Lecture 5: Conservation Laws

Sunday, September 13, 2020 4:18 PM

LAST TIME: NON-DIMENSIONALIZING & SIMILARITY

$L \rightarrow C_L$ $\qquad Re = \frac{\rho u_\infty c}{\mu}$

$D \rightarrow C_D$

$M \rightarrow C_m$ $\qquad M = \frac{u}{a}$

TODAY: CONSERVATION LAWS: Mass

Momentum

ENERGY

(E) CONSERVATION LAWS of AERODYNAMICS

↳ i.e., fluid mechanics (assumptions: air)

To PREDICT flow behavior and forces, we apply RULES the flow must follow.

Conservation of
- Mass $\dot{m}_{in} - \dot{m}_{out} = \dot{m}_{inside}$
- Momentum $F = ma$
- ENERGY $\delta w + \delta q = de$

In this lecture...

- we WON'T derive them ☹
- we WILL physically explain them ☺

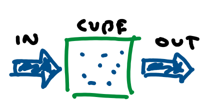
(1) CONSERVATION OF MASS

PRINCIPLE

Mass cannot be created or destroyed.

$$\dot{m}_{IN} - \dot{m}_{OUT} = \dot{m}_{CUBE}$$

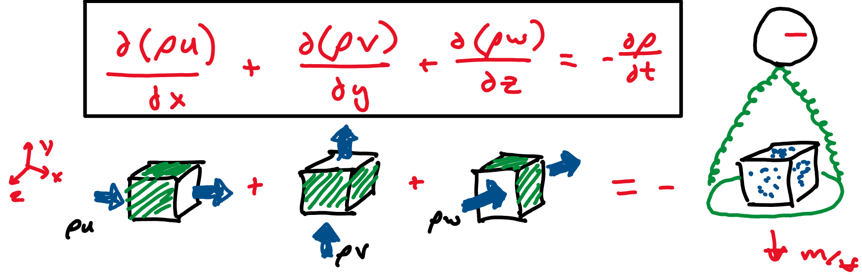
mass flow in ... mass flow out ... change in cube mass



In FLUIDS (EULERIAN P.O.V, differential form)

Cons. of mass (general)

$$\boxed{\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = -\frac{\partial \rho}{\partial t}}$$



Commonly, we assume the flow is INCOMPRESSIBLE $M<0.3$

$\rho = \text{const.}$

$\frac{\partial(\rho u)}{\partial t} = \rho \frac{\partial u}{\partial t}$

Cons. mass (incompressible)

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$

NOTE: can only make this assumption for:

- LIQUIDS
- SLOW GAS ($M<0.3$)

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(2) CONSERVATION OF MOMENTUM

PRINCIPLE

An object's CHANGE IN MOMENTUM must be due to FORCE.

$F = ma$. more generally: $F = \frac{d(mv)}{dt}$

In FLUIDS each direction (y, x, z) has an equation (x3)

MATERIAL DERIV.

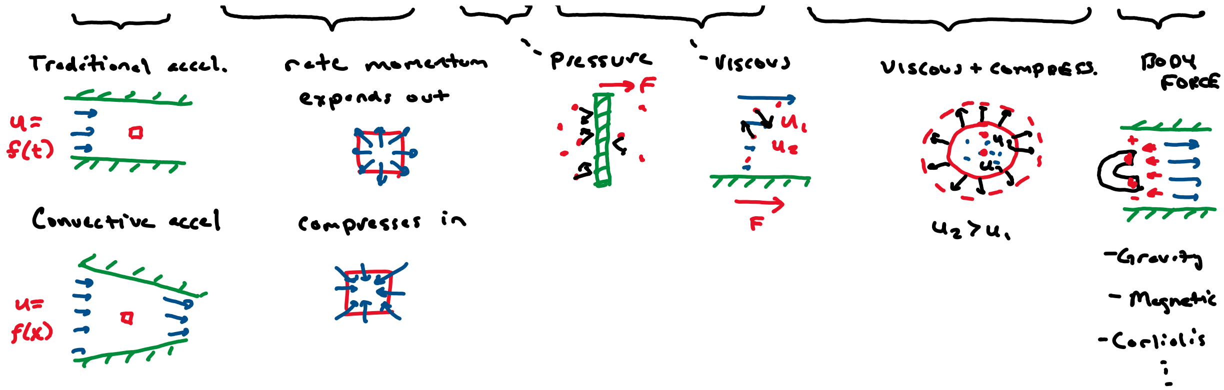
$$\frac{D(\cdot)}{Dt} = \frac{\partial(\cdot)}{\partial t} + u\frac{\partial(\cdot)}{\partial x} + v\frac{\partial(\cdot)}{\partial y} + w\frac{\partial(\cdot)}{\partial z}$$

Cons. of Momentum (general)

(x) $$\frac{D(\rho u)}{Dt} + \rho u\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) + \frac{1}{3}\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) + \rho a_x$$

(y) $$\frac{D(\rho v)}{Dt} + \rho v\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) + \frac{1}{3}\mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) + \rho a_y$$

(z) $$\frac{D(\rho w)}{Dt} + \rho w\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) + \frac{1}{3}\mu\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) + \rho a_z$$



Almost never used in full form for AERO

Common ASSUMPTIONS · (1) No body force $\rho a = 0$ → gas is light, gravity not important

(2) Inviscid $\mu = 0$ → Most flow force is pressure-based

(3) Steady $\frac{\partial(\,)}{\partial t} = 0$ → mostly objects at constant velocity

This gives us...

CONS. OF MOMENTUM (No body force, inviscid, steady)

$$\boxed{\begin{aligned}
&(x) \quad u\frac{\partial(\rho u)}{\partial x} + v\frac{\partial(\rho u)}{\partial y} + w\frac{\partial(\rho u)}{\partial z} + \rho u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial x} \\
&(y) \quad u\frac{\partial(\rho v)}{\partial x} + v\frac{\partial(\rho v)}{\partial y} + w\frac{\partial(\rho v)}{\partial z} + \rho v\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial y} \\
&(z) \quad u\frac{\partial(\rho w)}{\partial x} + v\frac{\partial(\rho w)}{\partial y} + w\frac{\partial(\rho w)}{\partial z} + \rho w\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z}
\end{aligned}}$$

"EULER" Equations (most common for Aero)

(... shortened to $\nabla \cdot (\rho u \vec{V}) = -\frac{\partial P}{\partial x}$)

Another less common Assumption: Incompressible ($M < 0.3$)

$\rho = \text{const.}$

$\frac{\partial(\rho u)}{\partial x} = \rho \frac{\partial u}{\partial x}$

If we make these assumption & consider flow on a STREAMLINE

$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$ ← from Cons. of Mass

$$\boxed{P_1 + \tfrac{1}{2}\rho u_1^2 = P_2 + \tfrac{1}{2}\rho u_2^2}$$ "BERNOULLI" Equation

(3) Conservation of Energy

Principle

Energy cannot be created/destroyed, only change form

$$\underbrace{\overbrace{\delta q}^{\text{heat}} + \underbrace{\delta w}_{\text{work}} = \underbrace{de}_{\text{change in energy}}}$$

(a.k.a., 1st Law of Thermodynamics)

For **Fluids**

Cons. of Energy (general)

$$\boxed{\underbrace{\frac{D}{Dt}\left[\rho\left(e+\frac{\vec{u}^2}{2}\right)\right]}_{(1)} + \underbrace{\rho\left(e+\frac{\vec{u}^2}{2}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)}_{(2)} = \underbrace{\rho\dot{q}}_{(3)} - \underbrace{\left(\frac{\partial(Pu)}{\partial x}+\frac{\partial(Pv)}{\partial y}+\frac{\partial(Pw)}{\partial z}\right)}_{(4)} + \rho\overbrace{\left(a_x u + a_y v + a_z w\right)}^{(5)} + \underbrace{\dot{Q}_{visc}}_{(6)} + \underbrace{\dot{W}_{visc.}}_{(7)}}$$

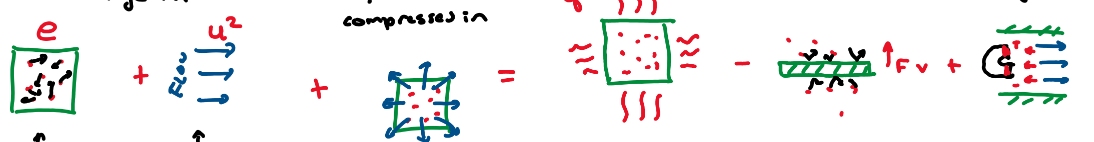
(1) Traditional + convective change in — e internal energy + u^2 kinetic energy (flow)

(2) rate energy expands out / compressed in

(3) heat in/out $\dot{q}$

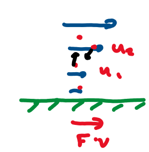
(4) work by pressure

(5) work by body force



= (3) − (4) F_v + (5)

(6+7) Work done $\dot{w}$ & Heat added $\dot{Q}$ by viscosity



Rarely used in full-form for **Aero** (thankfully)

Common Assumptions:
(1) Steady $\frac{\partial(\cdot)}{\partial t}=0$
(2) Inviscid $\mu=0$
(3) Adiabatic $\dot{q}=0$ (no heat addition)
(4) No Body Force $a=0$

Cons. of Energy (assumptions)

$$\boxed{u\frac{\partial}{\partial x}\left[\rho\left(e+\frac{\vec{u}^2}{2}\right)\right] + v\frac{\partial}{\partial y}\left[\rho\left(e+\frac{\vec{u}^2}{2}\right)\right] + w\frac{\partial}{\partial z}\left[\rho\left(e+\frac{\vec{u}^2}{2}\right)\right] = -\left(\frac{\partial(Pu)}{\partial x}+\frac{\partial(Pv)}{\partial y}+\frac{\partial(Pw)}{\partial z}\right)}$$

Also: Cons. of Energy ... 2 more equations for e & T ~temp.

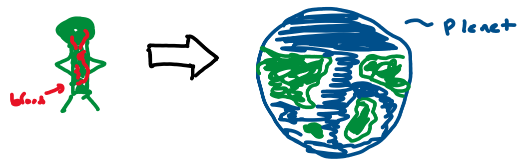
Note: Cons. of Energy needs

$$\boxed{e = \underbrace{C_v}_{\text{specific heat}} T}$$

$$\boxed{p = \rho \underbrace{R}_{\text{const}} T}$$

Theoretically...

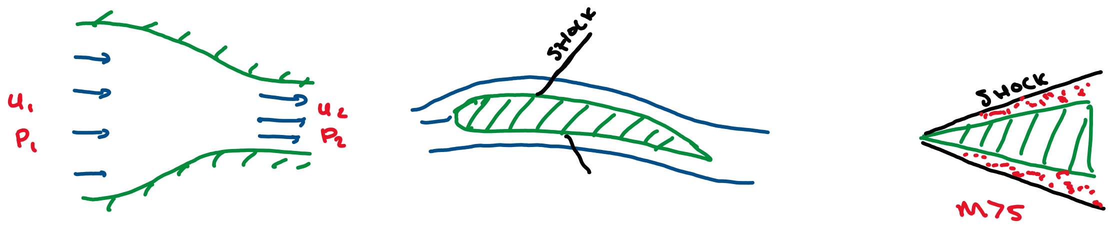
these FIVE eqns completely describe FLOW BEHAVIOR



Practically...

we use SIMPLIFIED versions of the eqn's for SPECIFIC FLOW SITUATIONS

(and sometimes they don't work)



Cons. of Mom. → Bernoulli eqn.

Cons. of Energy → Mach eqns

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