Lecture 7: Rotational and irrotational flow

Monday, September 21, 2020 4:39 PM

Last Time:

Path -
Streak -
Stream -
} lines for Flow Visualization

Stream function: $\psi(x,y) = C$, $u = \frac{\partial \psi}{\partial y}$, $v = \frac{\partial \psi}{\partial x}$ (incomp.)

Today: (1) flow rotation tools - Vorticity ζ & Circulation Γ

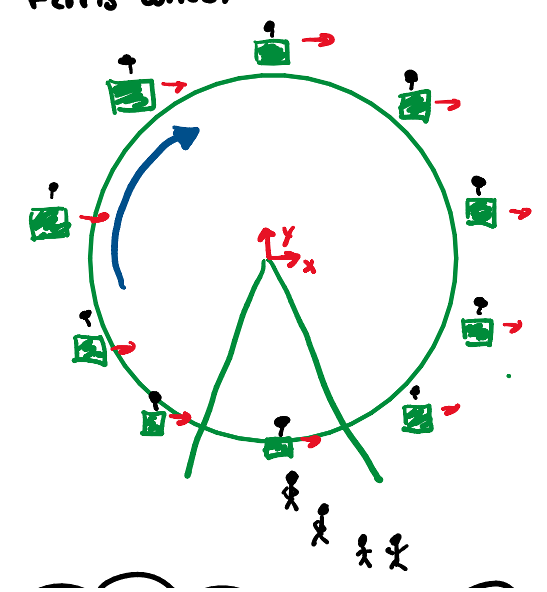
(2) irrotational flow tools - Velocity potential ϕ

(G) Rotation & Irrotation (?)

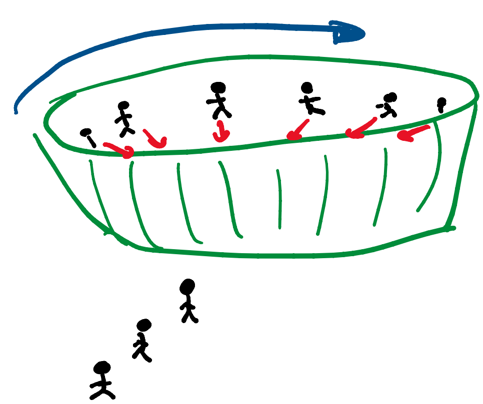
A major aspect of a flow is whether it can be considered: Rotational

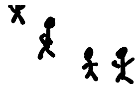
Non-fluids Example: The Fair! → 2 rides:

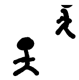
"Ferris Wheel"



"Gravitron"







You're going in a circle
but always facing the same way

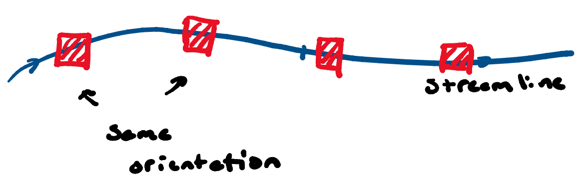
Irrotational

You're going in a circle
and changing the way you face

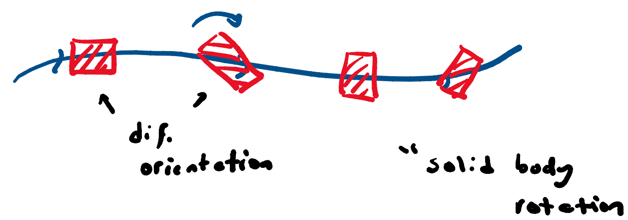
Rotational

For a FLUID, it's similar...

IRROTATIONAL

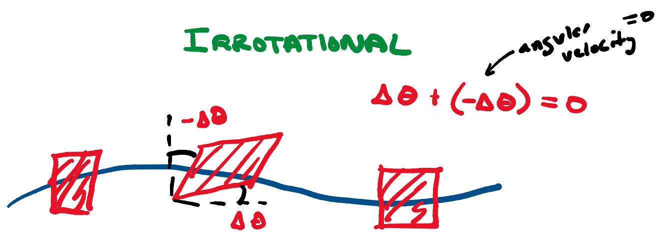


ROTATIONAL

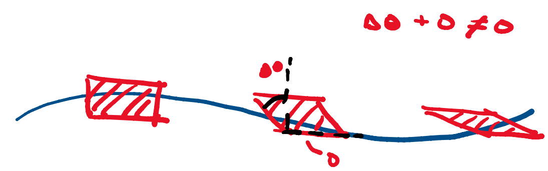


However, fluid elements can deform ...

IRROTATIONAL



ROTATIONAL



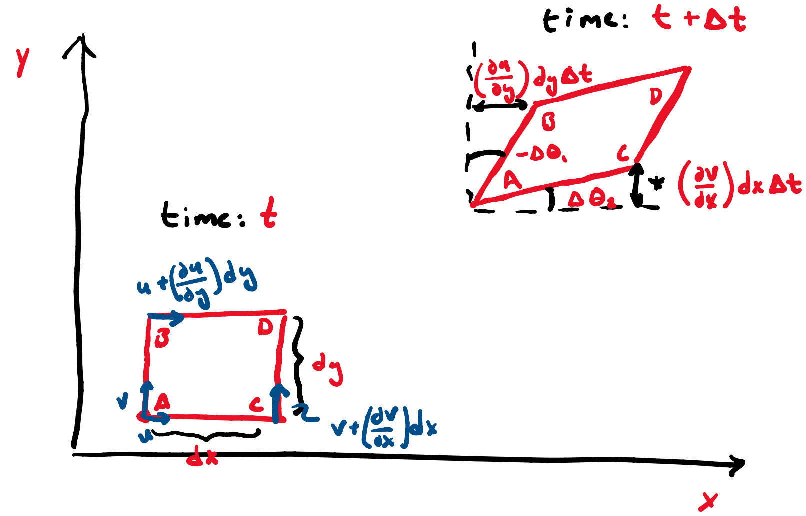
So FLUIDS undergo rotation in TRADITIONAL ways and NONTRADITIONAL ways

The official __measure__ of flow rotation is

(1) VORTICITY ζ

↳ rotational measure of FLUID FLOW

Let's define it more rigorously...



Note:

- y-distance A moves in Δt

$$= v\Delta t$$

- y-distance C moves in Δt

$$= \left(v + \frac{\partial v}{\partial x}dx\right)\Delta t$$

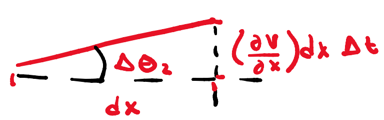
- Change in y between A & C after Δt...

$$= \left(v + \frac{\partial v}{\partial x}dx\right)\Delta t - v\,\Delta t$$

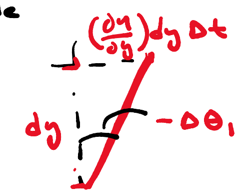
$$= \left(\frac{\partial v}{\partial x}dx\right)\Delta t \;*$$

Trig:

Bottom side



Left side



$$\tan(\Delta\theta_2) = \frac{\left(\frac{\partial v}{\partial x}\right)\cancel{dx}\,\Delta t}{\cancel{dx}} = \frac{\partial v}{\partial x}\Delta t$$

$$\tan(-\Delta\theta_1) = \frac{\left(\frac{\partial u}{\partial y}\right)\cancel{dy}\,\Delta t}{\cancel{dy}} = \frac{\partial u}{\partial y}\Delta t$$

if the angle is small: $\tan(\Delta\theta_2) \simeq \Delta\theta_2$

$$\Delta\theta_2 = \frac{\partial v}{\partial x}\Delta t \qquad\qquad \Delta\theta_1 = -\frac{\partial u}{\partial y}\Delta t$$

Assume <u>small</u> changes $\Delta \rightarrow d$

$$\frac{d\theta_2}{dt} = \frac{\partial v}{\partial x} \qquad\qquad \frac{d\theta_1}{dt} = -\frac{\partial u}{\partial y}$$

By definition, the ANGULAR VELOCITY is the average of these two

So...

$$\omega_z = \frac{1}{2}\left(\frac{d\theta_1}{dt} + \frac{d\theta_2}{dt}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

angular velocity

means in the x-y plane

In 3-dimensions (x, y, z) we get 3 components of ANGULAR VELOCITY $\omega_x, \omega_y, \omega_z$

$$\vec{\omega} = \frac{1}{2}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)i + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)j + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)k\right]$$

angular velocity vector

(... a.k.a. $\vec{\xi} = \nabla \times \vec{V}$)

and the VORTICITY is double $\vec{\omega}$

$$\boxed{\vec{\xi} = 2\vec{\omega}}$$

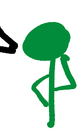
$$\begin{cases} \xi_x = \dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z} \\ \xi_y = \dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x} \\ \xi_z = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} \end{cases}$$

this is 2D

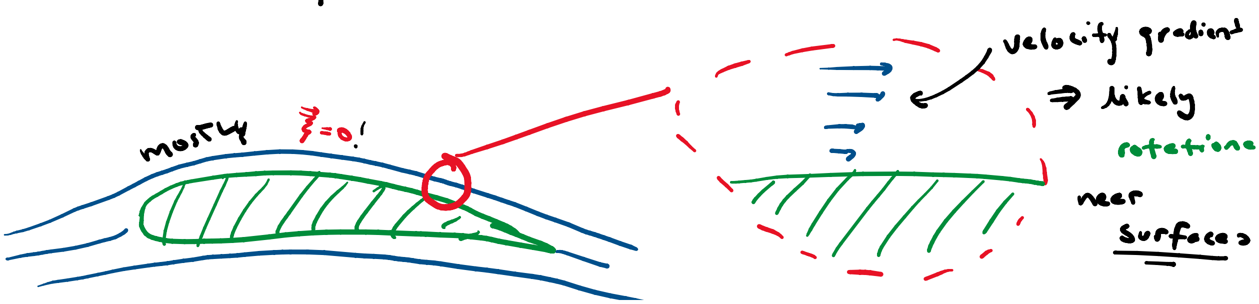
If $\vec{\xi} \neq 0$, FLOW is ROTATIONAL

If $\vec{\xi} = 0$, FLOW is IRROTATIONAL

when are my flows irrotational?



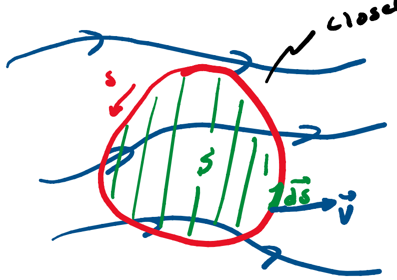
A lot of AERO analysis can use IRROTATIONAL



Sneak-peek: CIRCULATION Γ

used a lot in AERO and is related to vorticity.

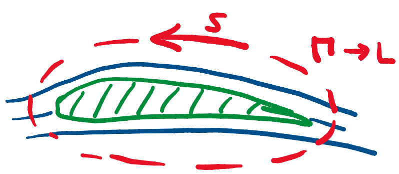
FLOW



Γ represents the flow "around the loop"

$$\boxed{\Gamma \equiv -\oint_C \vec{V}\cdot d\vec{s} = -\iint_S \underbrace{(\nabla\times\vec{V})}_{\xi \text{ vorticity}}\cdot dS}$$

Circulation is related to things like LIFT L



Stoke's theorem:

outer loop — surface

$$\oint A\cdot ds = \iint_S (\nabla\times A)\cdot dS$$

(2) VELOCITY POTENTIAL

ϕ

If flow is Irrotational $\xi = 0$ we can simplify our analysis greatly.

Note: much like the STREAM FUNCTION, ψ

we build ϕ

from above: $\xi = \nabla\times\vec{V} = 0$ → condition for IRROTATIONAL FLOW

we also know that: $\nabla\times(\nabla\phi) = 0$ → this is a vector identity we also know that: $\nabla \times (\nabla \phi) = 0$ → this is a vector identity

(ϕ ← scaler)

$$\Rightarrow \quad \boxed{\vec{V} = \nabla \phi}$$

For $\zeta = 0$ flow, a scaler function ϕ exists where velocity is the gradient of it. $\vec{V} = \nabla \phi$

CARTESIAN

(x)	$u = \dfrac{\partial \phi}{\partial x}$
(y)	$v = \dfrac{\partial \phi}{\partial y}$
(z)	$w = \dfrac{\partial \phi}{\partial z}$

CYLINDRICAL

$$u_r = \frac{\partial \phi}{\partial r}$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$$u_x = \frac{\partial \phi}{\partial x}$$

you may notice similarities between

VELOCITY POTENTIAL

$u = \dfrac{\partial \phi}{\partial x}$, $v = \dfrac{\partial \phi}{\partial y}$, $w = \dfrac{\partial \phi}{\partial z}$

- 3D
- $\zeta = 0$
- gradients same direction

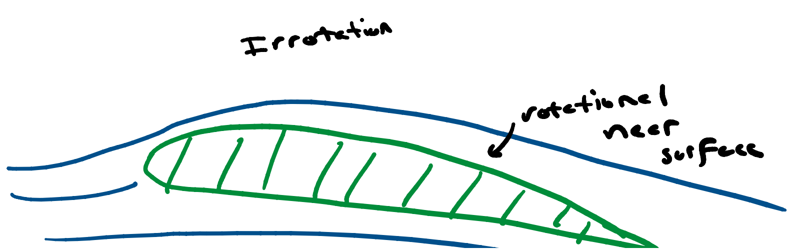
vs.

STREAM FUNCTION

$u = \dfrac{\partial \psi}{\partial y}$, $v = \dfrac{\partial \psi}{\partial x}$

- 2D
- $\zeta \neq 0$
- gradients opposite dir.

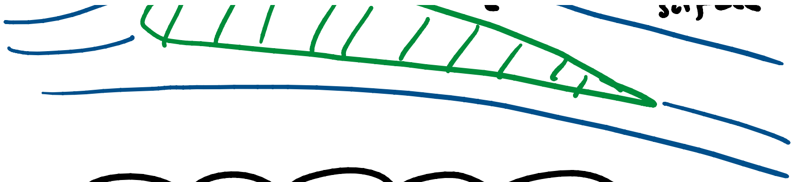
In <u>practice</u>, many flows are mostly IRROTATIONAL



Step 1: Irrotational analysis as first approximation

$u = \dfrac{\partial \phi}{\partial x} \; \ldots$

Step 2: Rotational analysis



Step 2: Rotational analysis added near boundary

$$\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} \dots\right.$$

Note: IRROTATIONAL implies INVISCID

$\zeta = 0$ $\qquad$ $\mu = 0$