SEQUENCE AND SERIES

1. DEFINITION

Sequence is a function whose domain is the set N of natural numbers.

Real Sequence : A sequence whose range is a subset of R is called a real sequence.

Series : If $a_1, a_2, a_3, a_4, \ldots\ldots\ldots, a_n, \ldots\ldots\ldots$ is a sequence, then the expression

$a_1 + a_2 + a_3 + a_4 + a_5 + \ldots\ldots + \ldots\ldots + a_n + \ldots\ldots$ is a series.

A series if finite or infinite according as the number of terms in the corresponding sequence is finite or infinite.

Progressions : It is not necessary that the terms of a sequence always follow a certain pattern or they are described by some explicit formula for the n^{th} term. Those sequences whose terms follow certain patterns are called progressions.

1.1 An Arithmetic Progression (AP)

AP is a sequence whose terms increase or decrease by a fixed number. This fixed number is called the common difference. If a is the first term & d the common difference, then AP can be written as a n^{th} term of this AP as $t_n = a + (n - 1)\,d$, where $d = a_n - a_{n-1}$.

The sum of the first n terms the AP is given by ;

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + \ell].$$

where ℓ is the last term.

Note...

Properties of Arithmetic Progression

(i) If each term of an A.P. is increased, decreased, multiplied or divided by the same non zero number, then the resulting sequence is also an AP.

(ii) 3 numbers in AP are $a - d, a, a + d$;

4 numbers in AP are $a - 3d, a - d, a + d, a + 3d$;

5 numbers in AP are $a - 2d, a - d, a, a + d, a + 2d$;

6 numbers in AP are $a - 5d, a - 3d, a - d, a + d, a + 3d; a + 5d$.

(iii) The common difference can be zero, positive or negative.

(iv) The sum of the two terms of an AP equidistant from the beginning & end is constant and equal to the sum of first & last terms.

(v) Any term of an AP (except the first) is equal to half the sum of terms which are equidistant from it. $a_n = 1/2\,(a_{n-k} + a_{n+k})$, $k < n$.

For $k = 1$, $a_n = (1/2)\,(a_{n-1} + a_{n+1})$;

For $k = 2$, $a_n = (1/2)\,(a_{n-2} + a_{n+2})$ and so on.

(vi) $t_r = S_r - S_{r-1}$

(vii) If a, b, c are in AP $\Rightarrow 2\ b = a + c$.

(viii) A sequence is an AP, iff its n^{th} terms is of the form $An + B$ i.e., a linear expression in n. The common difference in such a case is A i.e., the coefficient of n.

1.2 Geometric Progression (GP)

GP is a sequence of numbers whose first term is non zero & each of the succeeding terms is equal to the proceeding terms multiplied by a constant. Thus in a GP the ratio of successive terms is constant. This constant factor is called the COMMON RATIO of the series & obtained by dividing any term by that which immediately proceeds it. Therefore $a, ar, ar^2, ar^3\ ar^4, \ldots\ldots\ldots$ is a GP with a as the first term & r as common ratio.

(i) n^{th} term $= a\,r^{n-1}$

(ii) Sum of the I^{st} n terms i.e. $S_n = \dfrac{a(r^n - 1)}{r - 1}$, if $r \neq 1$.

(iii) Sum of an infinite GP when $|r| < 1$ when $n \to \infty$, $r^n \to 0$ if $|r| < 1$ therefore, $S_\infty = \dfrac{a}{1-r}\ (|r| < 1)$.

(iv) If each term of a GP be multiplied or divided by the same non-zero quantity, the resulting sequence is also a GP.

(v) Any 3 consecutive terms of a GP can be taken as $a/r, a, ar$; any 4 consecutive terms of a GP can be taken as $a/r^3, a/r, ar\ ar^3$ & so on.

(vi) If a, b, c are in GP $\Rightarrow b^2 = ac$.

Note...

Properties of Geometric Progressions

1. If all the terms of a GP be multiplied or divided by the same non–zero constant, then it remains a GP with the same common ratio.
2. The reciprocals of the terms of a given GP forms a GP.
3. If each term of a GP be raised to the same power, the resulting sequence also forms a G.P.
4. In a finite GP the product of the terms equidistant form the beginning and the end is always same and is equal to the product of the first and the last term.
5. Three non–zero numbers, a, b, c are in GP, if $b^2 = ac$.
6. If the terms of a given GP are chosen at regular intervals, then the new sequence so formed also forms a GP.
7. If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a GP of non–zero non–negative terms, then $\log a_1, \log a_2, \ldots \log a_n, \ldots$ is an AP and vice versa.

2. MEANS

2.1 Arithmetic Mean

If three terms are in AP then the middle term is called the AM between the other two, so if a, b, c, are in AP, b is AM of a & c.

AM for any n positive number $a_1, a_2, \ldots, a_n$ is ;

$$A = \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}$$

2.2 n-Arithmetic Means between Two Numbers

If a, b are any two given numbers & a, $A_1, A_2, \ldots, A_n$, b are in AP then $A_1, A_2, \ldots A_n$ are n AM's between a & b.

$$A_1 = a + \frac{b-a}{n+1}, A_2 = a + \frac{2(b-a)}{n+1}, \ldots, A_n = a + \frac{n(b-a)}{n+1}$$

$A_1 = a + d$, $A_2 = a + 2d$, ..., $A_n = a + nd$, where $d = \frac{b-a}{n+1}$

Note...

Sum of n AM's inserted between a & b is equal to n times the single AM between a & b i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single AM between a & b.

2.3 Geometric Mean

If a, b, c are in GP, b is the GM between a & c. $b^2 = ac$, therefore $b = \sqrt{ac}$; $a > 0, c > 0$.

2.4 n-Geometric Means between a & b

If a, b are two given numbers & a, $G_1, G_2, \ldots, G_n$, b are in GP. Then $G_1, G_2, G_3, \ldots, G_n$ are n GMs between a & b. $G_1 = a\,(b/a)^{1/n+1} = ar$, $G_2 = a\,(b/a)^{2/n+1} = ar^2$, ..., G_n a $(b/a)^{n/n+1} = ar^n$ where $r = (b/a)^{1/n+1}$

Note...

The product of n GMs between a & b is equal to the nth power of the single GM between a & b i.e. $\prod_{r=1}^{n} G_r = (G)^n$ where G is the single GM between a & b.

2.5 Arithmetic, Geometric and Harmonic means between two given numbers

Let A, G and H be arithmetic, geometric and harmonic means of two positive numbers a and b. Then,

$$A = \frac{a+b}{2}, G = \sqrt{ab} \text{ and } H = \frac{2ab}{a+b}$$

These three means possess the following properties

1. $A \geq G \geq H$
2. A, G, H form a GP i.e., $G^2 = AH$.
3. The equation having a and b as its roots is $x^2 - 2Ax + G^2 = 0$
4. If A, G, H are arithmetic, geometric and harmonic means between three given numbers a, b and c, then the equation having a, b, c as its roots is

$$x^3 - 3Ax^2 + \frac{3G^2}{H}x - G^3 = 0.$$

Note...

Some important properties of Arithmetic & Geometric Means between two quantities

1. If A and G are respectively arithmetic and geometric means between two positive quantities a and b, then the quadratic equation having a, b as its roots is $x^2 - 2Ax + G^2 = 0$.
2. If A and G be the AM and GM between two positive numbers, then the number are $A \pm \sqrt{A^2 - G^2}$.

3. SIGMA NOTATIONS

3.1 Theorems

(i) $\sum_{r=1}^{n}(a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$

(ii) $\sum_{r=1}^{n} k a_r = k\sum_{r=1}^{n} a_r$

(iii) $\sum_{r=1}^{n} k = k + k + k$........... n times = nk ; where k is a constant.

4. SUM TO n TERMS OF SOME SPECIAL SEQUENCES

4.1 Sum of first n natural numbers

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}.$$

4.2 Sum of the squares of first n natural numbers

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

4.3 Sum of the higher powers of first n natural numbers

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{k=1}^{n} k\right)^2$$

$$\sum_{k=1}^{n} k^4 = \frac{n}{30}(n+1)(2n+1)(3n^2+3n-1)$$

4.4 Sum of first n odd numbers

$$\sum_{k=1}^{n}(2k-1) = 1 + 3 + ... + (2n-1) = n^2$$

5. ARITHMETICO-GEOMETRIC SERIES

A series each term of which is formed by multiplying the corresponding term of an AP & GP is called the Arithmetico-Geometric Series. e.g. $1 + 3x + 5x^2 + 7x^3 +$............... Here, 1, 3, 5, ... are in AP & 1, x, x^2, x^3 .... are in GP.

5.1 Sum of n terms of an Arithmetico-Geometric Series

Let $S_n = a + (a + d) r + (a + 2 d) r^2 + ... + [a + (n - 1) d] r^{n-1}$

then $S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}, r \neq 1.$

5.2 Sum to Infinity

If $|r| < 1$ & $n \to \infty$

then $\lim\limits_{n\to\infty} r^n = 0.\ S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

6. HARMONIC PROGRESSION (HP)

A sequence is said to HP if the reciprocals of its terms are in AP.

If the sequence a_1, a_2, a_3, ..., a_n is an HP then $1/a_1, 1/a_2$, ..., $1/a_n$ is an AP & converse. Here we do not have the formula for the sum of the n terms of an HP. For HP whose first terms is a & second term is b, then n^{th} term is $t_n = \frac{ab}{b+(n-1)(a-b)}$

If a, b, c are in HP $\Rightarrow b = \frac{2ac}{a+c}$ or $\frac{a}{c} = \frac{a-b}{b-c}$.

7. HARMONIC MEAN

If a, b, c are in HP, b is the HM between a & c, then b = 2ac/[a + c].]