16
Simple Harmonic Motion

Periodic and Oscillatory (Vibratory) Motion

(1) A motion, which repeat itself over and over again after a regular interval of time is called a periodic motion

Revolution of earth around the sun (period one year), Rotation of earth about its polar axis (period one day), Motion of hour's hand of a clock (period 12-hour) *etc* are common example of periodic motion.

(2) Oscillatory or vibratory motion is that motion in which a body moves to and fro or back and forth repeatedly about a fixed point in a definite interval of time. In such a motion, the body is confined with in well-defined limits on either side of mean position. Oscillatory motion is also called as harmonic motion.

(i) Common examples are

(a) The motion of the pendulum of a wall clock

(b) The motion of a load attached to a spring, when it is pulled and then released.

(c) The motion of liquid contained in U-tube when it is compressed once in one limb and left to itself.

(d) A loaded piece of wood floating over the surface of a liquid when pressed down and then released executes oscillatory motion.

(ii) Harmonic oscillation is that oscillation which can be expressed in terms of single harmonic function (*i.e.* sine or cosine function). *Example* : $y = a\sin\omega t$ or $y = a\cos\omega t$

(iii) Non-harmonic oscillation is that oscillation which can not be expressed in terms of single harmonic function. It is a combination of two or more than two harmonic oscillations. *Example* : $y = a\sin\omega t + b\sin 2\omega t$.

Simple Harmonic Motion

(1) Simple harmonic motion is a special type of periodic motion, in which a particle moves to and fro repeatedly about a mean position.

(2) In linear S.H.M. a restoring force which is always directed towards the mean position and whose magnitude at any instant is directly proportional to the displacement of the particle from the mean position at that instant *i.e.* Restoring force $\propto$ Displacement of the particle from mean position.

$$F \propto -x \Rightarrow F = -kx$$

Where k is known as force constant. Its S.I. unit is Newton/meter and dimension is $[MT^{-2}]$.

(3) In stead of straight line motion, if particle or centre of mass of body is oscillating on a small arc of circular path, then for angular S.H.M.

Restoring torque $(\tau) \propto -$ Angular displacement (θ)

Some Important Definitions

(1) **Time period** (T) **:** It is the least interval of time after which the periodic motion of a body repeats itself.

S.I. unit of time period is second.

(2) **Frequency** (n) **:** It is defined as the number of oscillations executed by body per second. S.I unit of frequency is hertz (Hz).

(3) **Angular Frequency** (ω) **:** Angular frequency of a body executing periodic motion is equal to product of frequency of the body with factor 2π. Angular frequency $\omega = 2\pi n$

Its unit is *rad/sec.*

(4) **Phase** (ϕ) **:** Phase of a vibrating particle at any instant is a physical quantity, which completely express the position and direction of motion, of the particle at that instant with respect to its mean position.

In oscillatory motion the phase of a vibrating particle is the argument of *sine* or *cosine* function involved to represent the generalised equation of motion of the vibrating particle.

$$y = a\sin\theta = a\sin(\omega t + \phi_0)$$

here, $\theta = \omega t + \phi_0$ = phase of vibrating particle.

ϕ = Initial phase or epoch. It is the phase of a vibrating particle at t = 0.

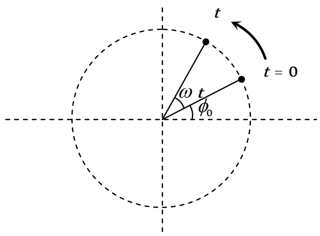


Fig. 16.2

(1) **Same phase :** Two vibrating particle are said to be in same phase, if the phase difference between them is an even multiple of π or path difference is an even multiple of $(\lambda / 2)$ or time interval is an even multiple of $(T / 2)$ because 1 time period is equivalent to 2π *rad* or 1 wave length (λ).

(2) **Opposite phase :** When the two vibrating particles cross their respective mean positions at the same time moving in opposite directions, then the phase difference between the two vibrating particles is 180°.

Opposite phase means the phase difference between the particle is an odd multiple of π (say π, 3π, 5π, 7π.....) or the path difference is an odd multiple of λ (say $\frac{\lambda}{2}, \frac{3\lambda}{2}$,.......) or the time interval is an odd multiple of $(T / 2)$.

(3) **Phase difference :** If two particles performs S.H.M and their equation are

$$y_1 = a\sin(\omega t + \phi_1) \text{ and } y_2 = a\sin(\omega t + \phi_2)$$

then phase difference $\Delta\phi = (\omega t + \phi_2) - (\omega t + \phi_1) = \phi_2 - \phi_1$

Displacement in S.H.M.

(1) The displacement of a particle executing S.H.M. at an instant is defined as the distance of particle from the mean position at that instant.

(2) Simple harmonic motion is also defined as the projection of uniform circular motion on any diameter of circle of reference.

(3) If the projection is taken on *y*-axis. then from the figure

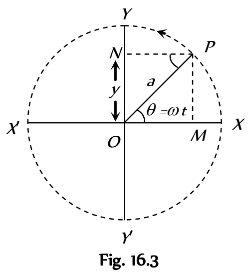


Fig. 16.3

$$y = a\sin\omega t = a\sin\frac{2\pi}{T}t = a\sin 2\pi n t = a\sin(\omega t \pm \phi)$$

(i) $y = a\sin\omega t$ when the time is noted from the instant when the vibrating particle is at mean position.

(ii) $y = a\cos\omega t$ when the time is noted from the instant when the vibrating particle is at extreme position.

(iii) $y = a\sin(\omega t \pm \phi)$ when the vibrating particle is ϕ phase leading or lagging from the mean position.

(4) If the projection of P is taken on X-axis then equations of S.H.M. can be given as

$$x = a\cos(\omega t \pm \phi) = a\cos\left(\frac{2\pi}{T}t \pm \phi\right) = a\cos(2\pi n t \pm \phi)$$

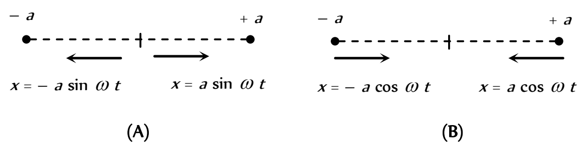


Fig. 16.4

(5) Direction of displacement is always away from the equilibrium position, particle either is moving away from or is coming towards the equilibrium position.

Velocity in S.H.M.

(1) Velocity of the particle executing S.H.M. at any instant, is defined as the time rate of change of its displacement at that instant.

(2) In case of S.H.M. when motion is considered from the equilibrium position, displacement $y = a\sin\omega t$

So $v = \frac{dy}{dt} = a\omega\cos\omega t = a\omega\sqrt{1 - \sin^2\omega t} = \omega\sqrt{a^2 - y^2}$

[As sin$\omega t = y/a$]

(3) At mean position or equilibrium position (y = 0 and $\theta = \omega t = 0$), velocity of particle is maximum and it is $v_{max} = a\omega$.

(4) At extreme position ($y = \pm a$ and $\theta = \omega t = \pi/2$), velocity of oscillating particle is zero *i.e.* v = 0.

(5) From $v = \omega\sqrt{a^2 - y^2} \Rightarrow v^2 = \omega^2(a^2 - y^2) \Rightarrow \frac{v^2}{\omega^2} = a^2 - y^2$

$$\Rightarrow \frac{v^2}{a^2\omega^2} + \frac{y^2}{a^2} = 1$$

This is the equation of ellipse. Hence the graph between v and y is an ellipse.

For ω = 1, graph between v and y is a circle.

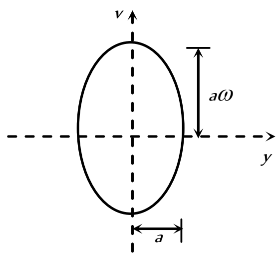


Fig. 16.5

(6) Direction of velocity is either towards or away from mean position depending on the position of particle.

Acceleration in S.H.M.

(1) The acceleration of the particle executing S.H.M. at any instant, is defined as the rate of change of its velocity at that instant. So acceleration

$$A = \frac{dv}{dt} = \frac{d}{dt}(a\omega\cos\omega t) = -\omega^2 a\sin\omega t = -\omega^2 y$$

[As $y = a\sin\omega t$]

(2) In S.H.M. as $|\text{Acceleration}| = \omega^2 y$ is not constant. So equations of translatory motion can not be applied.

(3) In S.H.M. acceleration is maximum at extreme position (at $y = \pm a$).

Hence $|A_{max}| = \omega^2 a$ when $|\sin\omega t| = \text{maximum} = 1$ *i.e.* at $t = \frac{T}{4}$ or $\omega t = \frac{\pi}{2}$. From equation (ii) $|A_{max}| = \omega^2 a$ when $y = a$.

(i) In S.H.M. acceleration is minimum at mean position

From equation (i) $A_{min} = 0$ when $\sin\omega t = 0$ *i.e.* at $t = 0$ or $t = \frac{T}{2}$ or $\omega t = \pi$. From equation (ii) $A_{min} = 0$ when $y = 0$

(ii) Acceleration is always directed towards the mean position and so is always opposite to displacement

i.e., $A \propto -y$

Graph between acceleration (A) and displacement (y) is a straight line as shown

Slope of the line = $-\omega$

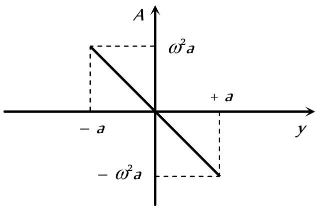


Fig. 16.6

Comparative Study of Displacement Velocity and Acceleration

(1) All the three quantities displacement, velocity and acceleration show harmonic variation with time having same period.

(2) The velocity amplitude is ω times the displacement amplitude

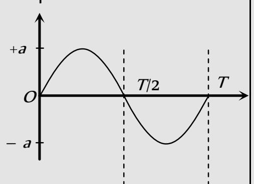
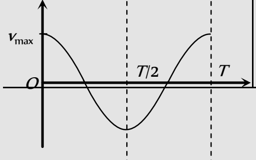
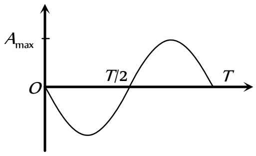
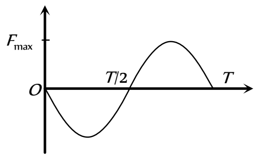
(3) The acceleration amplitude is ω^2 times the displacement amplitude

(4) In S.H.M. the velocity is ahead of displacement by a phase angle $\pi / 2$

(5) In S.H.M. the acceleration is ahead of velocity by a phase angle $\pi / 2$

(6) The acceleration is ahead of displacement by a phase angle of π

Table 16.1 : Various physical quantities in S.H.M. at different position :

Graph	Formula	At mean position	At extreme position
Displacement 	$y = a\sin\omega t$	$y = 0$	$y = \pm a$
Velocity 	$v = a\omega\cos\omega t$ $= a\omega\sin(\omega t + \frac{\pi}{2})$ or $v = \omega\sqrt{a^2 - y^2}$	$v_{max} = a\omega$	$v_{min} = 0$
Acceleration 	$A = -a\omega^2\sin\omega t$ $= a\omega^2\sin(\omega t + \pi)$ or $\lvert A\rvert = \omega^2 y$	$A_{min} = 0$	$\lvert A_{max}\rvert = \omega^2 a$
Force 	$F = -m\omega^2 a\sin\omega t$ or $F = m\omega^2 y$	$F_{min} = 0$	$F_{max} = m\omega^2 a$

Energy in S.H.M.

(1) **Potential energy :** This is an account of the displacement of the particle from its mean position.

(i) The restoring force $F = -ky$ against which work has to be done. Hence potential energy U is given by

$$U = \int dU = -\int dW = -\int_0^x F dx = \int_0^y ky\, dy = \frac{1}{2}ky^2 + U_0$$

where U_0 = Potential energy at equilibrium position.

If $U_0 = 0$ then $U = \frac{1}{2}m\omega^2 y^2$ [As $\omega^2 = k/m$]

(ii) Also $U = \frac{1}{2}m\omega^2 a^2\sin^2\omega t = \frac{1}{4}m\omega^2 a^2(1 - \cos 2\omega t)$

[As $y = a\sin\omega t$]

Hence potential energy varies periodically with double the frequency of S.H.M.

(iii) Potential energy maximum and equal to total energy at extreme positions

$$U_{max} = \frac{1}{2}ka^2 = \frac{1}{2}m\omega^2 a^2 \text{ when } y = \pm a;\ \omega t = \pi/2;\ t = \frac{T}{4}$$

(iv) Potential energy is minimum at mean position

$$U_{min} = 0 \quad \text{when } y = 0;\ \omega t = 0;\ t = 0$$

(2) **Kinetic energy :** This is because of the velocity of the particle

Kinetic Energy $K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(a^2 - y^2)$

[As $v = \omega\sqrt{a^2 - y^2}$]

(i) Also $K = \frac{1}{2}m\omega^2 a^2\cos^2\omega t = \frac{1}{4}m\omega^2 a^2(1 + \cos 2\omega t)$

[As $v = a\omega\cos\omega t$]

Hence kinetic energy varies periodically with double the frequency of S.H.M.

(ii) Kinetic energy is maximum at mean position and equal to total energy at mean position.

$$K_{max} = \frac{1}{2} m\omega^2 a^2 \text{ when } y = 0\,;\; t = 0\,;\; \omega t = 0$$

(iii) Kinetic energy is minimum at extreme position.

$$K_{min} = 0 \qquad \text{when } y = a\,;\;\; t = T/4\,,\; \omega t = \pi/2$$

(3) **Total mechanical energy :** Total mechanical energy always remains constant and it is equal to sum of potential energy and kinetic energy *i.e.* $E = U + K$

$$E = \frac{1}{2} m\omega^2 (a^2 - y^2) + \frac{1}{2} m\omega^2 y^2 = \frac{1}{2} m\omega^2 a^2$$

Total energy is not a position function.

(4) **Energy position graph**

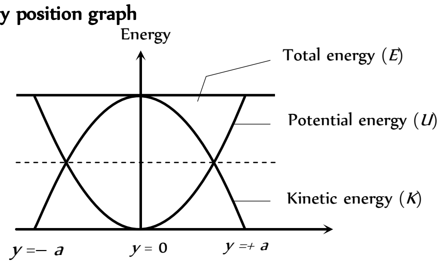


Fig. 16.7

(i) At $y = 0$; $U = 0$ and $K = E$

(ii) At $y = \pm a$; $U = E$ and $K = 0$

(iii) At $y = \pm \frac{a}{2}$; $U = \frac{E}{4}$ and $K = \frac{3E}{4}$

(iv) At $y = \pm \frac{a}{\sqrt{2}}$; $U = K = \frac{E}{2}$

Average Value of P.E. and K.E.

The average value of potential energy for complete cycle is given by

$$U_{average} = \frac{1}{T}\int_0^T U\,dt = \frac{1}{T}\int_0^T \frac{1}{2} m\,\omega^2 a^2 \sin^2(\omega t + \phi) = \frac{1}{4} m\,\omega^2 a^2$$

The average value of kinetic energy for complete cycle

$$K_{average} = \frac{1}{T}\int_0^T K\,dt = \frac{1}{T}\int_0^T \frac{1}{2} m\,\omega^2 a^2 \cos^2 \omega t\,dt = \frac{1}{4} m\,\omega^2 a^2$$

Thus average values of kinetic energy and potential energy of harmonic oscillator are equal and each equal to half of the total energy $K_{average} = U_{average} = \frac{1}{2} E = \frac{1}{4} m\omega^2 a^2$.

Differential Equation of S.H.M.

For S.H.M. (linear) Acceleration $\propto$ – (Displacement)

$$A \propto -y \quad \text{or} \quad A = -\omega^2 y \quad \text{or} \quad \frac{d^2 y}{dt^2} = -\omega^2 y$$

$$\text{or} \quad m\frac{d^2 y}{dt^2} + ky = 0 \qquad \left[\text{As } \omega = \sqrt{\frac{k}{m}}\,\right]$$

For angular S.H.M. $\quad \tau = -c\theta \quad$ and $\quad \frac{d^2\theta}{dt^2} + \omega^2\theta = 0$

where $\omega^2 = \frac{c}{I}$ [As c = Restoring torque constant and I = Moment of inertia]

How to Find Frequency and Time Period of S.H.M.

Step 1 : When particle is in its equilibrium position, balance all forces acting on it and locate the equilibrium position mathematically.

Step 2 : From the equilibrium position, displace the particle slightly by a displacement y and find the expression of net restoring force on it.

Step 3 : Try to express the net restoring force acting on particle as a proportional function of its displacement from mean position. The final expression should be obtained in the form.

$$F = -ky$$

Here we put – *ve* sign as direction of F is opposite to the displacement y. If a be the acceleration of particle at this displacement, we have $a = -\left(\frac{k}{m}\right) y$

Step 4 : Comparing this equation with the basic differential equation of S.H.M. we get $\omega^2 = \frac{k}{m} \Rightarrow \omega = \sqrt{\frac{k}{m}}$ or $n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

As ω is the angular frequency of the particle in S.H.M., its time period of oscillation can be given as $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

(i) In different types of S.H.M. the quantities m and k will go on taking different forms and names. In general m is called inertia factor and k is called spring factor.

Thus $T = 2\pi\sqrt{\frac{\text{Inertia factor}}{\text{Spring factor}}}$ or $n = \frac{1}{2\pi}\sqrt{\frac{\text{Spring factor}}{\text{Inertia factor}}}$

(ii) In linear S.H.M. the spring factor stands for force per unit displacement and inertia factor for mass of the body executing S.H.M. and in Angular S.H.M. k stands for restoring torque per unit angular displacement and inertial factor for moment of inertia of the body executing S.H.M.

For linear S.H.M.

$$T = 2\pi\sqrt{\frac{m}{k}} = \sqrt{\frac{m}{\text{Force/Displacement}}} = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}$$

Simple Pendulum

(1) An ideal simple pendulum consists of a heavy point mass body (bob) suspended by a weightless, inextensible and perfectly flexible string from a rigid support about which it is free to oscillate.

(2) But in reality neither point mass nor weightless string exist, so we can never construct a simple pendulum strictly according to the definition.

(3) Suppose simple pendulum of length l is displaced through a small angle θ from it's mean (vertical) position. Consider mass of the bob is m and linear displacement from mean position is x

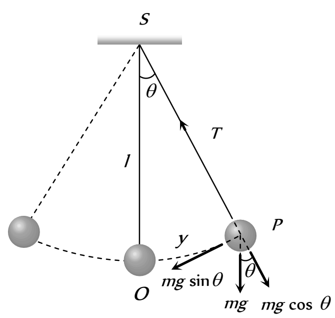


Fig. 16.8

Restoring force acting on the bob

$F = -mg\sin\theta$ or $F = -mg\theta = -mg\dfrac{x}{l}$

(When θ is small $\sin\theta \simeq \theta = \dfrac{\text{Arc}}{\text{Length}} = \dfrac{OP}{l} = \dfrac{x}{l}$)

$\therefore \dfrac{F}{x} = \dfrac{-mg}{l} = k$ (Spring factor)

So $T = 2\pi\sqrt{\dfrac{\text{Inertia factor}}{\text{Spring factor}}} = 2\pi\sqrt{\dfrac{m}{mg/l}} = 2\pi\sqrt{\dfrac{l}{g}}$

Factor Affecting Time Period of Simple Pendulum

(1) **Amplitude :** The period of simple pendulum is independent of amplitude as long as its motion is simple harmonic. But if θ is not small, *sin* $\theta \neq \theta$ then motion will not remain simple harmonic but will become oscillatory. In this situation if θ_0 is the amplitude of motion. Time period

$$T = 2\pi\sqrt{\frac{l}{g}}\left[1 + \frac{1}{2^2}\sin^2\left(\frac{\theta_0}{2}\right) + ...\right] \approx T_0\left[1 + \frac{\theta_0{}^2}{16}\right]$$

(2) **Mass of the bob :** Time period of simple pendulum is also independent of mass of the bob. This is why

(i) If the solid bob is replaced by a hollow sphere of same radius but different mass, time period remains unchanged.

(ii) If a girl is swinging in a swing and another sits with her, the time period remains unchanged.

(3) **Length of the pendulum :** Time period $T \propto \sqrt{l}$ where *l* is the distance between point of suspension and center of mass of bob and is called effective length.

(i) When a sitting girl on a swinging swing stands up, her center of mass will go up and so *l* and hence *T* will decrease.

(ii) If a hole is made at the bottom of a hollow sphere full of water and water comes out slowly through the hole and time period is recorded till the sphere is empty, initially and finally the center of mass will be at the center of the sphere. However, as water drains off the sphere, the center of mass of the system will first move down and then will come up. Due to this *l* and hence *T* first increase, reaches a maximum and then decreases till it becomes equal to its initial value.

(iii) Different graphs

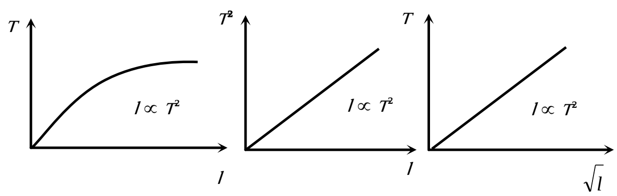


Fig. 16.9

(4) **Effect of *g* :** $T \propto \dfrac{1}{\sqrt{g}}$ *i.e.* as *g* increase *T* decreases.

(i) As we go high above the earth surface or we go deep inside the mines the value of *g* decrease, hence time period of pendulum (*T*) increases.

(ii) If a clock, based on simple pendulum is taken to hill (or on any other planet), *g* will decrease so *T* will increases and clock will become slower.

(iii) Different graphs

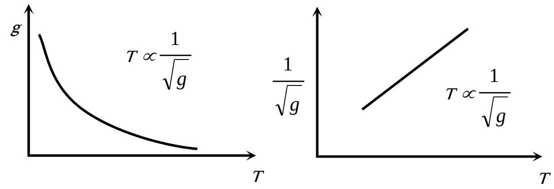


Fig. 16.10

(5) **Effect of temperature on time period :** If the bob of simple pendulum is suspended by a wire then effective length of pendulum will increase with the rise of temperature due to which the time period will increase.

$l = l_0(1 + \alpha\Delta\theta)$ (If $\Delta\theta$ is the rise in temperature, l_0 = initial length of wire, *l* = final length of wire)

$$\frac{T}{T_0} = \sqrt{\frac{l}{l_0}} = (1 + \alpha\Delta\theta)^{1/2} \approx 1 + \frac{1}{2}\alpha\Delta\theta$$

So $\dfrac{T}{T_0} - 1 = \dfrac{1}{2}\alpha\Delta\theta$ *i.e.* $\dfrac{\Delta T}{T} \approx \dfrac{1}{2}\alpha\Delta\theta$

Oscillation of Pendulum in Different Situations

(1) **Oscillation in liquid :** If bob a simple pendulum of density ρ is made to oscillate in some fluid of density σ (where $\sigma < \rho$) then time period of simple pendulum gets increased.

As thrust will oppose its weight hence $mg_{eff.} = mg -$ Thrust

or $g_{eff.} = g - \dfrac{V\sigma g}{V\rho}$ *i.e.* $g_{eff.} = g\left[1 - \dfrac{\sigma}{\rho}\right]$

$\Rightarrow \dfrac{g_{eff.}}{g} = \dfrac{\rho - \sigma}{\rho}$

$\Rightarrow \dfrac{T'}{T} = \sqrt{\dfrac{g}{g_{eff.}}} = \sqrt{\dfrac{\rho}{\rho - \sigma}} > 1$

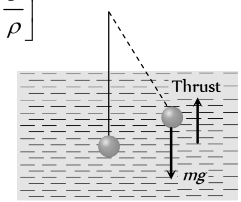


Fig. 16.11

(2) **Oscillation under the influence of electric field :** If a bob of mass *m* carries a positive charge *q* and pendulum is placed in a uniform electric field of strength *E*

(i) If electric field directed vertically upwards.

Effective acceleration

$g_{eff.} = g - \dfrac{qE}{m}$

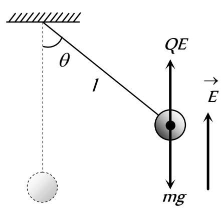


Fig. 16.12

So $T = 2\pi\sqrt{\dfrac{l}{g - \dfrac{qE}{m}}}$

(ii) If electric field is vertically downward then

$$g_{eff.} = g + \frac{qE}{m}$$

$$T = 2\pi\sqrt{\frac{l}{g + \dfrac{qE}{m}}}$$

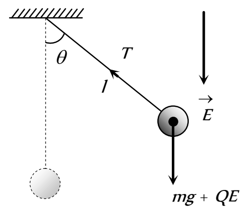


Fig. 16.13

(3) **Pendulum in a lift :** If the pendulum is suspended from the ceiling of the lift.

(i) If the lift is at rest or moving down ward /up ward with constant velocity.

$$T = 2\pi\sqrt{\frac{l}{g}}$$

and $n = \dfrac{1}{2\pi}\sqrt{\dfrac{g}{l}}$

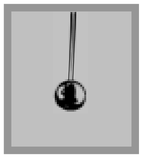


Fig. 16.14

(ii) If the lift is moving up ward with constant acceleration a

$$T = 2\pi\sqrt{\frac{l}{g+a}}$$

and $n = \dfrac{1}{2\pi}\sqrt{\dfrac{g+a}{l}}$

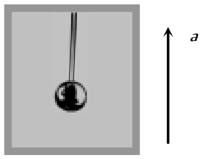


Fig. 16.15

Time period decreases and frequency increases

(iii) If the lift is moving down ward with constant acceleration a

$$T = 2\pi\sqrt{\frac{l}{g-a}}$$

and $n = \dfrac{1}{2\pi}\sqrt{\dfrac{g-a}{l}}$

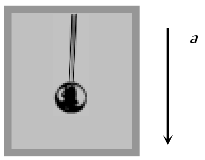


Fig. 16.16

Time period increase and frequency decreases

(iv) If the lift is moving down ward with acceleration $a = g$

$$T = 2\pi\sqrt{\frac{l}{g-g}} = \infty$$

and $n = \dfrac{1}{2\pi}\sqrt{\dfrac{g-g}{l}} = 0$

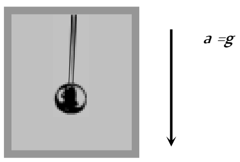


Fig. 16.17

It means there will be no oscillation in a pendulum.

Similar is the case in a satellite and at the centre of earth where effective acceleration becomes zero and pendulum will stop.

(4) **Pendulum in an accelerated vehicle :** The time period of simple pendulum whose point of suspension moving horizontally with acceleration a

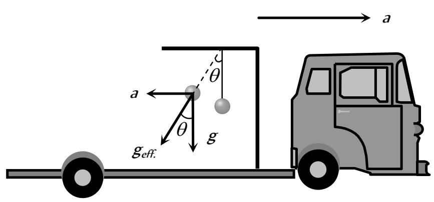


Fig. 16.18

In this case effective acceleration $g_{eff.} = \sqrt{g^2 + a^2}$

$$T = 2\pi\sqrt{\frac{l}{(g^2 + a^2)^{1/2}}} \quad \text{and} \quad \theta = \tan^{-1}(a/g)$$

If simple pendulum suspended in a car that is moving with constant speed v around a circle of radius r.

$$T = 2\pi\frac{\sqrt{l}}{\sqrt{g^2 + \left(\dfrac{v^2}{r}\right)^2}}$$

Some Other Types of Pendulum

(1) **Infinite length pendulum :** If the length of the pendulum is comparable to the radius of earth then

$$T = 2\pi\sqrt{\frac{1}{g\left[\dfrac{1}{l} + \dfrac{1}{R}\right]}}$$

(i) If $l << R$, then $\dfrac{1}{l} >> \dfrac{1}{R}$ so $T = 2\pi\sqrt{\dfrac{l}{g}}$

(ii) If $l >> R(\to \infty)$ then $\dfrac{1}{l} < \dfrac{1}{R}$

so $T = 2\pi\sqrt{\dfrac{R}{g}} = 2\pi\sqrt{\dfrac{6.4\times10^6}{10}} \cong 84.6$ *minutes*

and it is the maximum time period which an oscillating simple pendulum can have

(iii) If $l = R$ so $T = 2\pi\sqrt{\dfrac{R}{2g}} \cong 1 hour$

(2) **Second's Pendulum :** It is that simple pendulum whose time period of vibrations is two seconds.

Putting $T = 2$ *sec* and $g = 9.8 m/\sec^2$ in $T = 2\pi\sqrt{\dfrac{l}{g}}$ we get

$l = \dfrac{4\times9.8}{4\pi^2} = 0.993$ *m* = 99.3 *cm*

Hence length of second's pendulum is 99.3 *cm* or nearly 1 meter on earth surface.

For the moon the length of the second's pendulum will be 1/6 meter

[As $g_{moon} = \dfrac{g_{\text{Earth}}}{6}$]

(3) **Compound pendulum :** Any rigid body suspended from a fixed support constitutes a physical pendulum. Consider the situation when the body is displaced through a small angle θ. Torque on the body about O is given by

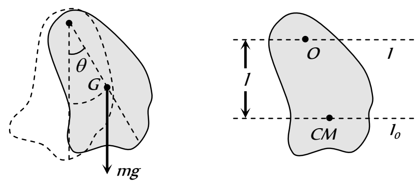


Fig. 16.19

$$\tau = mgl\sin\theta \quad ...(i)$$

where l = distance between point of suspension and centre of mass of the body.

If I be the M.I. of the body about O. Then $\tau = I\alpha$ ...(ii)

From (i) and (ii), we get $I\frac{d^2\theta}{dt^2} = -mgl\sin\theta$ as θ and $\frac{d^2\theta}{dt^2}$ are oppositely directed $\Rightarrow \frac{d^2\theta}{dt^2} = -\frac{mgl}{I}\theta$ since θ is very small

Comparing with the equation $\frac{d^2\theta}{dt^2} = -\omega^2\theta$, we get

$$\omega = \sqrt{\frac{mgl}{I}} \Rightarrow T = 2\pi\sqrt{\frac{I}{mgl}}$$

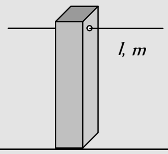
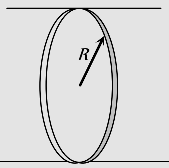
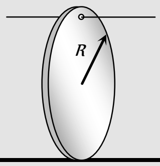
Also $I = I_{cm} + ml^2$ (Parallel axis theorem)

$= mk^2 + ml^2$ (where k = radius of gyration)

$$\therefore\ T = 2\pi\sqrt{\frac{mK^2 + ml^2}{mgl}} = 2\pi\sqrt{\frac{\frac{K^2}{l} + l}{g}} = 2\pi\sqrt{\frac{l_{eff}}{g}}$$

l_{eff} = Effective length of pendulum = Distance between point of suspension and centre of mass.

Table 16. 2: Some common physical pendulum

Body	Time period
Bar 	$T = 2\pi\sqrt{\frac{2l}{3g}}$
Ring 	$T = 2\pi\sqrt{\frac{2R}{g}}$
Disc 	$T = 2\pi\sqrt{\frac{3R}{2g}}$

Spring System

When a spring is stretched or compressed from its normal position (x = 0) by a small distance x, then a restoring force is produced in the spring because it obeys Hook's law

i.e. $F \propto -x \Rightarrow F = -kx$

where k is called spring constant.

(i) It's S.I. unit *Newton/metre*, C.G.S unit *Dyne/cm* and dimension is $[MT]$

(ii) Actually k is a measure of the stiffness/softness of the spring.

(iii) For massless spring constant restoring elastic force is same every where

(iv) When a spring compressed or stretched then work done is stored in the form of elastic potential energy in it.

(v) Spring constant depend upon radius and length of the wire used in spring.

(vi) The spring constant k is inversely proportional to the spring length.

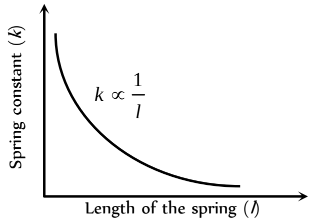


Fig. 16.20

$$k \propto \frac{1}{\text{Extension}} \propto \frac{1}{\text{Length of spring}}$$

That means if the length of spring is halved then its force constant becomes double.

(vii) When a spring of length l is cut in two pieces of length l_1 and l_2 such that $l_1 = nl_2$.

If the constant of a spring is k then spring constant of first part

$$k_1 = \frac{k(n+1)}{n}$$

Spring constant of second part $k_2 = (n+1)k$

and ratio of spring constant $\frac{k_1}{k_2} = \frac{1}{n}$

Spring Pendulum

A point mass suspended from a mass less spring or placed on a frictionless horizontal plane attached with spring (fig.) constitutes a linear harmonic spring pendulum

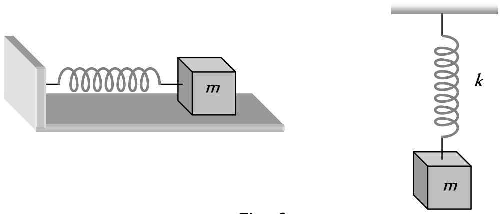


Fig. 16.21

Time period $T = 2\pi\sqrt{\frac{\text{Inertia factor}}{\text{Spring factor}}} = 2\pi\sqrt{\frac{m}{k}}$

and Frequency $n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

(1) Time period of a spring pendulum depends on the mass suspended

$\Rightarrow T \propto \sqrt{m}$ or $n \propto \frac{1}{\sqrt{m}}$ *i.e.* greater the mass greater will be the inertia and so lesser will be the frequency of oscillation and greater will be the time period.

(2) The time period depends on the force constant k of the spring *i.e.* $T \propto \frac{1}{\sqrt{k}}$ or $n \propto \sqrt{k}$

(3) Time of a spring pendulum is independent of acceleration due to gravity. That is why a clock based on spring pendulum will keep proper time every where on a hill or moon or in a satellite and time period of a spring pendulum will not change inside a liquid if damping effects are neglected.

(4) **Massive spring :** If the spring has a mass M and mass m is suspended from it, effective mass is given by $m_{eff} = m + \frac{M}{3}$. Hence

$$T = 2\pi\sqrt{\frac{m_{eff}}{k}}$$

(5) **Reduced mass :** If two masses of mass m_1 and m_2 are connected by a spring and made to oscillate on horizontal surface, the reduced mass m_r is given by $\frac{1}{m_r} = \frac{1}{m_1} + \frac{1}{m_2}$ so that

$$T = 2\pi\sqrt{\frac{m_r}{k}}$$

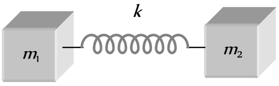


Fig. 16.22

(6) If a spring pendulum, oscillating in a vertical plane is made to oscillate on a horizontal surface, (or on inclined plane) time period will remain unchanged.

(7) Equilibrium position for a spring in a horizontal plain is the position of natural length of spring as weight is balanced by reaction. While in case of vertical motion equilibrium position will be $l + y_0$ with $ky_0 = mg$

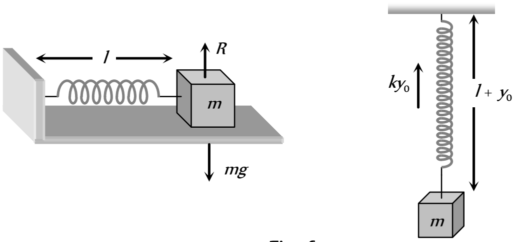


Fig. 16.23

If the stretch in a vertically loaded spring is y_0 then for equilibrium of mass m, $ky_0 = mg$ *i.e.* $\frac{m}{k} = \frac{y_0}{g}$

So that $$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{y_0}{g}}$$

Time period does not depends on '*g*' because along with *g*, y_0 will also change in such a way that $\frac{y_0}{g} = \frac{m}{k}$ remains constant

Oscillation of Spring Combination

(1) **Series combination :** If two springs of spring constants K_1 and K_2 are joined in series as shown then

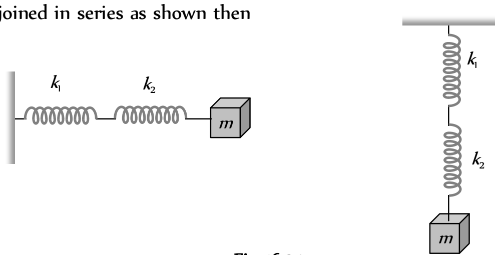


Fig. 16.24

(i) In series combination equal forces acts on spring but extension in springs are different.

(ii) Spring constants of combination

$$\frac{1}{k_s} = \frac{1}{k_1} + \frac{1}{k_2} \Rightarrow k_s = \frac{k_1 k_2}{k_1 + k_2}$$

(iii) If n springs of different force constant are connected in series having force constant k_1, k_2, k_3....... respectively then

$$\frac{1}{k_S} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \text{........}$$

If all spring have same spring constant then $k_S = \frac{k}{n}$

(iv) Time period of combination $T = 2\pi\sqrt{\frac{m}{k_S}} = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$

(2) **Parallel combination :** If the springs are connected in parallel as shown

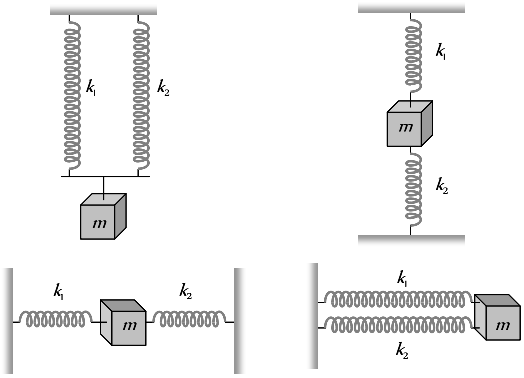


Fig. 16.25

(i) In parallel combination different forces acts on different springs but extension in springs are same

(ii) Spring constants of combination $k_P = k_1 + k_2$

(iii) If n springs of different force constant are connected in parallel having force constant k_1, k_2, k_3....... respectively then $k_P = k_1 + k_2 + k_3 + ...$

If all spring have same spring constant then $k_P = nk$

(iv) Time period of combination $T_P = 2\pi\sqrt{\frac{m}{k_P}} = 2\pi\sqrt{\frac{m}{(k_1 + k_2)}}$

Various Formulae of S.H.M.

(1) **S.H.M. of a liquid in U tube :** If a liquid of density ρ contained in a vertical U tube performs S.H.M. in its two limbs. Then time period

$$T = 2\pi\sqrt{\frac{L}{2g}} = 2\pi\sqrt{\frac{h}{g}}$$

where L = Total length of liquid column,

h = Height of undisturbed liquid in each limb (L=2h)

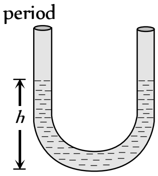


Fig. 16.26

(2) **S.H.M. of a floating cylinder :** If l is the length of cylinder dipping in liquid then

Time period $T = 2\pi\sqrt{\frac{l}{g}}$

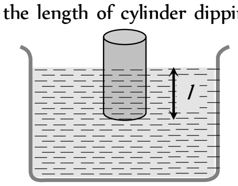


Fig. 16.27

(3) **S.H.M. of a small ball rolling down in hemi-spherical bowl**

$$T = 2\pi\sqrt{\frac{R - r}{g}}$$

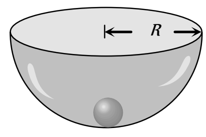


Fig. 16.28

R = Radius of the bowl

r =Radius of the ball

(4) **S.H.M. of a piston in a cylinder**

$$T = 2\pi\sqrt{\frac{Mh}{PA}}$$

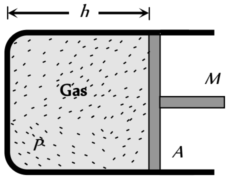


Fig. 16.29

M = mass of the piston

A = area of cross section

h = height of cylinder

P = pressure in a cylinder

(5) **S.H.M. of a body in a tunnel dug along any chord of earth**

$$T = 2\pi\sqrt{\frac{R}{g}} = 84.6 \text{ minutes}$$

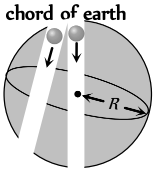


Fig. 16.30

(6) **Torsional pendulum :** In a torsional pendulum an object is suspended from a wire. If such a wire is twisted, due to elasticity it exert a restoring toque $\tau = C\theta$.

In this case time period is given by

$$T = 2\pi\sqrt{\frac{I}{C}}$$

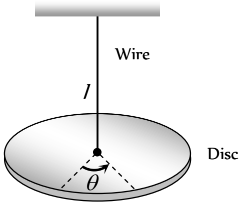


Fig. 16.31

where I = Moment of inertia a disc

C = Torsional constant of wire = $\dfrac{\pi\eta r^4}{2l}$

η = Modulus of elasticity of wire and r = Radius of wire

(7) **Longitudinal oscillations of an elastic wire :** Wire/string pulled a distance Δl and left. It executes longitudinal oscillations. Restoring force

$$F = -AY\left(\frac{\Delta l}{l}\right)$$

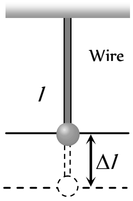


Fig. 16.32

Y = Young's modulus

A = Area of cross-section

Hence $T = 2\pi\sqrt{\dfrac{m}{k}} = 2\pi\sqrt{\dfrac{ml}{AY}}$

Free, Damped, Forced and Maintained Oscillations

(1) **Free oscillation**

(i) The oscillation of a particle with fundamental frequency under the influence of restoring force are defined as free oscillations

(ii) The amplitude, frequency and energy of oscillation remains constant

(iii) Frequency of free oscillation is called natural frequency because it depends upon the nature and structure of the body.

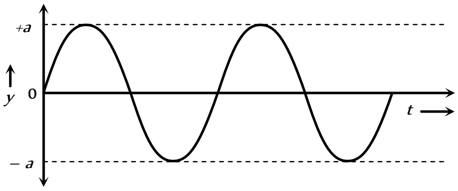


Fig. 16.33

(2) **Damped oscillation**

(i) The oscillation of a body whose amplitude goes on decreasing with time are defined as damped oscillation

(ii) In these oscillation the amplitude of oscillation decreases exponentially due to damping forces like frictional force, viscous force, hystersis *etc.*

(iii) Due to decrease in amplitude the energy of the oscillator also goes on decreasing exponentially

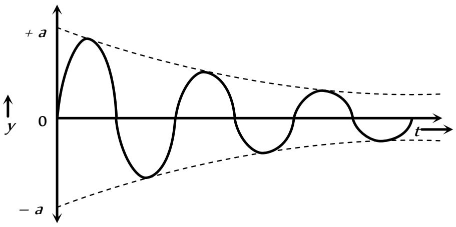


Fig. 16.34

(iv) The force produces a resistance to the oscillation is called damping force.

If the velocity of oscillator is v then

Dumping force $F_d = -bv$, b = damping constant

(v) Resultant force on a damped oscillator is given by

$$F = F_R + F_d = -Kx - Kv \Rightarrow \frac{md^2x}{dt^2} + b\frac{dx}{dt} + Kx = 0$$

(vi) Displacement of damped oscillator is given by

$x = x_m e^{-bt/2m}\sin(\omega' t + \phi)$ where ω' = angular frequency of the damped oscillator = $\sqrt{\omega_0^2 - (b/2m)^2}$

The amplitude decreases continuously with time according to

$$x = x_m e^{-(b/2m)t}$$

(vii) For a damped oscillator if the damping is small then the mechanical energy decreases exponentially with time as

$$E = \frac{1}{2}Kx_m^2 e^{-bt/m}$$

(3) **Forced oscillation**

(i) The oscillation in which a body oscillates under the influence of an external periodic force are known as forced oscillation

(ii) The amplitude of oscillator decrease due to damping forces but on account of the energy gained from the external source it remains constant.

(iii) Resonance : When the frequency of external force is equal to the natural frequency of the oscillator. Then this state is known as the state of resonance. And this frequency is known as resonant frequency.

(iv) While swinging in a swing if you apply a push periodically by pressing your feet against the ground, you find that not only the oscillations can now be maintained but the amplitude can also be increased. Under this condition the swing has forced or driven oscillation.

(v) In forced oscillation, frequency of damped oscillator is equal to the frequency of external force.

(vi) Suppose an external driving force is represented by

$F(t) = F_0 \cos \omega_d t$

The motion of a particle under combined action of

(a) Restoring force $(-Kx)$

(b) Damping force $(-bv)$ and

(c) Driving force $F(t)$ is given by $ma = -Kx - bv + F_0 \cos \omega_d t$

$$\Rightarrow m^2 \frac{d^2x}{d^2} + Kx + b\frac{dx}{dt} = F_0 \cos \omega_d t$$

The solution of this equation gives $x = x_0 \sin(\omega_d t + \phi)$ with amplitude $x_0 = \dfrac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2) + (b\omega/m)^2}}$ and $\tan\theta = \dfrac{(\omega^2 - \omega_0^2)}{b\omega/m}$

where $\omega_0 = \sqrt{\dfrac{K}{m}}$ = Natural frequency of oscillator.

(vii) **Amplitude resonance :** The amplitude of forced oscillator depends upon the frequency ω_d of external force.

When $\omega = \omega_d$, the amplitude is maximum but not infinite because of presence of damping force. The corresponds frequency is called resonant frequency (ω_0).

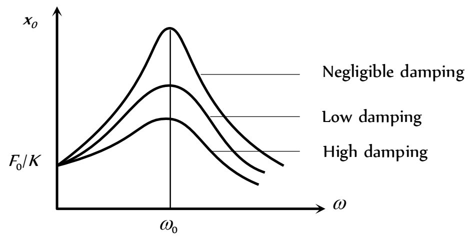


Fig. 16.35

(viii) **Energy resonance :** At $\omega = \omega_0$, oscillator absorbs maximum kinetic energy from the driving force system this state is called energy resonance.

At resonance the velocity of a driven oscillator is in phase with the driving term.

The sharpness of the resonance of a driven oscillator depends on the damping.

In the driven oscillator, the power input of the driving term in maximum at resonance.

(4) **Maintained oscillation :** The oscillation in which the loss of oscillator is compensated by the supplying energy from an external source are known as maintained oscillation.

Super Position of S.H.M's (Lissajous Figures)

If two S.H.M's act in perpendicular directions, then their resultant motion is in the form of a straight line or a circle or a parabola etc. depending on the frequency ratio of the two S.H.M. and initial phase difference. These figures are called Lissajous figures.

Let the equations of two mutually perpendicular S.H.M's of same frequency be

$$x = a_1 \sin\omega t \text{ and } y = a_2 \sin(\omega t + \phi)$$

then the general equation of Lissajou's figure can be obtained as

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1 a_2}\cos\phi = \sin^2\phi$$

For ϕ = 0° : $\dfrac{x^2}{a_1^2} + \dfrac{y^2}{a_2^2} - \dfrac{2xy}{a_1 a_2} = 0 \Rightarrow \left(\dfrac{x}{a_1} - \dfrac{y}{a_2}\right)^2 = 0$

$$\Rightarrow \frac{x}{a_1} = \frac{y}{a_2} \Rightarrow y = \frac{a_2}{a_1}x$$

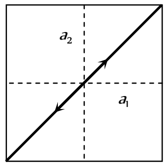
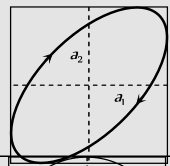
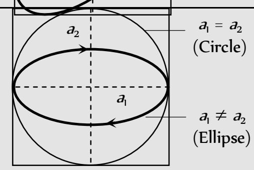
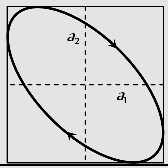
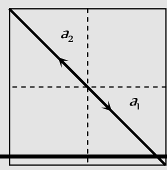


Fig. 16.36

This is a straight line passes through origin and it's slope is $\dfrac{a_2}{a_1}$.

Table 16.3 : Lissajou's figures in other conditions

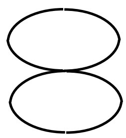
(with $\dfrac{\omega_1}{\omega_2} = 1$)

Phase diff.(ϕ)	Equation	Figure
$\dfrac{\pi}{4}$	$\dfrac{x^2}{a_1^2} + \dfrac{y^2}{a_2^2} - \dfrac{\sqrt{2}xy}{a_1 a_2} = \dfrac{1}{2}$	Oblique ellipse 
$\dfrac{\pi}{2}$	$\dfrac{x^2}{a_1^2} + \dfrac{y^2}{a_2^2} = 1$	
$\dfrac{3\pi}{4}$	$\dfrac{x^2}{a_1^2} + \dfrac{y^2}{a_2^2} - \dfrac{\sqrt{2}xy}{a_1 a_2} = \dfrac{1}{2}$	Oblique ellipse 
π	$\dfrac{x}{a_1} + \dfrac{y}{a_2} = 0$ $\Rightarrow y = -\dfrac{a_2}{a_1}x$	Straight line 

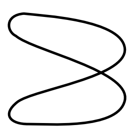
For the frequency ratio $\omega_1 : \omega_2 = 2 : 1$ the two perpendicular S.H.M's are

$$x = a_1 \sin(\omega t + \phi) \text{ and } y = a_2 \sin\omega t$$

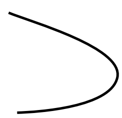
Different Lissajou's figures as follows



ϕ = 0, π, 2π
Figure of eight



ϕ = $\pi/4$, $3\pi/4$
Double parabola



ϕ = $\pi/2$
Parabola

$\phi = 5\pi/4,\ 7\pi/4$
Double parabola

$\phi = 3\pi/2$
Parabola

Fig. 16.37

Tips & Tricks

✍ Suppose a body of mass m vibrate separately with two different springs (of spring constants k_1 and k_2) with time period T_1 and T_2 respectively. $T_1 = 2\pi\sqrt{\frac{m}{k_1}}$ and $T_2 = 2\pi\sqrt{\frac{m}{k_2}}$

If the same body vibrates with series combination of these two springs then for the system time period $T = \sqrt{T_1^2 + T_2^2}$

If the same body vibrates with parallel combination of these two springs then time period of the system $T = \frac{T_1 T_2}{\sqrt{T_1^2 + T_2^2}}$

✍ The pendulum clock runs slow due to increase in its time period whereas it becomes fast due to decrease in time period.

✍ If infinite spring with force constant $k, 2k, 4k, 8k$ .... respectively are connected in series. The effective force constant of the spring will be $k/2$.

✍ Percentage change in time period with l and g.

If g is constant and length varies by n%. Then % change in time period $\frac{\Delta T}{T}\times 100 = \frac{n}{2}\times 100$

If l is constant and g varies by n%. Then % change in time period $\frac{\Delta T}{T}\times 100 = -\frac{n}{2}\times 100$

(Valid only for small percentage change say 5%).

✍ Suppose a spring of force constant k oscillates with time period T. If it is divided in to n equal parts then spring constant of each part will become nk and time period of oscillation of each part will become $\frac{T}{\sqrt{n}}$.

If these n parts connected in parallel then $k_{eff} = n^2 k$. So time period of the system becomes $T' = \frac{T}{n}$

✍ If a particle performs S.H.M. whose velocity is v_1 at a x_1 distance from mean position and velocity v_2 at distance x_2

$$\omega = \sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}}\ ;\ T = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$$

$$a = \sqrt{\frac{v_1^2 x_2^2 - v_2^2 x_1^2}{v_1^2 - v_2^2}}\ ;\ v_{max} = \sqrt{\frac{v_1^2 x_2^2 - v_2^2 x_1^2}{x_2^2 - x_1^2}}$$

✍ If $y_1 = a\sin\omega t$ and $y_2 = b\cos\omega t$ are two S.H.M. then by the superimposition of these two S.H.M. we get $\vec{y} = \vec{y_1} + \vec{y_2}$

$\Rightarrow\ y = a\sin\omega t + b\cos\omega t \Rightarrow y = A\sin(\omega t + \phi)$ this is also the equation of S.H.M.; where $A = \sqrt{a^2 + b^2}$ and $\phi = \tan^{-1}(b/a)$

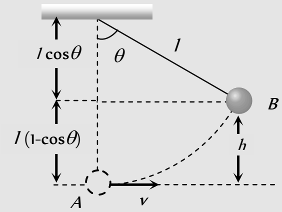
✍ In the absence of resistive force the work done by a simple pendulum in one complete oscillation is zero

✍ If θ is the angular amplitude of pendulum then

Height rises by the bob $h = l(1 - \cos\theta)$

Velocity at mean position

$$v = \sqrt{2gl(1-\cos\theta)}$$



Work done in displacement

$$W = U = mgl(1-\cos\theta)$$

K.E. at mean position

$$KE_{mean} = mgl(1-\cos\theta)$$

Tension in the string of pendulum

At mean position : $T_A(\max) = mg + \frac{mv^2}{l} = (3mg - 2mg\cos\theta)$

At extream position : $T = mg\cos\theta$