

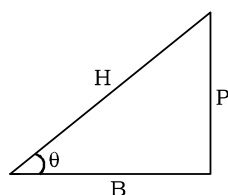
TRIGONOMETRY

TRIGONOMETRIC RATIOS & IDENTITIES

1. The meaning of Trigonometry

Tri Gon Metron
↓ ↓ ↓
3 sides Measure

Hence, this particular branch in Mathematics was developed in ancient past to measure 3 sides, 3 angles and 6 elements of a triangle. In today's time—trigonometric functions are used in entirely different shapes. The 2 basic functions are sine and cosine of an angle in a right-angled triangle and there are 4 other derived functions.



$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\operatorname{cosec} \theta$
$\frac{P}{H}$	$\frac{B}{H}$	$\frac{P}{B}$	$\frac{B}{P}$	$\frac{H}{B}$	$\frac{H}{P}$

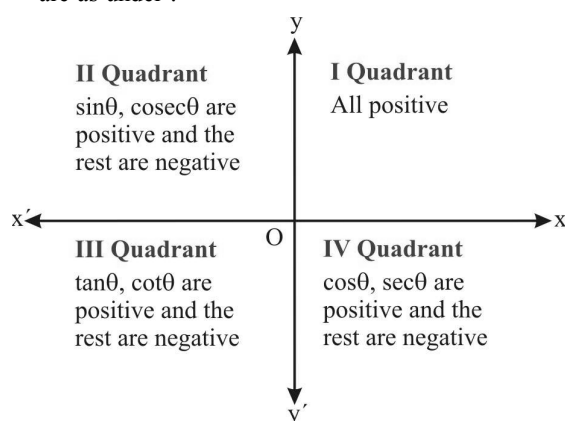
2. Basic Trigonometric Identities

- (a) $\sin^2 \theta + \cos^2 \theta = 1 : -1 \leq \sin \theta \leq 1; -1 \leq \cos \theta \leq 1 \quad \forall \theta \in \mathbb{R}$
- (b) $\sec^2 \theta - \tan^2 \theta = 1 : |\sec \theta| \geq 1 \quad \forall \theta \in \mathbb{R}$
- (c) $\operatorname{cosec}^2 \theta - \cot^2 \theta = 1 : |\operatorname{cosec} \theta| \geq 1 \quad \forall \theta \in \mathbb{R}$

Trigonometric Ratios of Standard Angles

T-Ratio ↓	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

The sign of the trigonometric ratios in different quadrants are as under :



3. Trigonometric Ratios of Allied Angles

Using trigonometric ratio of allied angles, we could find the trigonometric ratios of angles of any magnitude.

$$\sin(-\theta) = -\sin \theta \quad \cos(-\theta) = \cos \theta$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta \quad \cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$$

$$\tan(-\theta) = -\tan \theta \quad \cot(-\theta) = -\cot \theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta \quad \cot\left(\frac{\pi}{2} - \theta\right) = \tan \theta$$

$$\operatorname{cosec}(-\theta) = -\operatorname{cosec} \theta \quad \sec(-\theta) = \sec \theta$$

$$\sec\left(\frac{\pi}{2} - \theta\right) = \operatorname{cosec} \theta \quad \operatorname{cosec}\left(\frac{\pi}{2} - \theta\right) = \sec \theta$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos \theta \quad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta$$

$$\sin(\pi - \theta) = \sin \theta \quad \cos(\pi - \theta) = -\cos \theta$$

$$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot \theta \quad \cot\left(\frac{\pi}{2} + \theta\right) = -\tan \theta$$

$$\tan(\pi - \theta) = -\tan \theta \quad \cot(\pi - \theta) = -\cot \theta$$

$$\sec\left(\frac{\pi}{2} + \theta\right) = -\operatorname{cosec} \theta \quad \operatorname{cosec}\left(\frac{\pi}{2} + \theta\right) = \sec \theta$$

$$\sec(\pi - \theta) = -\sec \theta \quad \operatorname{cosec}(\pi - \theta) = \operatorname{cosec} \theta$$

$$\sin(\pi + \theta) = -\sin \theta \quad \cos(\pi + \theta) = -\cos \theta$$

$$\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos \theta \quad \cos\left(\frac{3\pi}{2} - \theta\right) = -\sin \theta$$

$$\tan(\pi + \theta) = \tan \theta \quad \cot(\pi + \theta) = \cot \theta$$

$$\tan\left(\frac{3\pi}{2} - \theta\right) = \cot \theta \quad \cot\left(\frac{3\pi}{2} - \theta\right) = \tan \theta$$

$$\sec(\pi + \theta) = -\sec \theta \quad \operatorname{cosec}(\pi + \theta) = -\operatorname{cosec} \theta$$

$$\sec\left(\frac{3\pi}{2} - \theta\right) = -\operatorname{cosec} \theta \quad \operatorname{cosec}\left(\frac{3\pi}{2} - \theta\right) = -\sec \theta$$

$$\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos \theta \quad \cos\left(\frac{3\pi}{2} + \theta\right) = \sin \theta$$

$$\sin(2\pi - \theta) = -\sin \theta \quad \cos(2\pi - \theta) = \cos \theta$$

$$\tan\left(\frac{3\pi}{2} + \theta\right) = -\cot \theta \quad \cot\left(\frac{3\pi}{2} + \theta\right) = -\tan \theta$$

$$\tan(2\pi - \theta) = -\tan \theta \quad \cot(2\pi - \theta) = -\cot \theta$$

$$\sec\left(\frac{3\pi}{2} + \theta\right) = \operatorname{cosec} \theta \quad \operatorname{cosec}\left(\frac{3\pi}{2} + \theta\right) = -\sec \theta$$

$$\sec(2\pi - \theta) = \sec \theta \quad \operatorname{cosec}(2\pi - \theta) = -\operatorname{cosec} \theta$$

$$\sin(2\pi + \theta) = \sin \theta \quad \cos(2\pi + \theta) = \cos \theta$$

$$\tan(2\pi + \theta) = \tan \theta \quad \cot(2\pi + \theta) = \cot \theta$$

$$\sec(2\pi + \theta) = \sec \theta \quad \operatorname{cosec}(2\pi + \theta) = \operatorname{cosec} \theta$$

4. Trigonometric Functions of Sum or Difference of Two Angles

$$(a) \sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$(b) \sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$(c) \cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$(d) \cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$(e) \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$(f) \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$(g) \cot(A + B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

$$(f) \cot(A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

$$(h) \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A + B) \cdot \sin(A - B)$$

$$(i) \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A + B) \cdot \cos(A - B)$$

$$(j) \tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

5. Multiple Angles and Half Angles

$$(a) \sin 2A = 2 \sin A \cos A ; \quad \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$

$$(b) \cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A ;$$

$$2 \cos^2 \frac{\theta}{2} = 1 + \cos \theta, \quad 2 \sin^2 \frac{\theta}{2} = 1 - \cos \theta$$

$$(c) \tan 2A = \frac{2 \tan A}{1 - \tan^2 A} ; \quad \tan \theta = \frac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$$

$$(d) \sin 2A = \frac{2 \tan A}{1 + \tan^2 A} ; \quad \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$(e) \sin 3A = 3 \sin A - 4 \sin^3 A$$

$$(f) \cos 3A = 4 \cos^3 A - 3 \cos A$$

$$(g) \tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

6. Transformation of Products into Sum or Difference of Sines & Cosines

$$(a) 2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$(b) 2 \cos A \sin B = \sin(A + B) - \sin(A - B)$$

$$(c) 2 \cos A \cos B = \cos(A + B) + \cos(A - B)$$

$$(d) 2 \sin A \sin B = \cos(A - B) - \cos(A + B)$$

7. Factorisation of the Sum or Difference of Two Sines or Cosines

$$(a) \sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$$

$$(b) \sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}$$

$$(c) \cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$$

$$(d) \cos C - \cos D = -2 \sin \frac{C+D}{2} \sin \frac{C-D}{2}$$

8. Important Trigonometric Ratios

$$(a) \sin n\pi = 0; \cos n\pi = (-1)^n; \tan n\pi = 0 \text{ where } n \in \mathbb{Z}$$

$$(b) \sin 15^\circ \text{ or } \sin \frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^\circ \text{ or } \cos \frac{5\pi}{12};$$

$$\cos 15^\circ \text{ or } \cos \frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75^\circ \text{ or } \sin \frac{5\pi}{12};$$

$$\tan 15^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3} = \cot 75^\circ;$$

$$\tan 75^\circ = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3} = \cot 15^\circ$$

$$(c) \sin \frac{\pi}{10} \text{ or } \sin 18^\circ = \frac{\sqrt{5}-1}{4} \text{ \&}$$

$$\cos 36^\circ \text{ or } \cos \frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$$

9. Conditional Identities

If $A + B + C = \pi$ then :

$$(i) \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$$

$$(ii) \sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

$$(iii) \cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$$

$$(iv) \cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$(v) \tan A + \tan B + \tan C = \tan A \tan B \tan C$$

$$(vi) \tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$$

$$(vii) \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cdot \cot \frac{B}{2} \cdot \cot \frac{C}{2}$$

$$(viii) \cot A \cot B + \cot B \cot C + \cot C \cot A = 1$$

10. Range of Trigonometric Expression

$$E = a \sin \theta + b \cos \theta$$

$$E = \sqrt{a^2 + b^2} \sin(\theta + \alpha), \left(\text{where } \tan \alpha = \frac{b}{a} \right)$$

$$E = \sqrt{a^2 + b^2} \cos(\theta - \beta), \left(\text{where } \tan \beta = \frac{a}{b} \right)$$

$$\text{Hence for any real value of } \theta, -\sqrt{a^2 + b^2} \leq E \leq \sqrt{a^2 + b^2}$$

11. Sine and Cosine Series

$$(a) \sin \alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots + \sin(\alpha + (n-1)\beta)$$

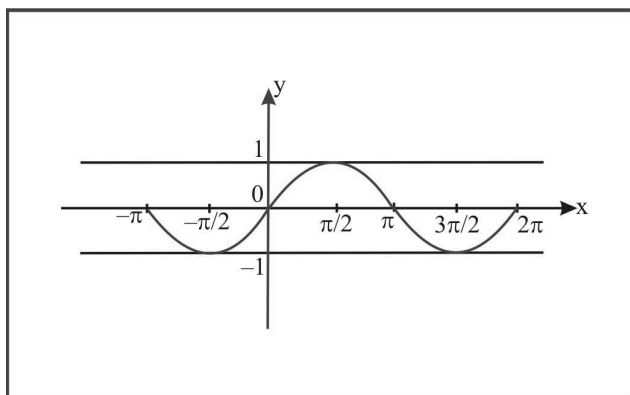
$$= \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \sin\left(\alpha + \frac{n-1}{2}\beta\right)$$

$$(b) \cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots + \cos(\alpha + (n-1)\beta)$$

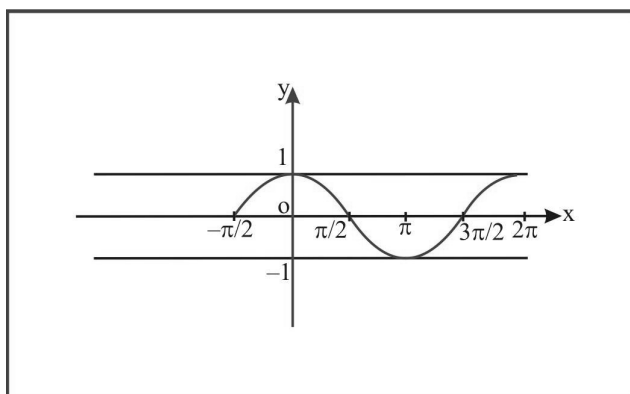
$$= \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \cos\left(\alpha + \frac{n-1}{2}\beta\right)$$

12. Graphs of Trigonometric Functions

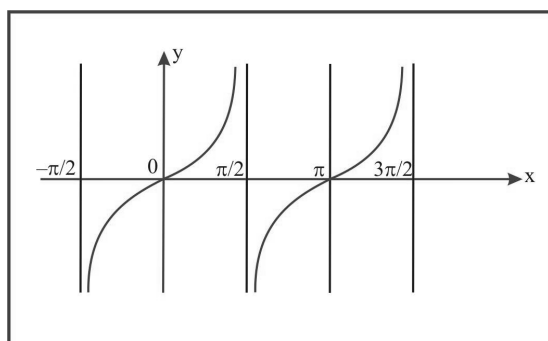
- (a) $y = \sin x$,
 $x \in \mathbb{R}; y \in [-1, 1]$



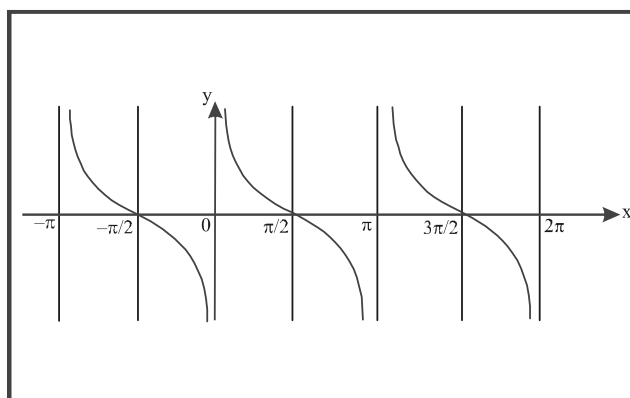
- (b) $y = \cos x$,
 $x \in \mathbb{R}; y \in [-1, 1]$



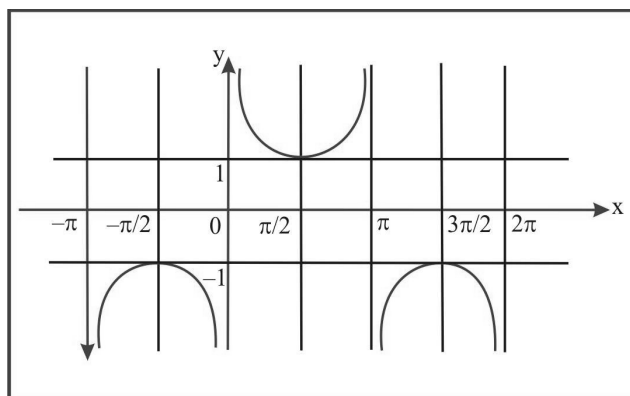
- (c) $y = \tan x$,
 $x \in \mathbb{R} - \left\{ (2n+1)\frac{\pi}{2}; n \in \mathbb{Z} \right\}; y \in \mathbb{R}$



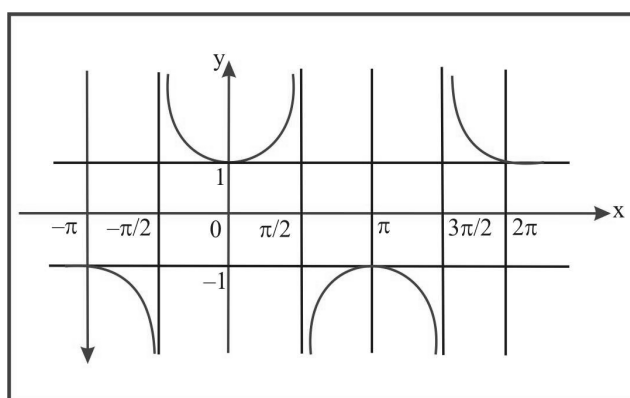
- (d) $y = \cot x$,
 $x \in \mathbb{R} - \{n\pi; n \in \mathbb{Z}\}; y \in \mathbb{R}$



- (e) $y = \operatorname{cosec} x$,
 $x \in \mathbb{R} - \{n\pi; n \in \mathbb{Z}\}; y \in (-\infty, -1] \cup [1, \infty)$



- (f) $y = \sec x$,
 $x \in \mathbb{R} - \left\{ (2n+1)\frac{\pi}{2}; n \in \mathbb{Z} \right\}; y \in (-\infty, -1] \cup [1, \infty)$



TRIGONOMETRIC EQUATIONS

13. Trigonometric Equations

The equations involving trigonometric functions of unknown angles are known as Trigonometric equations.

e.g., $\cos \theta = 0$, $\cos^2 \theta - 4 \cos \theta = 1$.

A solution of a trigonometric equation is the value of the unknown angle that satisfies the equation.

e.g., $\sin \theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$ or $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \dots$

Thus, the trigonometric equation may have infinite number of solutions and can be classified as :

- (i) Principal solution
- (ii) General solution

14. General Solution

Since, trigonometric functions are periodic, a solution generalised by means of periodicity of the trigonometrical functions. The solution consisting of all possible solutions of a trigonometric equation is called its general solution.

14.1 Results

1. $\sin \theta = 0 \Leftrightarrow \theta = n\pi$
2. $\cos \theta = 0 \Leftrightarrow \theta = (2n+1)\frac{\pi}{2}$
3. $\tan \theta = 0 \Leftrightarrow \theta = n\pi$

$$4. \sin \theta = \sin \alpha \Leftrightarrow \theta = n\pi + (-1)^n \alpha, \text{ where}$$

$$\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$5. \cos \theta = \cos \alpha \Leftrightarrow \theta = 2n\pi \pm \alpha, \text{ where } \alpha \in [0, \pi].$$

$$6. \tan \theta = \tan \alpha \Leftrightarrow \theta = n\pi + \alpha, \text{ where } \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$7. \sin^2 \theta = \sin^2 \alpha \Leftrightarrow \theta = n\pi \pm \alpha.$$

$$8. \cos^2 \theta = \cos^2 \alpha \Leftrightarrow \theta = n\pi \pm \alpha.$$

$$9. \tan^2 \theta = \tan^2 \alpha \Leftrightarrow \theta = n\pi \pm \alpha.$$

$$10. \sin \theta = 1 \Leftrightarrow \theta = (4n+1)\frac{\pi}{2}.$$

$$11. \cos \theta = 1 \Leftrightarrow \theta = 2n\pi.$$

$$12. \cos \theta = -1 \Leftrightarrow \theta = (2n+1)\pi.$$

$$13. \sin \theta = \sin \alpha \text{ and } \cos \theta = \cos \alpha \Leftrightarrow \theta = 2n\pi + \alpha.$$



1. Every where in this chapter 'n' is taken as an integer, if not stated otherwise.
2. The general solution should be given unless the solution is required in a specified interval or range.
3. α is taken as the principal value of the angle. (i.e., Numerically least angle is called the principal value).