ELSEVIER

31 July 1995

PHYSICS LETTERS A

Physics Letters A 203 (1995) 300-311

Strange attractors in a relaxation oscillator model for the dripping water faucet

G.I. Sánchez-Ortiz [a,1], A.L. Salas-Brito [b,2]

[a] *Department of Computing, Imperial College, London SW7 2BZ, UK*

[b] *Laboratorio de Sistemas Dinámicos, Departamento de Ciencias Básicas, Universidad Autónoma Metropolitana-Azcapotzalco, Apartado Postal 21-726, Coyoacán 04000 D.F., Mexico*

Received 27 April 1955; accepted for publication 19 May 1995
Communicated by C.R. Doering

Abstract

Numerical evidence for the existence of strange chaotic attractors is found in a variable-mass relaxation oscillator model for the dripping faucet. We exhibit bifurcation maps which illustrate the richness of the model dynamics; we also have found simple periodic attractors up to period-12. More complex attractors exist which, after computing their fractal dimension and maximal Lyapunov exponent, can be regarded as both strange and chaotic. We have found evidence of boundary crisis and intermittence, as recent experimental results have shown to occur in the actual dripping system.

PACS: 02.70+d; 47.20.Tg; 03.20.+i

The dripping water faucet, also called the leaky tap, has been studied experimentally in various ways [1-6] as one of the paradigmatic examples of chaotic behaviour. These studies have not been exhaustive since some unresolved questions on the system remain. It is not even known what the route (if there is only one) to chaos followed by the system is. Recent experiments of Sartorelli et al. [6] have exhibited the existence of boundary crisis, intermittence related to tangent bifurcations and intermittent behaviour between even periodic attractors. Little has been done though to model the dynamics of the dripping faucet, the only attempts we are aware of are the one-dimensional feedback-loop model of Austin [7], the electric analogue used by Bernhard [8] and the variable mass oscillator model of Shaw [9]. This paper presents results of numerical investigations of a new variable mass relaxation oscillator model for the dripping system. The model considered is an improvement of the model originally proposed by Shaw. The modification introduced allows for a closer qualitative correspondence with the experimental situation.

Let us note that, with the exception of Bernhard's model [8], none of the other models has been subjected to systematic studies. In particular, Shaw's oscillator model has never been thoroughly tested, looking for strange attractors or chaotic behaviour. Analogue simulations carried out by Shaw and collaborators [9] are the only pieces of evidence for the appropriateness of the model in capturing the essentials of the dripping faucet behaviour. This is surprising, since the dripping faucet has been used as a sort of "role model" in modelling magnetospheric processes

[1] E-mail: giso@doc.ic.ac.uk.
[2] E-mail: salas@lsd.uam.mx.

SSDI 0375-9601(95)00419-X

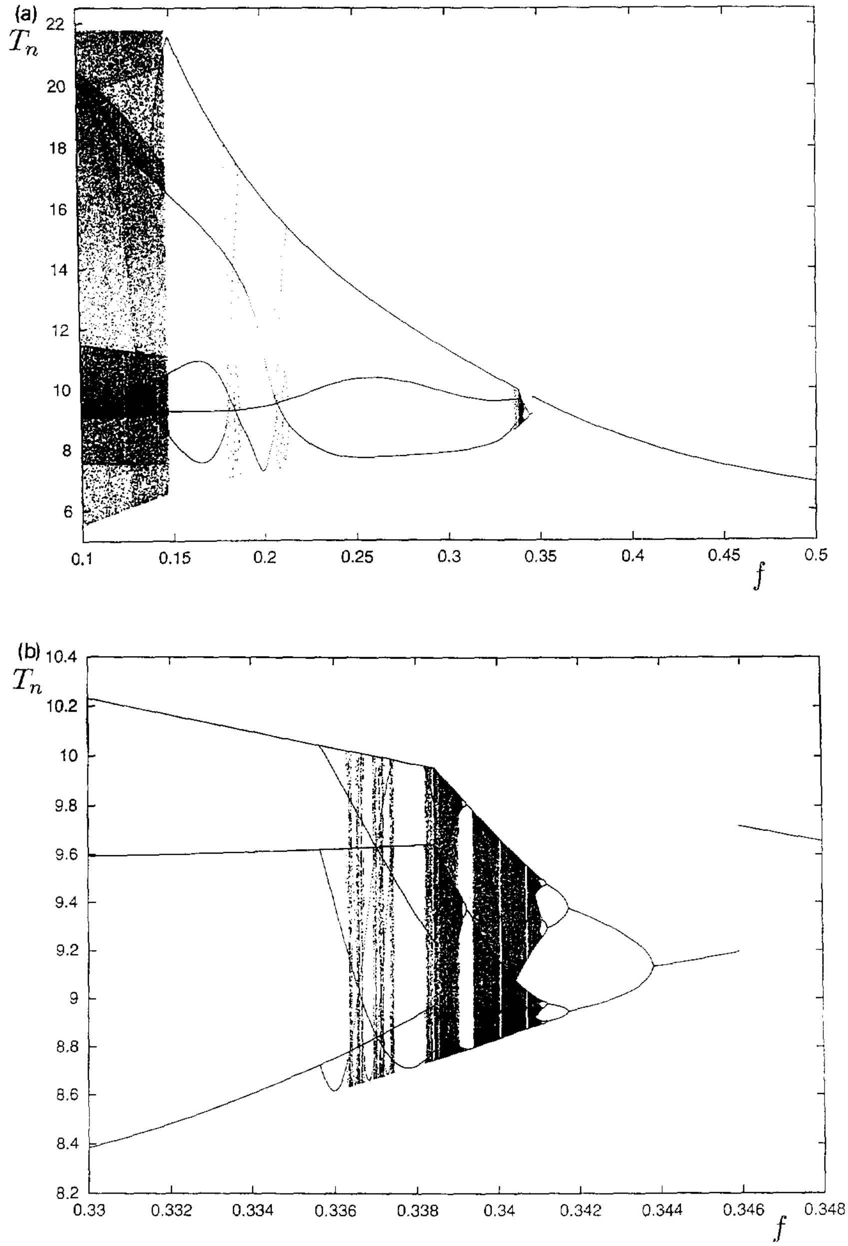


Fig. 1. A bifurcation diagram for the relaxation oscillator model for the leaky faucet. In all the bifurcation diagrams exhibited transients were discarded unless otherwise indicated. Notice the chaotic region at lower values of the "flow rate" f. (a) The attractors (in T_n) at $g = 0.32$ and $h = 7$ as f is varied. Periodic behaviour and crisis are easily spotted in some regions of the diagram. The conspicuous gap in the region around $f = 0.35$ suggests that the *gap road* to chaos may have some bearings with the model's behaviour although further analysis is required to fully understand this phenomenon. (b) Enlargement of the chaotic region at the center of (a). The gap, crisis and an inverse cascade can be discerned. This exhibits that in some regions of parameter space, the systems follow a period-doubling route to chaos.

[10], but also because the characteristic pattern of chaotic relaxation oscillations might be of interest on its own as well as for other applications. In this work, we try to partially remedy this situation by reporting studies of our modified oscillator model. The model conceives the drop as a kind of one-dimensional oscillator [11,12]. The mechanical elastic force in the oscillator is regarded as modelling the cohesive forces (basically the surface tension σ) of the liquid. Since the dripping is driven by the water flow into the hanging drop and by the related gradual increase of the drop's weight before the breaking off, we considered the oscillator as varying its mass at a constant rate f – which may be thought of as the water flow rate into the drop – and as subjected to a constant external force g. The basic non-dimensional, first-order equations of the model are then [9,13]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{x+y}{m} - g, \quad \frac{\mathrm{d}m}{\mathrm{d}t} = f, \tag{1}$$

where x is the position, y the velocity and m the mass of the oscillating drop. This set of equations has to be supplemented with a mechanism allowing for the detachment of a drop. This mechanism is quite simple: when the x-coordinate of the oscillator reaches a prefixed value x_c (analogous to the meniscus length), the mass of the oscillator is made to diminish abruptly, thence simulating a drop, by a quantity

$$\Delta m = hm(t)y(t), \tag{2}$$

where h is another constant and Δm is regarded as the mass of the falling drop. In the non-dimensional coordinates we use, we normalize to unity the value of x_c. In Eq. (2) resides the main difference with Shaw's model. For, in the original model, Δm was considered as proportional to y only, whereas we have made it also proportional to the current value of m. This modification has an evident physical basis, provides better control on the minimum size of the falling drops and allows greater flexibility in the admissible range of numerical values for the three (f, g, h) control parameters. The drop-detachment mechanism also provides the crucial nonlinear ingredient for both a threshold and a quick depletion of the mass accumulated thus allowing relaxation oscillations.

For gathering evidence of complex attractors in the model, we have numerically integrated Eqs. (1) together with condition (2). Whenever a dripping occurred, we recorded the lapse of time between the previous drop and the present one, i.e. we recorded the drip interval T_n (where n is the drop number), the mass M_n, the speed at the moment of the detachment Y_n, and the mass remaining in the nozzle Q_n – even though the remaining mass may be computed from the other data as $Q_n = Q_0 + f\sum_{i=1}^{n} T_i - \sum_{i=1}^{n} M_i$.

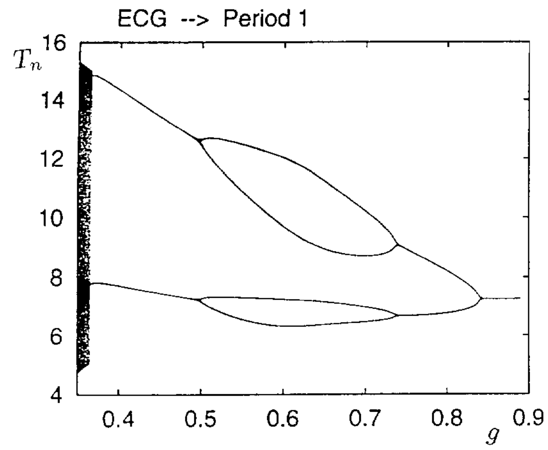


Fig. 2. Bifurcation diagram illustrating the transition from the chaotic attractor ECG, at $f = 0.1$ and $h = 7$, to period-2, -4, -2 and -1 attractors through inverse bifurcations, while varying g.

We have thoroughly searched for attractors in the region $0.05 < f < 2, 0.1 < g < 1, 5 < h < 11$ avoiding the zones of "saturation" (i.e. parameter ranges where the mass remaining after detachment is so big as to prevent further oscillations). Results of some of the computations are exhibited in the bifurcation diagrams of Figs. 1–3. As we fix two of the parameters and vary the third (for example, in Figs. 1 and 3 f is varied while g and h are kept constant), the system shows chaotic and periodic windows, cascades and crisis. The sudden changes in the attractors may be related to a boundary crisis as there are no appreciable changes in the mean values [20]. In some regions of the parameters there are chaotic and periodic windows intermingled at apparently any scale in the way fractals show substructure. This behaviour is worth a more detailed analysis that will be presented in a forthcoming article. The gaps appearing in the bifurcation diagrams may be considered as reinforcing the suggestion [14] that the so-called "gap" road to chaos [15,16] might be present in models of the leaky tap dynamics – mainly due to the discontinuities related to

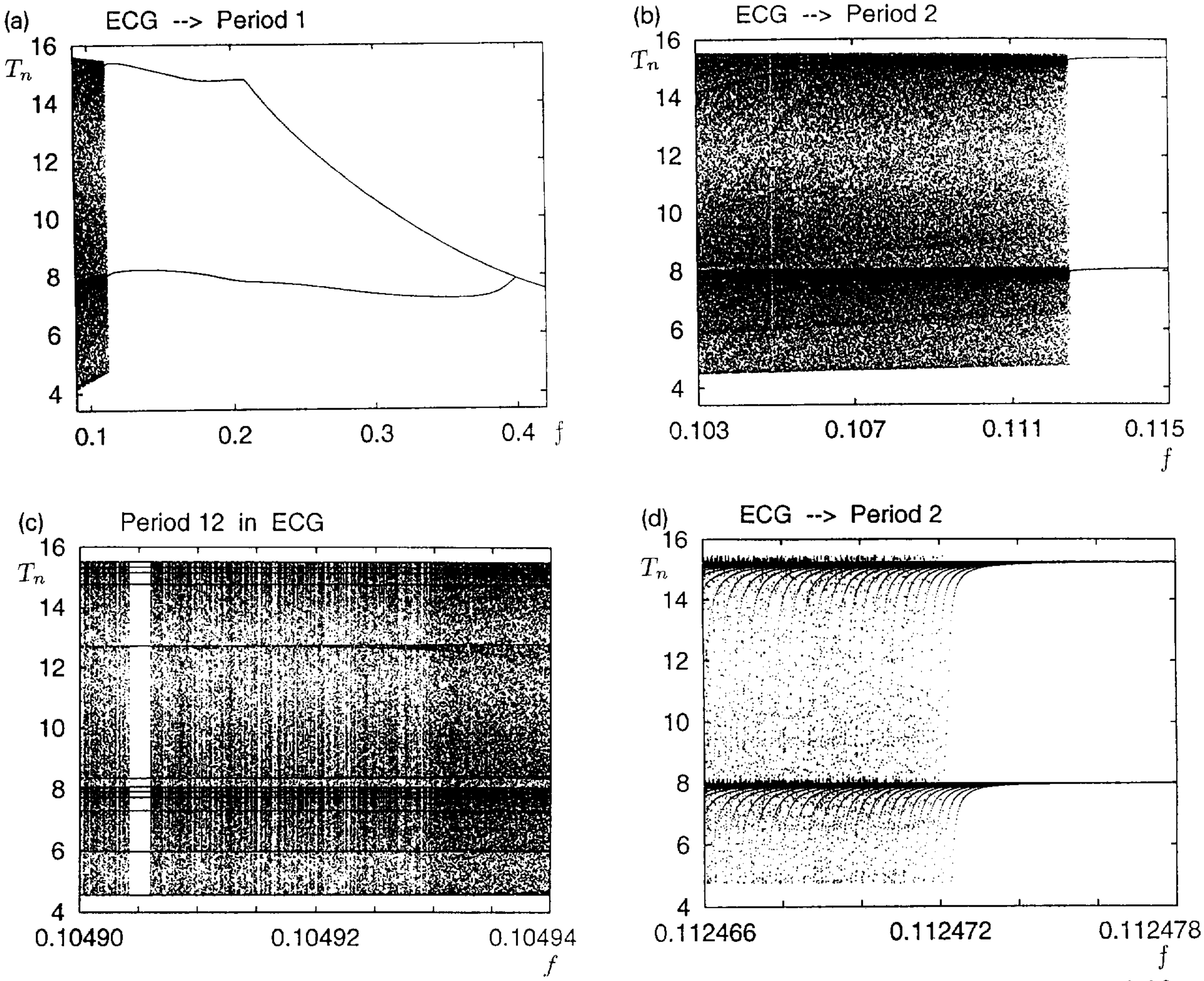


Fig. 3. Bifurcation diagram illustrating the transition from an instance of the chaotic attractor ECG at $g = 0.34$ and $h = 7$, to period-2 and -1 attractors, while varying f. (a) The whole interval $0.095 < f < 0.415$ in which the transition occurs. As in Fig. 1a, chaos occurs at low f values. (b) Blowup of a region of (a) clearly showing crisis and a periodic window. (c) Magnification of (b) showing crisis and the period-12 window in the region $0.10490 < f < 0.10494$. (d) Close-up of the small region ($0.112466 < f < 0.112478$) of (b) where the transition from chaos to a period-2 attractor occurs through tangent intermittence.

drop detachment. However, at present we are not certain how closely related the system behaviour and the gap road are. On the other hand, in Fig. 3d we show a region of parameter space where the transition from chaos to period-2 behaviour happens through tangent intermittence [17].

In what follows, we shall employ time-delay plots for reconstructing attractors from chaotic windows in analogous fashion to what is done with experimental data. Apart from periodic attractors, we have uncovered some qualitatively different types of complex attractors, from which we will show here a particularly interesting example. This suffices to ascertain the complex dynamics of the relaxation oscillator model for the dripping faucet and of the existence of crisis and intermittence in the attractors. A more complete discussion of the dynamics together with a catalogue of the different chaotic attractors we have discovered so far will be given in a more detailed paper [18].

A conspicuous feature of the model in the param-

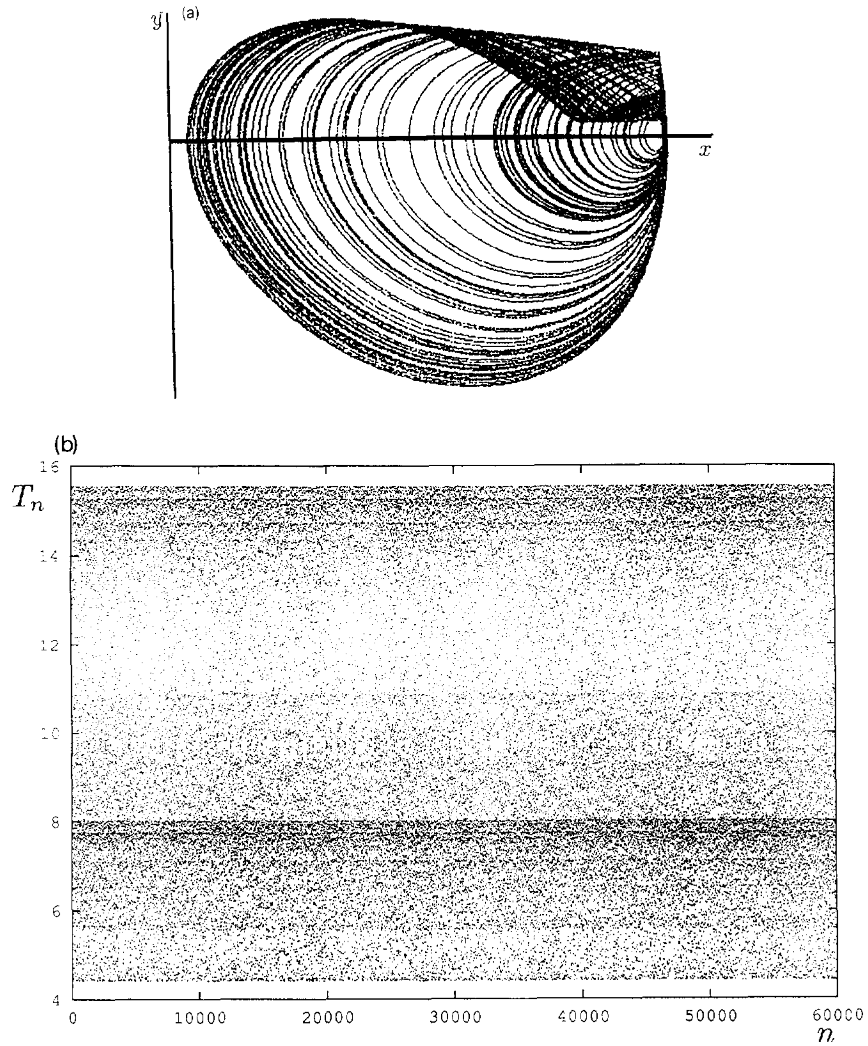


Fig. 4. The ECG attractor occurring at $f = 0.10$, $g = 0.34$, $h = 7.0$. (a) x–y projected phase-space trajectory corresponding to about 100 drops. (b) Time-series of drip intervals for ECG. Notice the two bands with higher density of points, corresponding to what we call a pseudo period-2 behaviour. (c) Power spectrum of the ECG attractor. Very broad peaks appear in the neighborhood of frequencies 1/8, 1/4, and 1/2. The Nyquist frequency is 1/2.

eter range investigated is the dominance of periodic attractors. For most values of (f, g, h), the system shows a quick convergence to cyclic attractors of period 1, 2, 3, 4, 5, 8 or 12 (some of these attractors are clearly evident in Figs. 1, 2 or 3). The last periodic attractor is not so dominant, though, as we manage to discover it only after analysing the bifurcation diagrams (see Fig. 3). We do find evidence of strange attractors, as is also quite clear from the figures. An example of this is presented in Fig. 4 where the attractor nicknamed ECG is shown both as a x–y projected trajectory (Fig. 4a) and as a time series of 35×10^4 of its drip intervals (Fig. 4b); we also exhibit there the power spectrum associated with the time series (Fig. 4c). This attractor is found at the parameter values $g = 0.34$, $h = 7.0$ and in the neighborhood of $f = 0.10$ (within the range of parameters exhibited in the bifurcation diagrams of Fig. 3). Notice the accu-

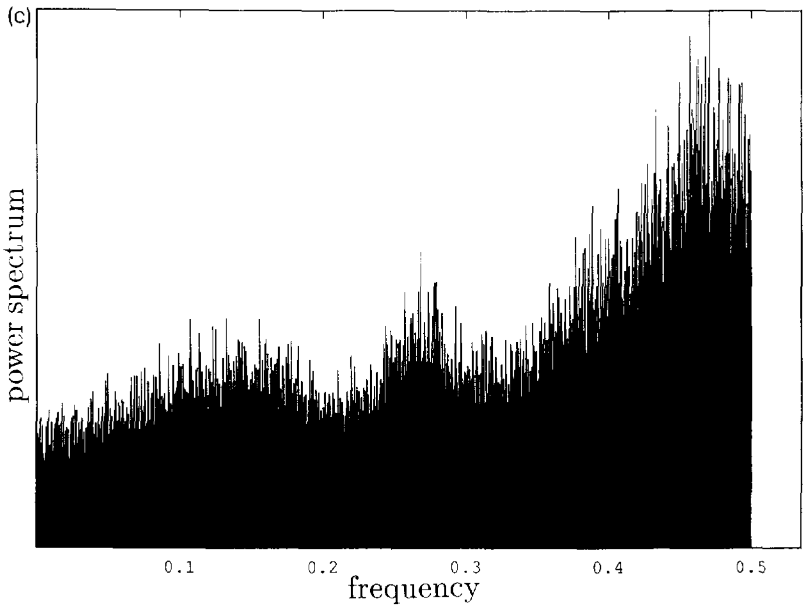


Fig. 4. Continued.

mulation of points around two values which is rather evident in the time series of ECG (Fig. 4b). If these were experimental results such a behaviour could be mistaken for a noisy period-2 attractor. A similar behaviour may be observed in experimental data; we have observed it in some data at our disposal [4,18] and, with some imagination, in the results of Cahalan et al. [5]. Evidence for this (we call it a pseudo period-2 behaviour) is also found in the power spectrum, shown in Fig. 4c, associated with this particular attractor. This power spectrum was calculated using not only the points shown in Fig. 2, but using drip intervals totalling 1.3×10^5. Notice the broad peak near 1/2 which corresponds to the dense accumulation of points mentioned before and the even broader peaks at 1/4 and 1/8 which may indicate a similar behavior for a somewhat hidden pseudo period-8 behavior. The crises occurring when either f or g vary whereas the other two parameters are fixed are shown in Figs. 2 and 3, respectively. The transition exhibited in Fig. 3d, from chaotic to period-2 behaviour, occurs through tangent intermittence [17].

As a further check for the existence of attractors at these parameter values, Figs. 5 and 6 show different time-delayed reconstructions of 3×10^5 drip intervals for the ECG attractor. Fig. 5 exhibits a T_{n+1} versus T_n two-dimensional reconstruction together with the blowup of two of its regions. The existence of substructure is rather evident there, this pinpoints the fractal nature of the attractors which thus may be called strange. The Hausdorff dimension associated with the ECG attractor was found to be $D_H = 1.24 \pm 0.07$. In these reconstructions is also noticeable the existence of crisis, i.e. regions where chaotic attractors suddenly become periodic attractors [20].

The ECG attractor was also reconstructed using the variables M_i and Q_i (Fig. 6) and 3D time-delay plots. Notice that the same kind of substructure is present in both the masses and the drip-interval reconstructions, this in spite of their apparently different global structure. The Hausdorff dimension calculated for these reconstructions ($D_H = 1.26 \pm 0.07$) serves as confirmation of the fractal behaviour of ECG. The fractal dimensions given here were calculated using the Grassberger and Procaccia algorithm [21], as implemented by Oropeza-López [22], using 1×10^5 points.

We have also found a sort of, let us say, "unstable" behaviour in some periodic attractors where, for example, a seemingly period-4 attractor eventually and gradually evolved toward a period-2 attractor (Fig. 7). Since the final behaviour is a stable period-2 attractor, we have to say that the initial was the only tran-

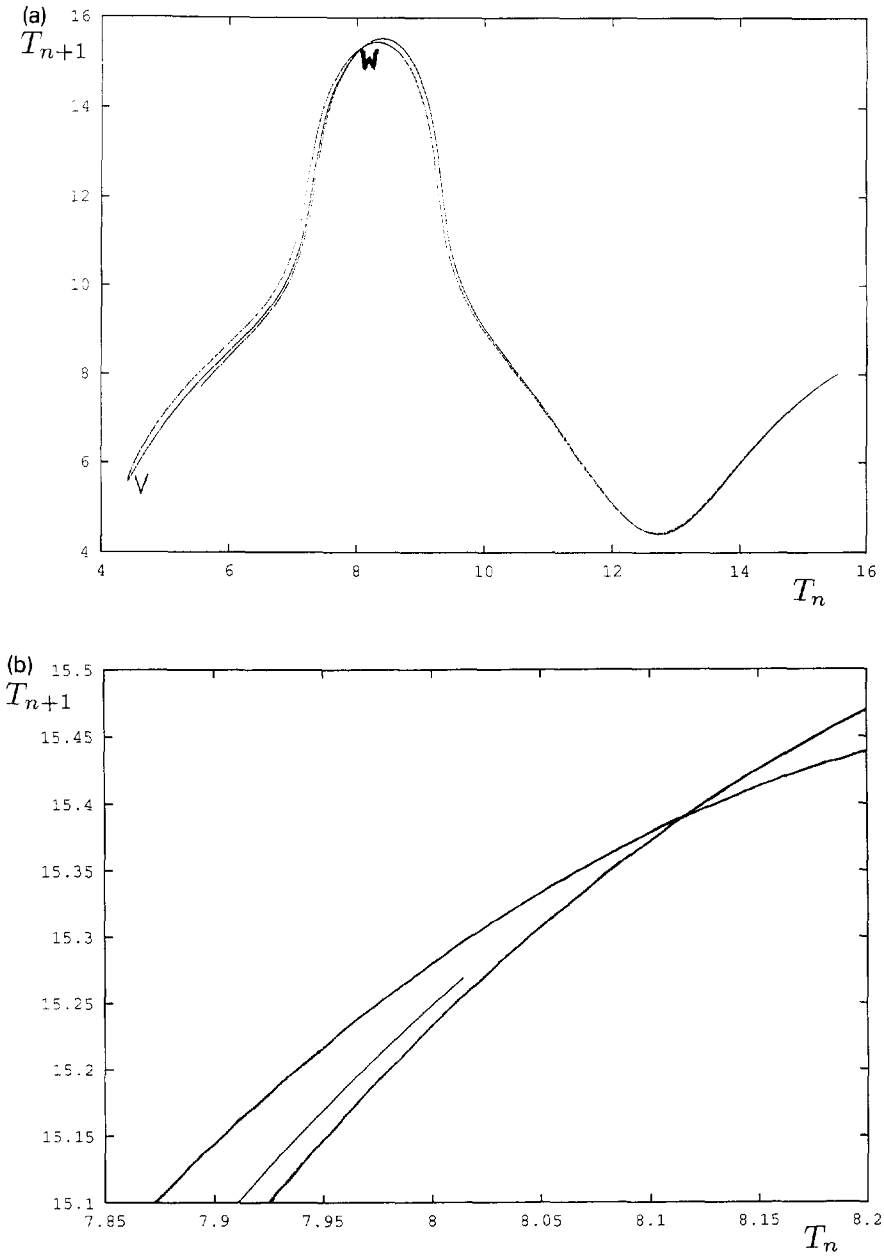
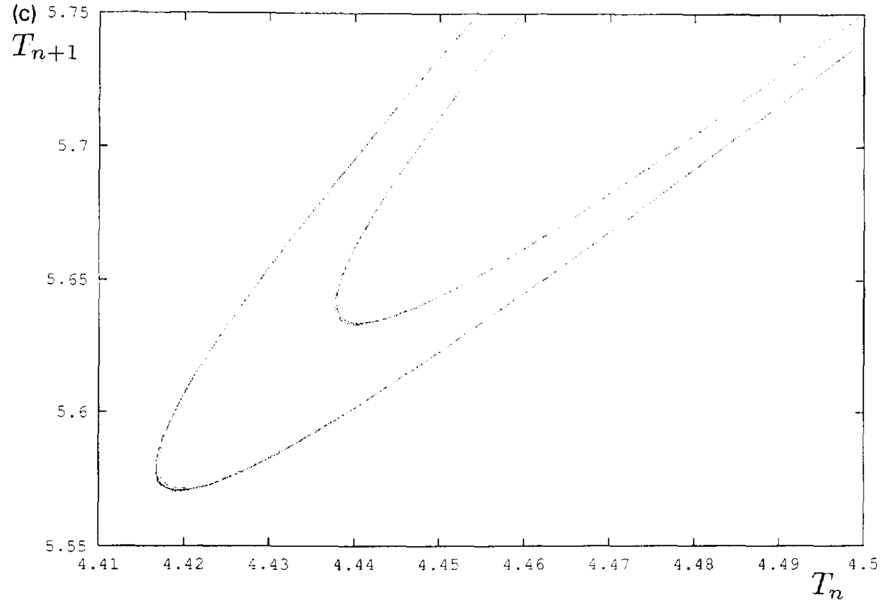


Fig. 5. Return map reconstructions of the ECG attractor of Fig. 4. 3×10^5 points were used for the reconstruction. The fractal dimension of ECG is $\simeq 1.3$. (a) Time-delay plot of drip intervals. (b) Blowup of the W region of (a). (c) Blowup of region V. (d) Blowup of a small region of (c) near the tip, exhibiting the substructure of the set.



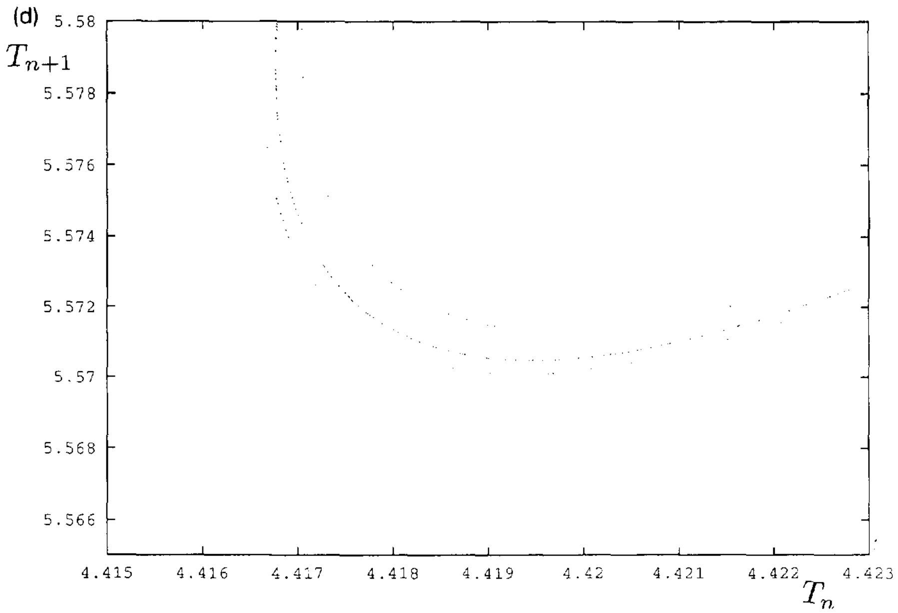


Fig. 5. Continued.

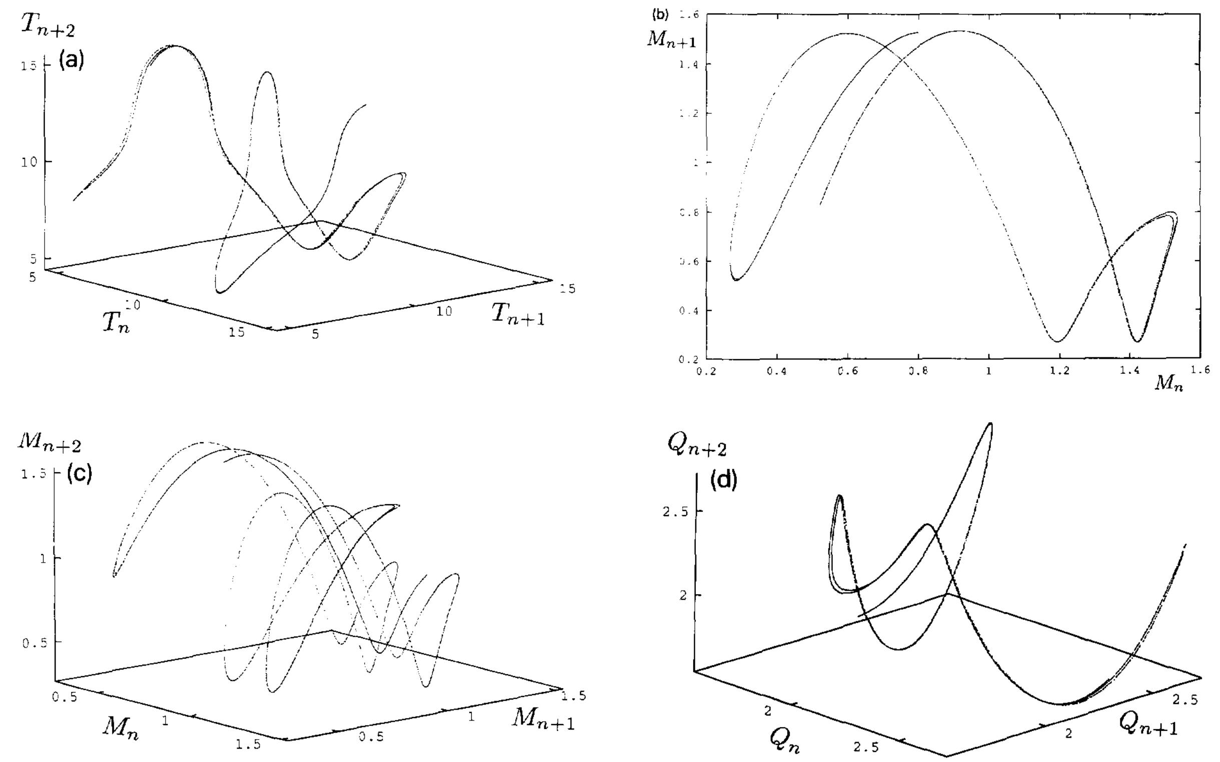


Fig. 6. Other reconstructions of the ECG attractor of Fig. 4. The ECG attractor has an estimated Hausdorff dimension $D_H \simeq 1.3$ and a maximal Lyapunov exponent $\lambda_+ \simeq 0.9$ bit/drop. (a) 3D reconstruction using drip intervals. (b) 2D time-delay plot reconstruction using the detached-mass data M_i. (c) 3D time-delay plot using the detached mass data M_i. (d) 3D time-delay plot using the hanging mass data Q_i.

sient state which seemed to be periodic. This process illustrates the soft transition that takes place between period-1, -2 and -4 attractors shown in Fig. 2, and also has stimulated the search for a region of parameter space where intermittence between even periodic attractors may be found, as recent experimental results have shown to occur in the actual dripping system [6]. In fact, we have also found a sort of intermittent version of the chaotic attractor ECG, in which brief states with periodic behaviour of periods 2, 3 and 6 alternate in an irregular manner.

Another kind of intermittence, similar to that reported in Ref. [6] – see Figs. 12 and 14 in Ref. [6] – , has been found in the model dynamics. This is illustrated by Fig. 8 where intermittence between chaos and period-5 behaviour is exhibited. But the agreement between Fig. 8 and the results of Ref. [6] is qualitative and, at this time, based mostly on resemblances. Further investigations are needed to ascertain whether this is the same kind of behaviour or not.

Since fractal properties of attractors cannot guarantee chaotic behaviour, we have evaluated the maximum Lyapunov exponent λ_+ for the ECG attractor. We did this using two methods. The first one uses the equations of the model to follow the evolution of a small separation vector $\xi(t)$ between nearby trajectories, carefully handling the discontinuity [13] and using

$$\lambda_+ = \lim_{t\to\infty} \frac{1}{t} \log \frac{||\xi(t)||}{||\xi(0)||}, \tag{3}$$

to evaluate the maximum Lyapunov exponent. In this way we were able to get a value certainly above zero. However, since some practical problems occurred during the computation and we were not completely sure of the reliability of the value obtained, we employed another method using again an experimental-data approach. We analysed the series of drip-interval and

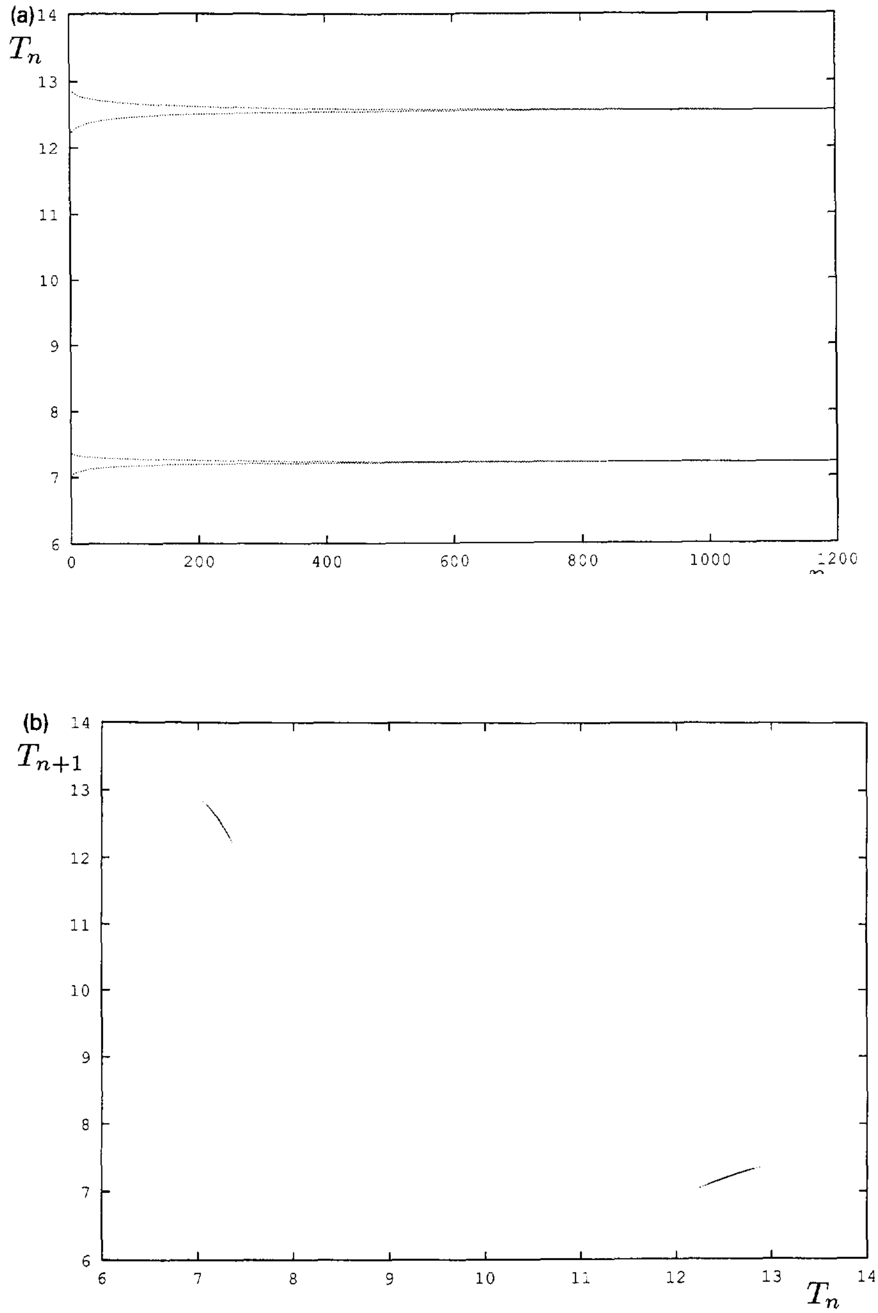


Fig. 7. Two views of the transition from period-4 to period-2 behaviour occurring in the model at parameter values $f = 0.1$, $g = 0.5$, $h = 7$. As the final period-2 behaviour is stable, the initial state is a transient which only seems to be periodic. (a) Series of drip intervals. (b) Time-delay plot.

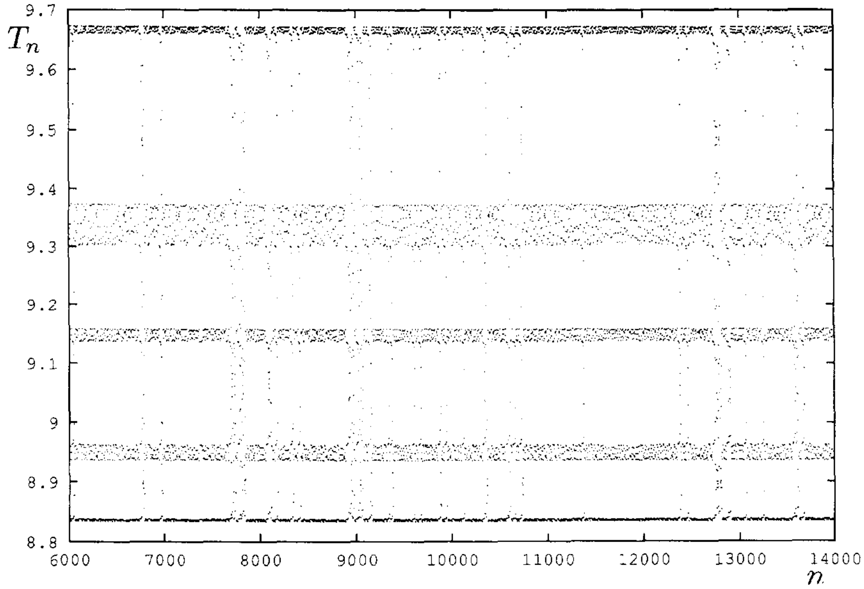


Fig. 8. Tangent intermittence between chaos and noisy period-5 behaviour is exhibited here (f = 0.338, g = 0.325, h = 10). A similar behaviour has been found in the experimental system [6].

mass data employing the algorithm of Wolf et al. [23] to estimate the value of λ_+. In this way, we were able to obtain the value $\lambda_+ \simeq 0.9$ bits/drop for the ECG attractor. Though we cannot consider the numbers given here as definitive, we do think that our results offer a strong numerical evidence of the existence of strange chaotic attractors in the relaxation oscillator model for the dripping water faucet. We are now trying to convert the system (1) to an equivalent mapping which perhaps might be the best way of modelling the system and studying the system further on.

We conclude then that the variable-mass relaxation oscillator presented here is a reasonable model for the dripping faucet. Though, obviously, not all the results of the numerical studies of the model have experimental counterparts, others can reproduce qualitative characteristics of the actual system. On the similar features, our results show that the periods do not increase in the way one would expect if it were following the period-doubling route to chaos. The model as well as the experimental system can reach a chaotic state through tangent intermittence. In fact, the route to chaos followed by the model seems to depend on the region of the space of parameters. It is also the case that, contrary to earlier expectations, chaos tends to be greater at low values of the "flow rate" f. The same is also true of the experimental realizations of the dripping faucet [1–6,19]. The model also exhibits attractors similar to some found in the experimental system. In particular, the attractor ECG has experimental counterparts found in several studies [1–6,18]. However, there are also obvious differences. One of the most conspicuous is the absence of closed-loop attractors as those reported in Ref. [4]. We think the model is worth further analysis [18].

This work has been partially supported by CONACYT (grant 4846-E9406) and by the Fundación Ricardo J. Zevada. Very helpful discussions with H.N. Núñez-Yépez and all the support she offered are also gratefully acknowledged. GISO thanks T.H. Seligman and F. Leyvraz for their enlightening academic advice. This work has benefitted from comments and suggestions of G. Tlahui, Q. Gris, K. Quiti, G. Tavi, Q. Mikei, Q. Motita and M. Ek.

References

[1] P. Martien, S.C. Pope, P.L. Scott and R.S. Shaw, Phys. Lett. A 110 (1985) 399.

[2] H.N. Núñez-Yépez, A.L. Salas-Brito, C. Vargas and L. Vicente, Eur. J. Phys. 10 (1989) 99.
[3] H.N. Núñez-Yépez, C. Carbajal, A.L. Salas-Brito, C.A. Vargas and L. Vicente, in: Nonlinear phenomena in fluids, solids and other complex systems, eds. P. Cordero and B. Nachtergaele (Elsevier, Amsterdam, 1991) p. 467.
[4] X. Wu and Z.A. Schelly, Physica D 40 (1989) 433.
[5] R.F. Cahalan, H. Leidecker and G.D. Cahalan, Comp. Phys. 3 (1990) 368.
[6] J.C. Sartorelli, W.M. Gonçalves and R.D. Pinto, Phys. Rev. E 5 (1994) 3963.
[7] J. Austin, Phys. Lett. A 155 (1991) 148.
[8] P.A. Bernhard, Chaos 2 (1992) 183.
[9] R. Shaw, The dripping faucet as a model chaotic system (Aerial Press, Santa Cruz, 1984).
[10] D.N. Baker, A.J. Klimas, R.L. McPherron and J. Buchner, Geophys. Res. Lett. 17 (1990) 41.
[11] E. Becker, W.H. Hiller and T.A. Kowalewski, J. Fluid Mech. 27 (1991) 189.
[12] J.W.S. Rayleigh, The theory of sound (Dover, New York, 1945) §364
[13] G.I. Sánchez-Ortiz, El grifo goteante: estudio numérico de un modelo mecánico, Senior Thesis (FCUNAM, México D.F., 1991).
[14] C. Tsallis, personal communication.
[15] M.C. de Sousa-Vieira, E. Lazo and C. Tsallis, Phys. Rev. A 35 (1987) 945.
[16] M.C. de Sousa-Vieira and C. Tsallis, Europhys. Lett. 9 (1989) 119.
[17] P. Manneville and Y. Pomeau, Physica D 1 (1980) 219.
[18] G.I. Sánchez-Ortiz and A.L. Salas-Brito, Physica D (1995), to appear.
[19] A.L. Salas-Brito and C. Vargas, unpublished data.
[20] C. Grebogi, E. Ott and J.A. Yorke, Physica D 7 (1983) 181.
[21] P. Grassberger and I. Procaccia, Phys. Rev. Lett. 50 (1983) 346.
[22] A.M. Oropeza-López, Dimensión fractal de acelerogramas de sismos registrados en el valle de México, Senior Thesis (FCUNAM, México D.F., 1991).
[23] A. Wolf, J.B. Swift, H.L. Swinney and J.A. Vastano, Physica D 16 (1985) 285.