Functions HL: Challenging Questions

Worksheet # 2.3

Topics Covered

1. More Challenging questions on Functions

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1 HL Only Functions Challenging Questions

1. Without using a calculator, solve the inequality $\frac{2x^2-x}{x^2+3x-4} > 1$.

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2. (i) On the same axes, sketch the graphs of $y = 2 + \frac{a}{x}$ and $y = 2 - |x|$, where a is a constant such that $1 < a < 2$.

(ii) Hence, or otherwise, solve the inequality $2 + \frac{a}{x} < 2 - |x|$.

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3. (GDC ALLOWED) The function f is defined by

$$\mathrm{f} : x \mapsto x^2 - mx, \quad x \in \mathbb{R}, x \geq \frac{m}{2},$$

where m is a positive constant.

(i) Find $\mathrm{f}^{-1}(x)$ and write down the domain of f^{-1}.

(ii) Sketch on the same diagram the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$, showing clearly the graphical relationship between the two graphs.

(iii) Find the value of m such that the curves in part (ii) intersect at the point where $x = 4$.

In the rest of the question, the value of m is given to be 1 . The function g is defined by

$$\mathrm{g} : x \mapsto \ln x, \quad x \in \mathbb{R}, \quad x \geq \sqrt{\mathrm{e}}.$$

(iv) Find an expression for $\mathrm{fg}(x)$ and hence, or otherwise, find the exact value of $(\mathrm{fg})^{-1}(2)$.

(v) Solve the inequality $\mathrm{fg}(x) > 5 - 0.1x$.

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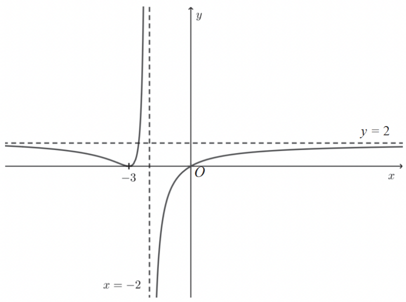
4. (a) The diagram shows the graph of $y = \mathrm{f}(x)$. The curve passes through the origin, has a minimum point at $(-3,0)$ and its equations of asymptotes are $x = -2$ and $y = 2$.



Sketch the following graphs on separate axes.

(i) $y = \frac{1}{\mathrm{f}(x)}$

(ii) $y = \mathrm{f}(-|x|)$

(iii) $y = (f(x))^2$

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5. The curve C has equation $y = \frac{x^2+ax+6}{x-2}$, where a is a constant. It is given that the line $y = x+5$ is an asymptote of C.

(i) Show that $a = 3$.

(ii) Prove algebraically that y cannot lie between -1 and 15 .

(iii) Sketch the curve C, stating clearly the coordinates of any points of intersection with the axes, the coordinates of any stationary points and the equations of asymptotes.

(iv) Hence, solve the inequality $\frac{x^2+3x+6}{x-2} \geq -x-3$.

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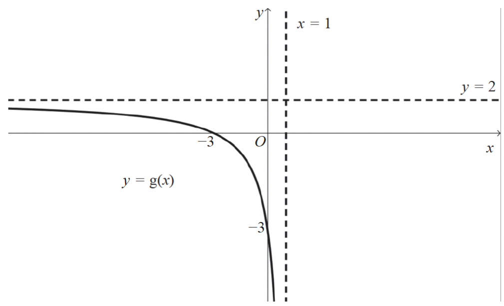
6. The function f is defined as follows

$$f : x \mapsto x - \sqrt{x^2+1}, \quad x \in \mathbb{R}$$

The graph of function g with domain $(-\infty, 1)$ is given in the diagram below. It has asymptotes $x = 1$ and $y = 2$, and cuts the axes at $(0,-3)$ and $(-3,0)$.

(a) (i) Show that f^{-1} exists.



(ii) Show that gf exists and find the range of gf.

(b) Sketch the graph of $y = \mathrm{g}(-x-1)$.

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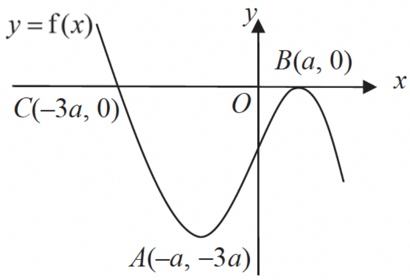
7. (a) The diagram shows the curve $y = \mathrm{f}(x)$, where a is a positive constant.

The curve has a minimum point at $A(-a, -3a)$, a maximum point at $B(a, 0)$ and cuts the x-axis at the point $C(-3a, 0)$.



Sketch, labelling each graph clearly and showing the coordinates of the points corresponding to A, B and C whenever possible, the graphs of

(i) $y = 3\mathrm{f}(x-a)$,

(ii) $y = \mathrm{f}\left(\left|\frac{x}{2}\right|\right)$,

(iii) $y = \frac{1}{\mathrm{f}(x)}$.

(iv) $y = (f(x))^2$

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8. Functions f and g are defined by $\mathrm{f} : x \mapsto \ln(x-a), \quad x \in \mathbb{R}, a < x < a+1$, where a is a positive constant, $\mathrm{g} : x \mapsto \frac{1}{x^2+1}, \quad x \in \mathbb{R}, x \leq 2$.

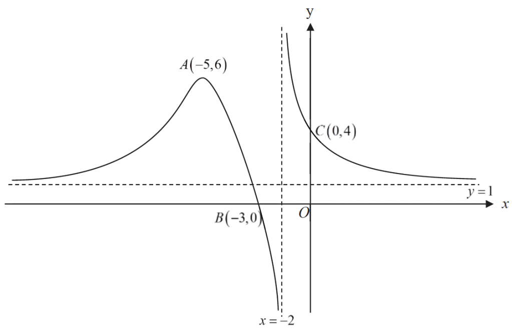
(i) Show that the composite function gf exists.

(ii) Find gf in a similar form.

(iii) Find the range of gf, showing your working clearly.

9. (a) State a sequence of transformations that will transform the curve with equation $y = x^2$ onto the curve with equation $y = 3 + \frac{1}{4}x^2$.

(b) In the diagram, the graph of $y = \mathrm{f}(x)$ has a maximum turning point at $A(-5, 6)$ and axial intercepts at $B(-3, 0)$ and $C(0, 4)$. The lines $x = -2$ and $y = 1$ are the asymptotes of the graph.



Sketch, on separate diagrams, the graphs of

(i) $y = \mathrm{f}(3x + 5)$,

(ii) $y = \mathrm{f}(-|x|)$,

(iii) $y = (f(x))^2$

stating clearly, in each case, the equations of any asymptotes and the coordinates of the points corresponding to A, B and C.

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10. The function f is defined by

$$\mathrm{f} : x \mapsto \frac{x^2 - 8x + 28}{4x - 32} \quad \text{for } x \in \mathbb{R}, \quad x \neq 8$$

(i) Find the exact x-coordinates of the turning points of $y = \mathrm{f}(x)$.

(ii) Sketch the graph of $y = \mathrm{f}(x)$, labelling clearly the equations of the asymptotes and coordinates of axial intercepts and turning points.

For the rest of the question, the domain of f is restricted to $8 < x \leq a, x \in \mathbb{R}$, where a is a positive constant such that the function f^{-1} exists.

(iii) State the exact greatest value of a.

(iv) Using the value of a found in part (iii), find $\mathrm{f}^{-1}(x)$ and write down the domain of f^{-1}.

(v) Sketch, on a single diagram, the graphs of $y = \mathrm{f}(x), y = \mathrm{f}^{-1}(x)$ and $y = x$, showing the relationship between the graphs.

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11. Functions f and g are defined by

$$\mathrm{f} : x \mapsto 1 - \lambda x^2, \quad x \in \mathbb{R}, x < -1,$$
$$\mathrm{g} : x \mapsto 2 - \mathrm{e}^{1-x}, \quad x \in \mathbb{R}, x1$$

where λ is a positive constant.

(i) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}.

(ii) Show that gf exists and find the range of gf, giving your answer in terms of λ.

The function h is defined by

$$\mathrm{h} : x \mapsto 1 - \lambda x^2, \quad x \in \mathbb{R}, x < k$$

where k is a constant.

Determine the set of values of k for which the range of gh is the same as the range of g.

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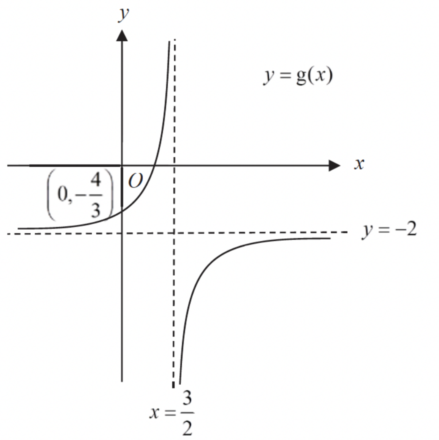
12. The diagram below shows the graph of $y = \mathrm{g}(x)$, where $\mathrm{g}(x) = \frac{ax+b}{2x+c}$.

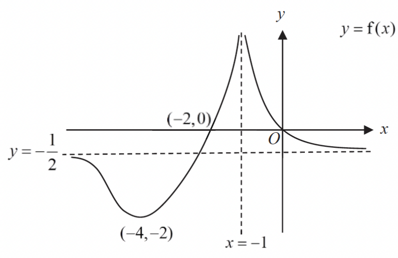


Determine the values of a, b and c.

It is also given that $\mathrm{g}(x) = \mathrm{f}\left(\frac{1}{2}x - 1\right)$. State a sequence of 2 transformations that will map the graph of $y = \mathrm{g}(x)$ to the graph of $y = \mathrm{f}(x)$.

Find $\mathrm{f}(x)$.

13.



The diagram above shows the graph of $y = \mathrm{f}(x)$. The curve passes through $(0,0)$ and $(-2,0)$, and has a minimum point at $(-4,-2)$. The curve has asymptotes $x = -1$ and $y = -\frac{1}{2}$.

(a) State the coordinates of the turning point of the curve $y = 1 - 2\mathrm{f}(x)$.

(b) On separate diagrams, sketch the graphs of (i) $y = \mathrm{f}(|x|)$, [2]

(ii) $y = \frac{1}{\mathrm{f}(x)}$,

(iii) $y = \mathrm{f}'(x)$.

(iv) $y = (f(x))^2$

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14. A curve $y = \mathrm{f}(x)$ undergoes, in succession, the following transformations.

A : A translation of 1 unit in the negative x-direction.

B : A reflection about the y-axis.

C : A scaling parallel to the y-axis with scale factor of 2 .

The equation of the resulting curve is $y = \mathrm{g}(x)$, where $\mathrm{g}(x) = \frac{4x-1}{x^2+3}$.

Find $\mathrm{f}(x)$.

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15. The function f is defined as follows.

$$\mathrm{f} : x \mapsto x^2 - 3x + 2, \quad x \in \mathbb{R}, \quad x \geq \frac{3}{2}$$

(i) Find $\mathrm{f}^{-1}(x)$ and write down the domain and range of f^{-1}.

The function g is defined as follows.

$$\mathrm{g} : x \mapsto \ln x, \quad x \in \mathbb{R}, \quad x > 0$$

(ii) Explain why the composite function gf^{-1} exists.

(iii) Find $\mathrm{gf}^{-1}(x)$, stating the domain and range of gf^{-1}.

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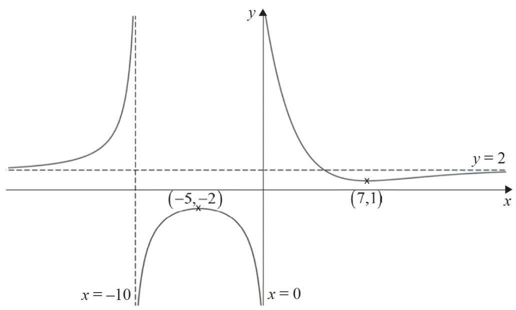
16. The diagram shows the curve $y = \mathrm{g}(x)$ with asymptotes $y = 2, x = -10$ and $x = 0$.

The maximum point and the minimum point of the curve are $(-5, -2)$ and $(7, 1)$ respectively.



On separate diagrams, sketch the graphs of

(i) $y = \frac{1}{\mathrm{g}(x)}$,

(ii) $y = \mathrm{g}'(x)$,

(iii) $y = (f(x))^2$

stating clearly the equations of any asymptotes, the coordinates of any turning points and any points of intersection with the x - and y-axes.
