the matrix has 8 elements then all possible order of the matrix can be found as below.
  - Find the Factors of 8 → 1, 2, 4, 8
  - Pair up them in all possible ways → (1,8), (2, 4), (8,1), (4, 2)

Types of a Matrix

1. **Column Matrix**
   - A matrix having only one column and any number of rows

- Eg: $A = \begin{bmatrix} 0 \\ \sqrt{5} \\ -2 \\ 3/2 \end{bmatrix}$ of order 4 x 1
- General Form : A= $[a_{ij}]_{m \times 1}$, order of the matrix is m x 1

2. Row Matrix

- A matrix having only one row and any number of columns
- Eg: $A = \begin{bmatrix} \frac{-3}{4} & \sqrt{7} & 3 & 5 \end{bmatrix}$ of order 1 x 4
- General Form = A= $[a_{ij}]_{1 \times n}$, Order of the matrix is 1 x n

3. Square Matrix

- A matrix of order m x n, such that m = n
- Eg: $A= \begin{bmatrix} 3 & -1 & 0 \\ \frac{3}{2} & 3\sqrt{2} & 1 \\ 4 & 3 & -1 \end{bmatrix}$ of order 3 x 3
- General Form = A= $[a_{ij}]_{m \times m}$, Order of the matrix is m

4. Diagonal Matrix

- A square matrix is said to be a diagonal matrix if all its non-diagonal elements are zero
- Eg: $A = \begin{bmatrix} -1.1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ of order 3x3
- General Form = A= $[b_{ij}]_{m \times m}$, Order of the matrix is *m* where $b_{ij} = 0$, if $i \neq j$

5. Scalar Matrix

- A diagonal matrix is said to be a scalar matrix if its diagonal elements are equal
- Eg: $A = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$ of order 3x3
- General Form = A = $[b_{ij}]_{m \times m}$, Order of the matrix is *m* where
  - $b_{ij} = 0$, if $I \neq j$
  - $b_{ij} = k$, if i = j for k = constant

6. Identity Matrix

- A square matrix in which elements in the diagonal are all 1 and rest are all zero
- Eg: $A= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ of order 3x3
- General Form = A= $[b_{ij}]_{m \times m}$, Order of the matrix is m where

- b_{ij} =0, if i≠j
- b_{ij} =1, if i = j

7. Zero Matrix

- A matrix is said to be zero matrix or null matrix if all its elements are zero
- Eg: A = $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ of order 3 x 3
- Denoted by O

8. Equal Matrices

- Two matrices are said to be equal if
  - they are of the same order
  - each element of A is equal to the corresponding element of B
- Eg: A = $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ and B= $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ are equal matrices
- Using the concept of Equal matrices, we can find the unknown values of any matrix.

Eg:If $\begin{bmatrix} x+3 & z+4 & 2y-4 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$. Find x,y,z,a,b,c

We can equate the corresponding terms from the given matrices:

x + 3 = 0	z + 4 = 6	2y – 7 = 3y – 2
a – 1 = – 3	0 = 2c + 2	b – 3 = 2b + 4,

Simplifying, we get
a = – 2, b = – 7, c = – 1, x = – 3, y = –5, z = 2

Operation on Matrices

Addition of Matrices

- Let A and B be two matrices each of order m x n. Then, the sum of matrices A + B is defined only if matrices A and B are of same order.
  - If A = $[a_{ij}]_{m \times n}$ and B = $[b_{ij}]_{m \times n}$
    Then, A + B = $[a_{ij} + b_{ij}]_{m \times n}$
- General Format

  IF $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

  Then, we define $A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+a_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$
- Example

$$\begin{pmatrix} 5 & 2 \\ 4 & 9 \\ 10 & -3 \end{pmatrix} + \begin{pmatrix} -11 & 0 \\ 7 & 1 \\ -6 & -8 \end{pmatrix} = \begin{pmatrix} 5+(-11) & 2+0 \\ 4+7 & 9+1 \\ 10+(-6) & -3+(-8) \end{pmatrix} = \begin{pmatrix} -6 & 2 \\ 11 & 10 \\ 4 & -11 \end{pmatrix}$$

- When adding two matix A & B, if the order is not same then A + B is not defined

Properties of Addition of Matrices

- If $A = [a_{ij}]$, $B = [b_{ij}]$and $C = [c_{ij}]$are three matrices of order m x n, then
    - Commutative Law
        - A + B = B + A
    - Associative Law
        - (A + B) + C = A + (B + C)
    - Existence of Additive Identity
        - A zero matrix (0) of order m x n (same as of A), is additive identity, if A + 0 = A = 0 + A
    - Existence of Additive Inverse
        - If A is a square matrix, then the matrix (- A) is called additive inverse, if A + (– A) = 0 = (- A) + A
            - – A is the additive inverse of A or negative of A.
    - Cancellation Law
      A + B = A + C ⇒ B = C (left cancellation law)
      B + A = C + A ⇒ B = C (right cancellation law)

Subtraction of Matrices

- Let A and B be two matrices of the same order, then subtraction of matrices, A – B, is defined as $A - B = [a_{ij} - b_{ij}]_{n \times n}$, where $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$

- Eg: If A= $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 6 \end{bmatrix}$ and B = $\begin{bmatrix} 0 & -4 & 3 \\ 9 & -4 & -3 \end{bmatrix}$
    - Here Order of A is 3 x 2 and Order of B is 3 x 2 therefore, subtraction is possible

$$A - B = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 0 & -4 & 3 \\ 9 & -4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -1-0 & 2-(-4) & 0-3 \\ 0-9 & 3-(-4) & 6-(-3) \end{bmatrix}$$

Then $= \begin{bmatrix} -1 & 2+4 & -3 \\ -9 & 3+4 & 6+3 \end{bmatrix} = \begin{bmatrix} -1 & 6 & -3 \\ -9 & 7 & 9 \end{bmatrix}$

Multiplication of of Matrices

- **Multiplication of a Matrix by a Scalar**

- Let A = $\left[a_{ij}\right]_{m\times n}$ be a matrix and k be any scalar. Then, the matrix obtained by multiplying each element of A by k is called the scalar multiple of A by k and is denoted by kA, given as kA= $\left[ka_{ij}\right]_{m\times n}$

- Eg: If A = $\begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix}$. Find 3A
  - Here A is the mtrix of order 3 x3 and k =3, constant
  - Hence 3A = $3\begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 4.5 \\ 3\sqrt{5} & 21 & -9 \\ 6 & 0 & 15 \end{bmatrix}$

- Properties of Scalar Multiplication If A and B are matrices of order m x n, then

  - k(A + B) = kA + kB
  - $(k_1 + k_2)A = k_1A + k_2A$
  - k_1k_2A = k1(k2A) = k_2(k1A)
  - (- k)A = – (kA) = k(– A) also called as negative of a matrix

- **Multiplication of two Matrices**

  - Consider two matries A and B, then for the multiplication to e possible umber of columns in A should be equal to the number of rows in B.
  - If A = $[a_{ij}]$ of order mxn and B = $[b_{ij}]$ of order n xp then the product C = $[c_{ij}]$will be a patrix of the oder m x p
  - Eg: A=$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 3 & -1 \end{bmatrix}$ of order 3 x 2, B = $\begin{bmatrix} 0 & 3 \\ -2 & -1 \\ 0 & 4 \end{bmatrix}$ of order 2 × 3

    Then A×B =

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & -1 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ -2 & -1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1\cdot 0+0\cdot(-2)+(-2)\cdot 0 & 1\cdot 3+0\cdot(-1)+(-2)\cdot 4 \\ 0\cdot 0+3\cdot(-2)+(-1)\cdot 0 & 0\cdot 3+3\cdot(-1)+(-1)\cdot 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0 & 3+0-8 \\ 0-6+0 & 0-3-4 \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ -6 & -7 \end{bmatrix}$$

  - Note:
    - If AB is defined, then BA need not be defined.
    - If A, B are, respectively m × n, k × l matrices, then both AB and BA are defined if and only if n = k and l = m.

- If both A and B are square matrices of the same order, then both AB and BA are defined.
- If AB and BA are both defined, it is not necessary that AB = BA.
- If the product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix

Properties of Multiplication of Matrices

If $A = [a_{ij}]$, $B = [b_{ij}]$and $C = [c_{ij}]$are three matrices of order m x n, then

- **Commutative Law**
  - $AB \neq BA$
- **Associative Law**
  - $(A B)C = A(BC)$
- **Distributive Law**
  - $A (B+C) = AB + AC$
  - $(A+B) C = AC + BC$, whenever both sides of equality are defined.
- **Existence of Multiplicative Identity**
  - For every square matrix A, there exist an identity matrix of same order such that IA = AI = A
- **Cancellation Law**
  - If A is non-singular matrix, then
    - $AB = AC \Rightarrow B = C$ (Left cancellation law)
    - $BA = CA \Rightarrow B = C$ (Right cancellation law)
    - AB = 0, does not necessarily imply that A = 0 or B = 0 or both A and B =

Transpose of a Matrix

- Let $A = [a_{ij}]m \times n$, be a matrix of order m x n. Then, the n x m matrix obtained by interchanging the rows and columns of A is called the transpose of A and is denoted by ′or AT.
- $A' = AT = [a_{ij}]n \times m$
- Eg: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}, A^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$

Properties of Transpose

- $(A')' = A$
- $(A + B)' = A' + B'$
- $(AB)' = B'A'$
- $(kA)' = kA'$
- $(AN)' = (A')N$
- $(ABC)' = C' B' A'$

Symmetric & Skew Symmetric Matrices

- A square matrix $A = [a_{ij}]$ is said to be symmetric if $A' = A$

- Eg: A = $\begin{bmatrix} \sqrt{3} & 2 & 3 \\ 2 & -1.5 & -1 \\ 3 & -1 & 1 \end{bmatrix}$. Here A′ = A

- A square matrix A = $[a_{ij}]$ is said to be skew symmetric matrix if A′ = – A
  - Eg:A= $\begin{bmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{bmatrix}$, A′ = -A
- Note:
  - If A is Symmetric then, $[a_{ij}] = [a_{ji}]$
  - If A is Skew Symmetric then, $[a_{ij}] = -[a_{ji}]$
  - All the diagonal elements of a skew symmetric matrix are zero.

- **Theorem 1**
  - For any square matrix A with real number entries, A + A′ is a symmetric matrix and A – A′ is a skew symmetric matrix.
  - Proof

    Let B = A + A′, then

    B′ = (A + A′)′

    = A′ + (A′)′ (as (A + B)′ = A′ + B′)

    = A′ + A (as (A′)′ = A)

    = A + A′ (as A + B = B + A)

    = B

    Therefore B = A + A′ is a symmetric matrix

    Now let C = A – A′

    C′ = (A – A′)′ = A′ – (A′)′ (Why?)

    = A′ – A (Why?)

    = – (A – A′) = – C

    Therefore C = A – A′ is a skew symmetric matrix
- **Theorem 2**
  - Any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.
  - Proof

    Let A be a square matrix, then we can write

    $$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

    From the Theorem 1, we know that (A + A′) is a symmetric matrix and (A – A′) is a skew symmetric matrix.

    Since for any matrix A, (kA)′ = kA′, it follows that $\frac{1}{2}(A + A')$ is symmetric matrix and $\frac{1}{2}(A - A')$ is skew symmetric matrix. Thus, any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

Elementary Operation (Transformation) of a Matrix

- The six operations (transformations) on a matrix, three of which are due to rows and three due to columns
  - Interchanging any two rows (or columns), denoted by $R_i \Leftrightarrow R_j$ or $C_i \Leftrightarrow C_j$
  - Multiplication of the element of any row (or column) by a non-zero quantity and denoted by $R_i \Leftrightarrow kR_j$ or $C_i \Leftrightarrow kC_j$
  - Addition of constant multiple of the elements of any row to the corresponding element of any other row, denoted by $Ri \rightarrow Ri + kRj$ or $Ci \rightarrow Ci + kCj$ OR $R_i \Leftrightarrow R_i + k R_j$ or $C_i \Leftrightarrow C_i + k C_j$

Invertible Matrices

- If A is a square matrix of order m, and if there exists another square matrix B of the same order m, such that AB = BA = I, then B is called the inverse matrix of A and it is denoted by A^{-1}. In that case A is said to be invertible
- Eg:

  $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ be two matrices.

  $$AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

  $$= \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

  $$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

  Note
  - A rectangular matrix does not possess inverse matrix, since for products BA and AB to be defined and to be equal, it is necessary that matrices A and B should be square matrices of the same order.
  - If B is the inverse of A, then A is also the inverse of B

Inverse of a Square Matrix

- Let A be a square matrix of order n, then a square matrix B, such that AB = BA = I, is called inverse of A, denoted by A^{-1}
- $AA^{-1} = A^{-1}A = 1$

Theorem 3 (Uniqueness of inverse)

- Inverse of a square matrix, if it exists, is unique.

  i.e Let $A = [a_{ij}]$ be a square matrix of order m and B and C be two inverses of A.
  To prove : B = C

- **Proof**:

Since B is the inverse of A

AB = BA = I ... (1)
Since C is also the inverse of A
AC = CA = I ... (2)
Thus B = BI = B (AC) = (BA) C = IC = C

Theorem 4

- If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.
- **Proof :**
  From the definition of inverse of a matrix, we have
  $(AB)(AB)^{-1} = 1$
  $A^{-1}(AB)(AB)^{-1} = A^{-1} I$ (Pre multiplying both sides by A^{-1})

  $\Rightarrow (A^{-1}A) B (AB)^{-1} = A -1$ (Since $A^{-1} I = A^{-1}$)

  $\Rightarrow IB(AB)^{-1} = A^{-1}$
  $\Rightarrow B(AB)^{-1} = A^{-1}$
  $\Rightarrow B^{-1} B (AB)^{-1} = B^{-1} A^{-1}$
  $\Rightarrow I (AB)-^{-1} = B^{-1} A^{-1}$
  Hence, $(AB)^{-1} = B^{-1} A^{-1}$