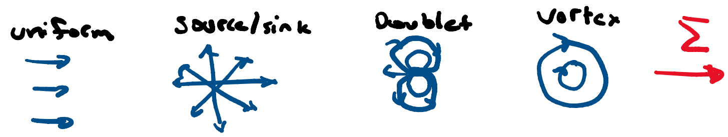
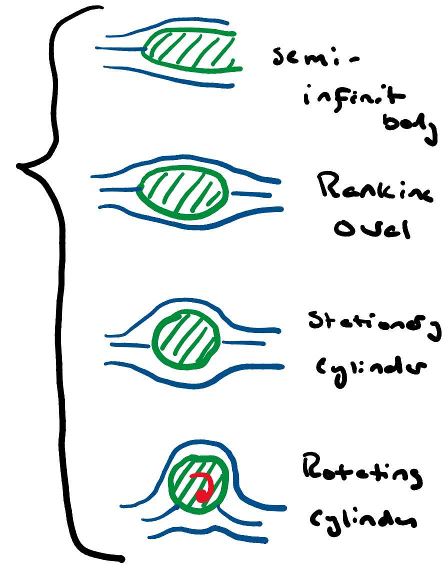
Lecture 10: Kuta-Joukowski Theorem

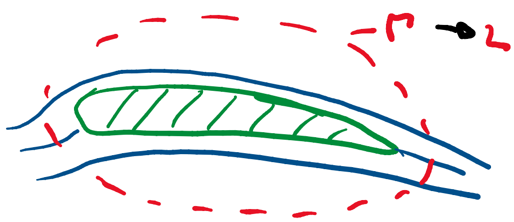
Sunday, October 4, 2020 8:05 PM

LAST TIME: ELEMENTARY FLOWS

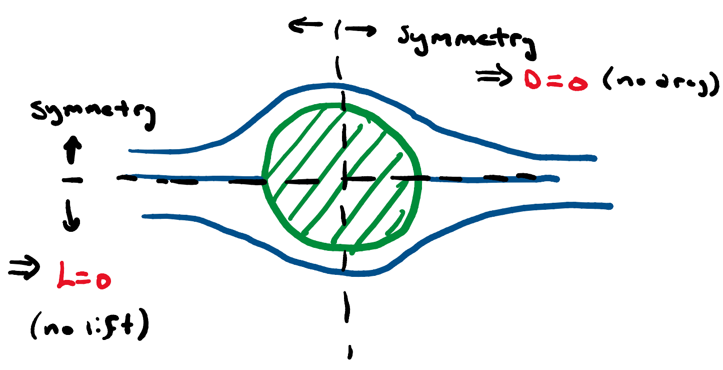


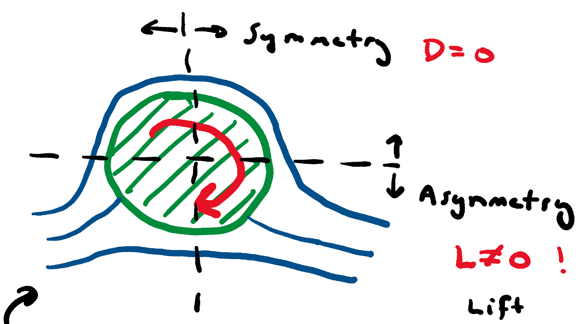


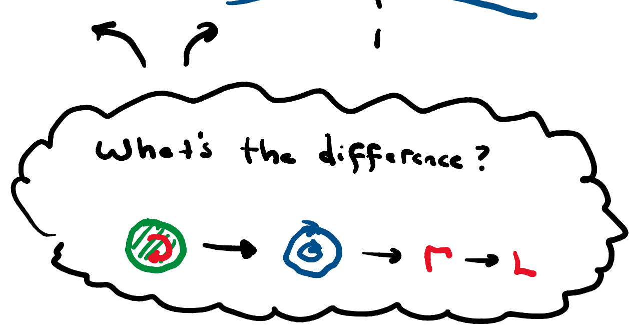
TODAY: KUTTA-JOUKOWSKI theorem $L' = \rho u_\infty \Gamma$



Recall, the STATIONARY & ROTATING CYLINDER from ELEMENTAL FLOWS







German mathematician

Russian physicist

(J) KUTTA-JOUKOWSKI THEOREM

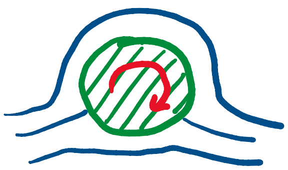
how circulation indicates LIFT

$L' = \rho u_\infty \Gamma$

how circulation indicates LIFT

$L' = \rho u_\infty \Gamma$

Back to the ROTATING CYLINDER



VELOCITY:

$$u_r = \left(1-\frac{R^2}{r^2}\right)u_\infty \cos\theta$$

(cylindrical coordinates)

$$u_\theta = -\left(1+\frac{R^2}{r^2}\right)u_\infty \sin\theta - \frac{\Gamma}{2\pi r}$$

Let's try to calculate LIFT

velo. press. lift

$$\vec{u} \xrightarrow{1} P \xrightarrow{2} L$$

instead of P, let's use its coefficient

STEP 1: Find PRESSURE from VELOCITY

Pressure coefficient: $C_p = \dfrac{\Delta P}{\frac{1}{2}\rho u_\infty^2}$

~ pressure dif. from atmosphere $(P - P_\infty) = \Delta P$

~ dynamic pressure

Let's use BERNOULLI to get ΔP in terms of u

✓ ρ = const.

✓ $\mu = 0$

$$P_1 + \frac{1}{2}\rho u_1^2 = P_2 + \frac{1}{2}\rho u_2^2$$

set $P_1, u_1 \rightarrow P_\infty, u_\infty$ ~ freestream values

$$P_1 + \tfrac{1}{2}\rho u_1^2 = P_2 + \tfrac{1}{2}\rho u_2^2 \qquad \text{set} \quad P_1, u_1 \rightarrow P_\infty, u_\infty$$

freestream values

$$P_\infty + \tfrac{1}{2}\rho u_\infty^2 = P + \tfrac{1}{2}\rho u^2$$

$$\underbrace{P - P_\infty}_{\Delta P} = \tfrac{1}{2}\rho\left(u_\infty^2 - u^2\right)$$

Use this ΔP in above C_p

$$C_p = \frac{\Delta P}{\frac{1}{2}\rho u_\infty^2} = \frac{\cancel{\frac{1}{2}}\cancel{\rho}\left(u_\infty^2 - u^2\right)}{\cancel{\frac{1}{2}}\cancel{\rho}\, u_\infty^2} = \frac{u_\infty^2 - u^2}{u_\infty^2}$$

we know this

$$\boxed{C_p = 1 - \left(\frac{u}{u_\infty}\right)^2}$$

need this to get LIFT

NOTE: Velocity at SURFACE is $u = u_\theta$ at $r = R$

STEP 2: find LIFT from PRESSURE

Recall, we can get FORCES by integrating PRESSURE / SHEAR over surface.

trailing edge

$$N' = -\int_{LE}^{TE}\left(P_u\cos\theta + \cancel{\tau_u\sin\theta}\right)ds_u + \int_{LE}^{TE}\left(P_\ell\cos\theta - \cancel{\tau_\ell\sin\theta}\right)ds_\ell$$

inviscid

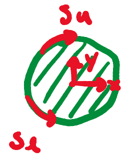
leading edge

$$A' = \int_{LE}^{TE}\left(-P_u\sin\theta + \cancel{\tau_u\cos\theta}\right)ds_u + \int_{LE}^{TE}\left(P_\ell\sin\theta + \cancel{\tau_\ell\cos\theta}\right)ds_\ell$$

s_u

lift' normal'

Fo...

For us 

therefore $dx = \cos\theta \, d\theta$ and $\overset{\text{lift}'}{L'} = \overset{\text{normal}'}{N'}$ (cylinder coords aligned w/ free stream)

$dy = -\sin\theta \, d\theta$ $\qquad \underset{\text{drag}'}{D'} = \underset{\text{axial}'}{A'}$

$$L' = -\int_{LE}^{TE} P_u \, dx + \int_{LE}^{TE} P_l \, dx = \int_{LE}^{TE} (P_l - P_u) \, dx$$

$$D' = \int_{LE}^{TE} P_u \, dy - \int_{LE}^{TE} P_l \, dy = \int_{LE}^{TE} (P_u - P_l) \, dy$$

We want NON-DIMENSIONAL so $L' \rightarrow c_l$

$D' \rightarrow c_d$

$P \rightarrow c_p$

NOTE: our streamwise reference length is D (diameter)

$$c_l = \frac{1}{D} \int_{LE}^{TE} (c_{p_l} - c_{p_u}) \, dx$$

$$c_d = \frac{1}{D} \int_{LE}^{TE} (c_{p_u} - c_{p_l}) \, dy$$

CARTESIAN

transform

$y = R \sin\theta$

$dy = R \cos\theta \, d\theta$

$x = R \cos\theta$

$dx = -R \sin\theta \, d\theta$

$$c_l = -\frac{1}{2} \int_{\pi}^{2\pi} c_{p_l} \sin\theta \, d\theta + \frac{1}{2} \int_{\pi}^{0} c_{p_u} \sin\theta \, d\theta$$

$$c_d = \frac{1}{2} \int_{\pi}^{0} c_{p_u} \cos\theta \, d\theta - \frac{1}{2} \int_{\pi}^{2\pi} c_{p_l} \cos\theta \, d\theta$$

CYLINDRICAL

Note that $c_{p_u} = c_{p_l}$ therefore integrate c_p around whole circle $0 : 2\pi$

$$\int_{\pi}^{0} c_{p_u}(\ldots) + \int_{\pi}^{2\pi} c_{p_l}(\ldots) = \int_{0}^{2\pi} c_p(\ldots)$$

Let's get our C_p in full form

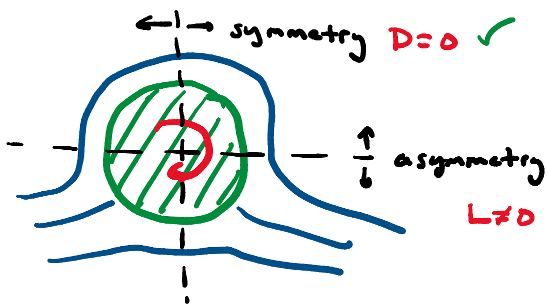
velo at surf.

$$C_p = 1 - \left(\frac{u_\theta}{U_\infty}\right)^2 \quad \text{where} \quad \underset{(r=R)}{u_\theta} = -2U_\infty \sin\theta - \frac{\Gamma}{2\pi R}$$

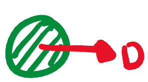
$$C_p = 1 - \left(-2\sin\theta - \frac{\Gamma}{2\pi R U_\infty}\right)^2$$

$$= 1 - 4\sin^2\theta - \frac{2\Gamma\sin\theta}{\pi R U_\infty} - \left(\frac{\Gamma}{2\pi R U_\infty}\right)^2 \quad *$$

Ready to solve for LIFT & DRAG (finally...)



DRAG



$$C_d = -\frac{1}{2}\int_0^{2\pi} C_p \cos\theta \, d\theta$$

$$= -\frac{1}{2}\int_0^{2\pi}\left[\cancelto{0}{1} - 4\cancelto{0}{\sin^2\theta} + \cancelto{0}{\frac{2\Gamma\sin\theta}{\pi R U_\infty}} + \cancelto{0}{\left(\frac{\Gamma}{2\pi R U_\infty}\right)^2}\right]\cos\theta\, d\theta$$

MATH NOTE:

$$\int_0^{2\pi} \cos\theta\, d\theta = 0$$

$$\int_0^{2\pi} \sin^2\theta\cos\theta\, d\theta = 0$$

$$\int_0^{2\pi} \sin\theta\cos\theta\, d\theta = 0$$

LIFT

$$C_l = -\frac{1}{2}\int_0^{2\pi} C_p \sin\theta\, d\theta$$

not zero!

$$= -\frac{1}{2}\int_0^{2\pi}\left[\cancelto{0}{1} - \cancelto{0}{4\sin^2\theta} + \frac{2\Gamma\sin\theta}{\pi R U_\infty} + \cancelto{0}{\left(\frac{\Gamma}{2\pi R U_\infty}\right)^2}\right]\sin\theta\, d\theta$$

MATH NOTE:

$$\int_0^{2\pi} \sin\theta\, d\theta = 0$$

$$\int_0^{2\pi} \sin^3\theta\, d\theta = 0$$

$$\int_0^{2\pi} \sin^2\theta\, d\theta$$

$$\int_0^{2\pi} \sin\theta\cos\theta\, d\theta = 0$$

$$\boxed{C_d = 0}$$

$$\int_0^{2\pi} \sin^2\theta\, d\theta = \pi$$

const.

$$C_l = \frac{1}{2}\int_0^{2\pi} \frac{2\Gamma}{\pi R U_\infty} \sin^2\theta\, d\theta$$

$$\boxed{C_l = \frac{\Gamma}{R U_\infty}}$$

So, the LIFT on our ROTATING CYLINDER is

our length scale

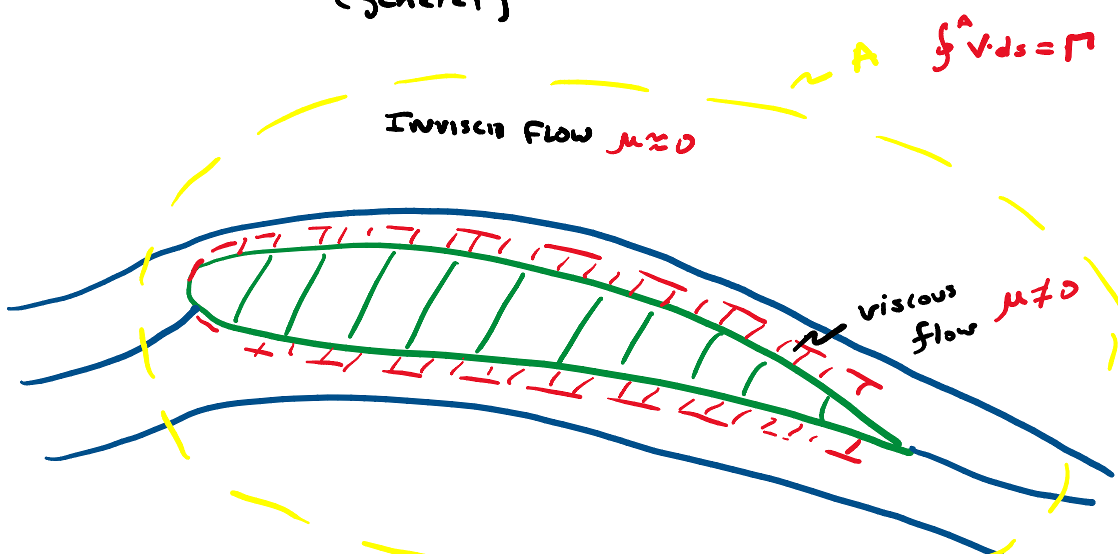
$$L' = \frac{1}{2}\rho U_\infty^2 D\, C_l = \cancel{\frac{1}{2}}\rho U_\infty^{\cancel{2}} (\cancel{2R}) \frac{\Gamma}{\cancel{R}\cancel{U_\infty}}$$

$$\boxed{L' = \rho U_\infty \Gamma}$$

KUTTA-JOUKOWSKI THEOREM

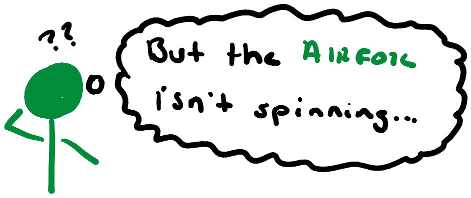
Although it appeared for ROTATING CYLINDERS

it also works for AERODYNAMIC BODIES!
(general)



As long as all the flow Γ is accounted for

$$L' = \rho U_\infty \Gamma \checkmark$$



— the CURVATURE (camber) and ANGLE cause FLOW rotation, $\Rightarrow \Gamma \neq 0$

Keep in mind – Γ doesn't make lift, it just indicates it exists

LIFT/DRAG are caused by PRESSURE/SHEAR, CIRCULATION is the foot print

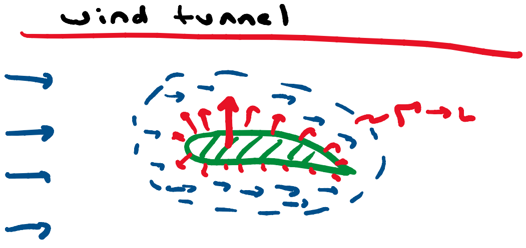
$$\begin{tikzcd} P \,\&\, \tau \arrow[r] & L \arrow[r] & \Gamma \arrow[l, bend right, dashed, "\text{Kutta-Joukowski}"'] \end{tikzcd}$$

In practice you will see KUTTA-JOUKOWSKI principles often.

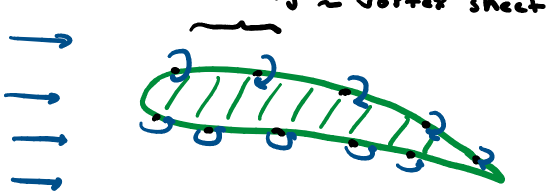
Experiments

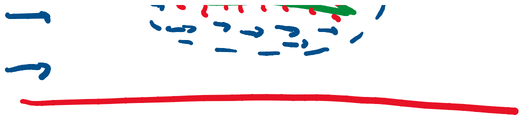
SIMULATIONS

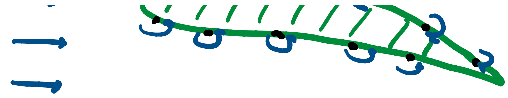
wind tunnel



Body ≈ vortex sheet







To get LIFT you can

(1) measure directly

(2) integrate pressure

* (3) Get Γ from $\vec{u}$

"VORTEX PANEL METHOD"

$\Gamma \rightarrow L$