Lecture 8: Bernoulli and Laplace

Sunday, September 27, 2020 1:40 PM

LAST TIME: VORTICITY $\vec{\xi} = \nabla \times \vec{V}$ $\Rightarrow$ Rotational: $\vec{\xi} \neq 0$

Irrotational: $\vec{\xi} = 0$

ϕ: $u = \frac{\partial \phi}{\partial x}$

$v = \frac{\partial \phi}{\partial y}$

TODAY: INVISCID & INCOMPRESSIBLE flows: Governing Equations

$\mu \approx 0$ $\qquad \rho = \text{const.}$

Bernoulli: $P + \frac{1}{2}\rho u^2 = \text{const.}$

Laplace: $\nabla^2 \phi = 0$, $\nabla^2 \psi = 0$

(H) INVISCID & INCOMPRESSIBLE flows (a.k.a. "ideal" or "perfect" fluids)

$\mu \approx 0$ $\qquad \rho = \text{const.}$

Today's focus: the Governing Equations

Is Inviscid = Irrotational?

... not exactly.

To make ▨ rotate, you need SHEAR STRESS

Conservation of Momentum - x (incomp.) $\rho = \text{const.}$

viscous

body force

$$\rho \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \rho g_x$$

viscous

body force

$$\rho \frac{Du}{Dt} = \underbrace{-\frac{\partial P}{\partial x} + \mu \Big(\frac{\partial^2 u}{\partial x^2}}_{\text{Normal stress}} + \underbrace{\Big(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \Big)}_{\text{Shear stress}} \Big) + \rho a_x$$

- If flow is INVISCID $\mu = 0$, the SHEAR STRESS terms go away.

$\Rightarrow$ nothing causes rotation 

- But, flow can come in w/ rotation & still be INVISCID

INVISCID $\neq$ IRROTATIONAL ... (technically)

when to assume

INVISCID : (1) away from boundaries (outside b-layer) 

(2) high $Re = \frac{\text{inertia}}{\text{viscous}}$ forces

(3) Low velocity gradients  $\int dy$ / du

INCOMP. : (1) low speed $M < 0.3$

(2) liquids 

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(1) BERNOULLI'S EQUATION

$P + \frac{1}{2}\rho u^2 = \text{const.}$

The most famous eqn. in AERODYNAMICS

$$P + \tfrac{1}{2}\rho u^2 = \text{const.}$$

The **most** famous eqn. in AERODYNAMICS

MAIN ASSUMPTIONS:

inviscid	incomp.	no body force	steady
$\mu \approx 0$	$\rho = \text{const.}$	$\rho g = 0$	$\frac{\partial(\cdot)}{\partial t} = 0$

Let's "derive" Bernoulli:

Start w/ Cons. Momentum (incomp.)

steady, inviscid, no body force

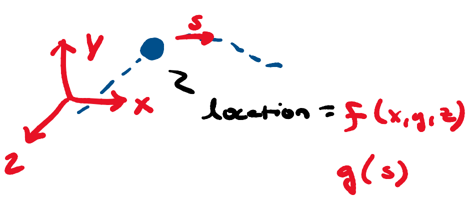
$$(x) \quad \rho\left(\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \cancel{\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)} + \cancel{\rho g_x}$$

$$(x) \quad \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x}$$

$$(y) \quad \rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y}$$

$$(z) \quad \rho\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z}$$

CARTESIAN SPACE



$x, y, z \rightarrow s$

$u, v, w \rightarrow U$ (sometimes V)

$\Rightarrow$ Along a streamline (Bernoulli:)

$$\rho U \frac{dU}{\cancel{ds}} = -\frac{dP}{\cancel{ds}}$$

$$\int \rho U\, dU = -\int dP$$

$$\tfrac{1}{2}\rho U^2 = -P + \text{const.}$$

BERNOULLI EQN!

$$\boxed{P + \tfrac{1}{2}\rho U^2 = \text{const}}$$

(along Streamline)

Note: this is **not** the exact derivation exact derivation

EXACT DERIVATION:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}$$

multiply by dx

$$u\frac{\partial u}{\partial x}dx+v\frac{\partial u}{\partial y}dx+w\frac{\partial u}{\partial z}dx=-\frac{1}{\rho}\frac{\partial p}{\partial x}dx \qquad (1)$$

If along STREAMLINE: $u\,dz-w\,dx=0$, $v\,dx-u\,dy=0$ $\Rightarrow$ $dx=\frac{u}{w}dz$, $dx=\frac{u}{v}dy$ $\qquad (2)$

Plug (2) into (1)

$$u\frac{\partial u}{\partial x}dx+u\frac{\partial u}{\partial y}dy+u\frac{\partial u}{\partial z}dz=-\frac{1}{\rho}\frac{\partial p}{\partial x}dx \qquad (3)$$

From CALCULUS if $u=f(x,y,z)$ then $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$

So (3) becomes

$$u\,du=-\frac{1}{\rho}\frac{\partial p}{\partial x}dx \qquad \text{a.k.a.} \qquad \frac{1}{2}d(u^2)=-\frac{1}{\rho}\frac{\partial p}{\partial x}dx \qquad (x)$$

$$\frac{1}{2}d(v^2)=-\frac{1}{\rho}\frac{\partial p}{\partial y}dy \qquad (y)$$

$$\frac{1}{2}d(w^2)=-\frac{1}{\rho}\frac{\partial p}{\partial z}dz \qquad (z)$$

Add up (x), (y), & (z) eqn's

U^2 dP

$$\frac{1}{2}\, d\big(\underbrace{u^2+v^2+w^2}_{\mathcal{U}^2}\big) = -\frac{1}{\rho}\underbrace{\left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right)}_{dP}$$

$$\frac{1}{2}\, d(\mathcal{U}^2) = -\frac{1}{\rho}\, dP \quad \text{a.k.a.} \quad \mathcal{U}\, d\mathcal{U} = -\frac{1}{\rho}\, dP \qquad (4)$$

Integrate both sides of (4)

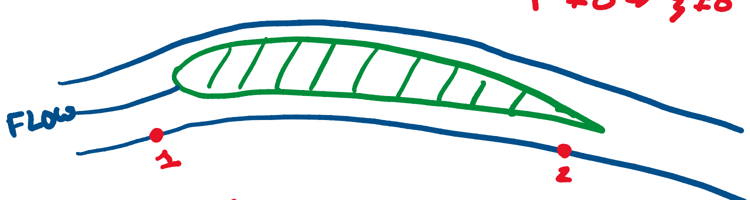
$$\rho \int \mathcal{U}\, d\mathcal{U} = -\int dP \;\Rightarrow\; \frac{1}{2}\rho\,\mathcal{U}^2 = -P + \text{const.}$$

Bernoulli Eqn.

$$\boxed{P + \frac{1}{2}\rho\,\mathcal{U}^2 = \text{const.}} \quad \text{(along streamline)}$$

If Rotational $\xi \neq 0$

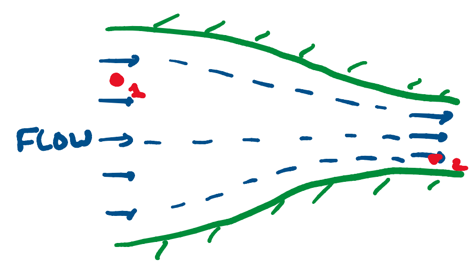
$\Gamma \neq 0 \Rightarrow \xi \neq 0$



$$P_1 + \frac{1}{2}\rho\,\mathcal{U}_1^2 = P_2 + \frac{1}{2}\rho\,\mathcal{U}_2^2$$

Bernoulli only works along streamline

If Irrotational $\xi = 0$



$$P_1 + \frac{1}{2}\rho\,\mathcal{U}_1^2 = P_2 + \frac{1}{2}\rho\,\mathcal{U}_2^2$$

Bernoulli: works everywhere in FLOW

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(2) Laplace's Equation(s)

The most famous eqn. in Math Physics

- electromagnetism
- heat
- fluids
- ⋮

$$\nabla^2 f = 0$$

$f(x,y,z)$

For Fluids $f = \phi$ (velo. potent.) & ψ (stream function)

Main Assumptions:

inviscid, incomp., irrotational *

$\mu = 0$ $\rho = \text{const.}$ $\vec{\xi} = 0$

There is Laplace Eqn. for Velo. Potential & Stream function

$\nabla^2 f = 0$ ϕ ψ

Velocity Potential:

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z} \qquad (1)$$

Since flow is Incompressible

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{(cons. of mass)} \qquad (2)$$

↳ incomp.

Combining (1) & (2)

Laplace Eqn. ϕ (cartesian)

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0}$$

(a.k.a. ... $\nabla^2 \phi = 0$)

(cylindrical)

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial x^2} = 0}$$

Stream Function:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \qquad (3)$$

Since flow is Irrotational $\vec{\xi} = 0$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad (4)$$ ← irrotational condition 2D

combine (3) & (4)

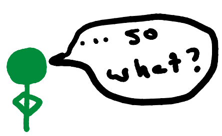
$$\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = 0$$

LAPLACE EQN. ψ

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0}$$

(a.k.a. ... $\nabla^2 \psi = 0$)

For IRROTATIONAL & INCOMPRESSIBLE flow the Velo. Pot. (ϕ) & Stream function (ψ) satisfy the LAPLACE EQN. ($\nabla^2 f = 0$)



LAPLACE's EQN. is popular throughout MATH + PHYSICS

⇒ solutions already exist! → Potential theory: harmonic functions

(i.e., PHYSICS gave us a TOOLBOX)

In practice, you use Bernoulli a lot

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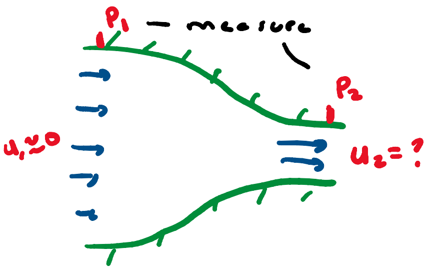
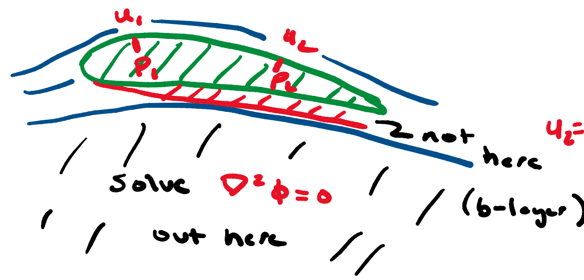
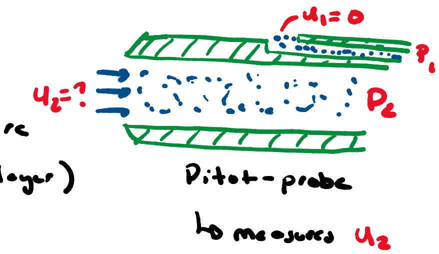
WIND TUNNELS



AIRFOILS



INSTRUMENTATION



You might also find LAPLACE in theory / simulations