CHEMYST
NAVIGATE IB WITH US

HL: Complex Numbers-Argand planes introduction

Worksheet # 6.1

Topics Covered

1. HL: Argand diagrams intro with basic complex numbers operations with problems related to polynomials

2. HL: De Moivre's theorem introduction and Drill questions

(Really important to convert one form of complex number to another quickly using fast trignonometry concepts)

(Refer to worksheet # 2.5 for more Basic introduction)

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1 Forms of complex numbers and Argand plane questions

1. Find the following complex numbers in polar form and check your answers by GDC.

a) $2+2\mathrm{i}$

b) $\frac{3}{2}\mathrm{i}$

c) $-4-3\mathrm{i}$

d) $21-20\mathrm{i}$

e) $-1+\sqrt{3}\mathrm{i}$

f) $-\frac{4}{3}\mathrm{i}$

g) $\frac{\sqrt{2}}{3}-\frac{\sqrt{2}}{4}i$

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2. Given the complex number $z=\frac{7}{12}\mathrm{e}^{\mathrm{i}\frac{\pi}{9}}$, find n polar form:

$a)-z$ b) z^* c) $-z^*$

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3. Multiply the following complex numbers in Euler's form and polar form.

a) $z_1=3\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$ and $z_2=5\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}$

b) $z_3=11\,\mathrm{cis}\,210°$ and $z_4=23\,\mathrm{cis}\,315°$

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4. Given the numbers $z_1=4$ cis $120°$ and $z_2=3+3i$, find:

a) z_1 in Cartesian form

b) z_2 in polar form

c) $z_1\times z_2$ in both forms.

Hence, find the exact value of:

d) $\cos 165°$

e) $\tan 165°$.

5. Given the numbers $z_1 = \text{cis}\,\frac{3\pi}{4}$ and $z_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}$, find:

a) z_1 in Cartesian form b) z_2 in polar form c) $z_1 \times z_2$ in both forms.

Hence find the exact values of:

d) $\sin\frac{17\pi}{12}$ e) $\tan\frac{17\pi}{12}$.

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6. Given the number $z = r\text{e}^{\text{i}\theta}$, find the values of r and θ so that:

a) $z(\sqrt{3} - \text{i})$ is a real number less than 3

b) $z(-1+i)$ is an imaginary number with a modulus greater than 4 .

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7. Calculate the following product:

$$\left(\sin\frac{\pi}{12} + i\cos\frac{\pi}{12}\right)\left(\sin\frac{\pi}{6} + i\cos\frac{\pi}{6}\right) \times \left(\sin\frac{\pi}{4} + i\cos\frac{\pi}{4}\right)$$

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8. Let $z_1 = 1 + \text{i}$ and $z_2 = 1 - \sqrt{3}\text{i}$.

Find the following in Euler's form:

a) $\frac{z_1}{z_2}$ b) $\frac{-z_2*}{z_1}$ c) $\frac{1}{z_1 z_2}$ d) $\frac{-z_1*}{(z_1 z_2)*}$

Express, in polar form:

a) $\frac{3}{2+2i}$ b) $\frac{4-4i}{-1+\sqrt{3}i}$ c) $\frac{\sqrt{15}-\sqrt{5}i}{\sqrt{2}+\sqrt{6}i}$

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9. Given the numbers $z_1 = 5\,\text{cis}\,60^\circ$ and $z_2 = 3 + 3i$, find:

a) z_1 in Cartesian form b) z_2 in polar form c) $\frac{z_1}{z_2}$ in both forms.

Hence find the exact value of:

d) $\cos 15^\circ$ e) $\sin 15^\circ$ & $\tan 15^\circ$

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10. Write in the form $a + bi$ where $a, b \in \mathbb{Q}$: a $\frac{3+4i}{1-3i}$ b $\frac{3}{i}\left(\frac{1}{\sqrt{5}} - \frac{2i}{\sqrt{5}}\right)^2$

11. Suppose $\frac{z+2}{z-2} = i$. Find z in the form $a+bi$ where $a, b \in \mathbb{R}$.

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12. Solve for z : $z^2 - z + 1 + i = 0$

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13. Find the exact values of $x, y \in \mathbb{R}$ such that:

a) $(3-2i)(x-yi) = -i$

b) $(x+yi)^2 - (x-yi)^2 = x - y + 16i$

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14. Suppose $z = iz^*$ where $z = x + iy$ and $x, y \in \mathbb{R}$.

Deduce that $x = y$.

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15. Prove that $(zw)^* = z^* w^*$.

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16. Show that, for any complex number $z \neq 0$, $\frac{z}{z^*} + \frac{z^*}{z}$ is always real.

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17. Simplify the expression $(w + 3z^*) + (z - w^*)^*$ using the properties of conjugates.

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18. z and w are complex numbers such that $\frac{w}{z} = 1 + i$ and $w - 2z^* = -1 - 5i$. Find z and w.

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19. Find $\sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n$.

20.Suppose $z = 2 + i$ and $w = 3 - 2i$. Find:

a) $2z + w$

b) $w^* - z$

c) $z^* + 2w + 2i$.

Illustrate your answers on separate Argand diagrams.

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21. Find the complex number z that satisfies the equation $\frac{10}{z} + \frac{15}{z^*} = 5 + 2i$ given $|z| = \sqrt{5}$.

z is a complex number where $|z| = 1$ and $\arg z \in \left[0, \frac{\pi}{2}\right]$.

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22. Given that $\arg\left(\frac{z}{z+2}\right) = \frac{\pi}{4}$, find $|z+2|$.

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23 .On an Argand plane, points P, Q, and R represent the complex numbers z_1, z_2, and z_3 respectively.

If $i(z_3 - z_2) = z_1 - z_2$, what can be deduced about triangle PQR ?

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24. Illustrate on an Argand diagram the complex numbers z satisfying:

a) $|z+3-2i| = 2$ b) $|z-i| > 1$ c) $\arg(z+1) = -\frac{\pi}{4}$ d) $\frac{\pi}{4} \leq \arg(z-i) < \frac{\pi}{2}$

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25. If $z = r\operatorname{cis}\theta$, write z^4, $\frac{1}{z}$, and iz^* in polar form.

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26. Use the properties of cis to simplify the following. Convert your answer to exact Cartesian form if possible.

a) $2\operatorname{cis}\frac{\pi}{7}\operatorname{cis}\frac{6\pi}{7}$ b) $\left(\operatorname{cis}\frac{5\pi}{12}\right)^2$ C) $\frac{\sqrt{8}\operatorname{cis}\frac{3\pi}{16}}{\sqrt{2}\operatorname{cis}\left(-\frac{5\pi}{16}\right)}$ d) $\operatorname{cis}(\theta + 15\pi)$

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27. Suppose

$$z = \sqrt{3} + i$$

and

$$w = 2 - 2i$$

.

a) Write z and w in polar form.

b) Hence find zw in polar form.

c) Describe the transformation to z when it is multiplied by w.

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28. Let $z = \frac{-1+i\sqrt{3}}{4}$ and $w = \frac{\sqrt{2}+i\sqrt{2}}{4}$.

a) Write z and w in the form $r\operatorname{cis}\theta$ where $r > 0$ and $-\pi < \theta \leq \pi$.

b) Show that $zw = \frac{1}{4}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$.

c) Hence find the exact values of $\cos\frac{11\pi}{12}$ and $\sin\frac{11\pi}{12}$.

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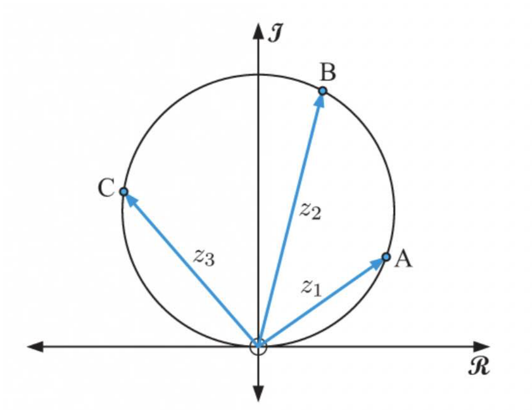
29. If $z = \cos\theta + i\sin\theta$ where $0 < \theta < \frac{\pi}{4}$, find the modulus and argument of $1 - z^2$.

30.



Points O, A, B, and C lie on a circle. Suppose z_1 represents $\overrightarrow{OA}$, z_2 represents $\overrightarrow{OB}$, and z_3 represents $\overrightarrow{OC}$

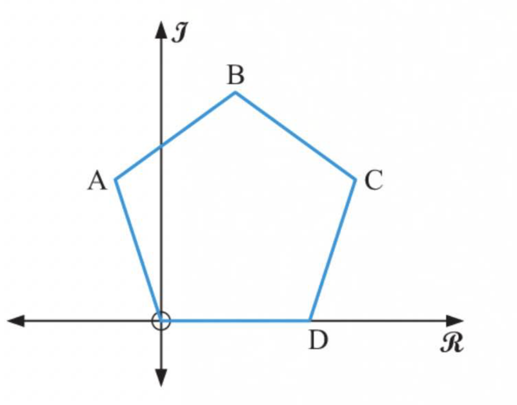
a) What vectors are represented by $z_1 - z_2$ and $z_3 - z_2$?

b) Hence find the value of $\arg\left(\frac{z_3}{z_1}\right) + \arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right)$.

31.

Using the sum and product of roots, find the real quadratic equations with roots $3\,\text{cis}\,\dfrac{5\pi}{6}$ and $3\,\text{cis}\,\dfrac{7\pi}{6}$.

32.



OABCD is a regular pentagon with side length 1 . Let $z_1 \equiv \overrightarrow{OA}, z_2 \equiv \overrightarrow{OB}$ $z_3 \equiv \overrightarrow{OC}$, and $z_4 \equiv \overrightarrow{OD}$

a Write in polar form:

i) z_1 ii) $z_2 - z_1$ iii) z_3

b) Find the smallest positive integer n such that z_2^n is a real number.

c) Show that $z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0$.

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33. Write:

a) $\sqrt{3} + i$ in polar form and Euler form

b) $2\operatorname{cis}\frac{5\pi}{6}$ in Cartesian form and Euler form

c) $5e^{-i\frac{\pi}{4}}$ in Cartesian form and polar form.

34. a) Express $1 + i$ and $\sqrt{3} - i$ in the form $re^{i\theta}$

b) Hence write $z = \frac{-1-i}{\sqrt{3}-i}$ in the form $re^{i\theta}$.

c) Find the smallest positive integer n such that z^n is a real number.

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35. a) Write $z = \frac{1+i\sqrt{3}}{1+i}$ in the form $r\operatorname{cis}\theta, r > 0$.

b) Hence find the smallest positive value of n such that z^n is:

i) real ii) purely imaginary.

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36. Write $z = \frac{-1+5i}{2+3i}$ in polar form. Hence show that $z^{12} = -64$.

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37. Use De Moivre's theorem to find the exact value of:

a $\left(\sqrt{5}\operatorname{cis}\frac{\pi}{8}\right)^6$

b $(\sqrt{3} - i)^5$

c $(\sqrt{2} + i\sqrt{6})^{\frac{1}{2}}$

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38. a) Find the cube roots of $-27i$ and display them on an Argand diagram, labelling them z_1, z_2, and z_3.

b) Show that $z_2 z_3 = z_1^2$, where z_1 is any of the cube roots found in **a**.

c) What is the value of $z_1 z_2 z_3$?

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39. a) Find the cube roots of $-2-2i$, and display them on an Argand diagram.

b) By considering the sum of the cube roots, show that $\cos\frac{\pi}{4} + \cos\frac{5\pi}{12} + \cos\frac{13\pi}{12} = 0$.

40.

$$2z^3 + pz^2 + qz + 16 = 0, p \in \mathbb{R}, q \in \mathbb{R}$$

The above cubic equation has roots α, β and γ, where γ is real.

It is given that $\alpha = 2(1 + \mathrm{i}\sqrt{3})$.

a) Find the other two roots, β and γ.

b) Determine the values of p and q.

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41. Solve the equation

$$2z^4 - 14z^3 + 33z^2 - 26z + 10 = 0, z \in \mathbb{C}$$

given that one of its roots is $3+i$.

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42. The following complex numbers are given.

$z_1 = 2 - 2\text{i}$, $\quad z_2 = \sqrt{3} + \text{i}$ $\quad$ and $\quad z_3 = a + b\text{i}$ where $a \in \mathbb{R}, b \in \mathbb{R}$.

a) If $|z_1 z_3| = 16$, find the modulus z_3.

b) Given further that $\arg\left(\frac{z_3}{z_2}\right) = \frac{7\pi}{12}$, determine the argument of z_3.

c) Find the values of a and b, and hence show $\frac{z_3}{z_1} = -2$.

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43. The following cubic equation is given

$$z^3 + 2z^2 + az + b = 0$$

where $a \in \mathbb{R}, b \in \mathbb{R}$.

One of the roots of the above cubic equation is $1 + \text{i}$.

a) Find the real root of the equation.

b) Find the value of a and the value of b.

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44. The complex conjugate of z is denoted by $\bar{z}$. Solve the equation

$$\frac{2z + 3\text{i}(\bar{z} + 2)}{1 + \text{i}} = 13 + 4\text{i}$$

giving the answer in the form $x + \text{i}y$, where x and y are real numbers.

45.Find the value of x and the value of y in the following equation, given further that $x \in \mathbb{R}, y \in \mathbb{R}$.

$$\frac{x}{1 + \text{i}} = \frac{1 - 5\text{i}}{3 - 2\text{i}} + \frac{y}{2 - \text{i}}$$

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46. It is given that

$$z + 2\text{i} = \text{i}z + k, k \in \mathbb{R} \quad \text{and} \quad \frac{w}{z} = 2 + 2\text{i}, \text{Im}\, w = 8$$

Determine the value of k.

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47. The complex numbers z and w are defined as

$z = 3 + \mathrm{i}$ and $w = 1 + 2\mathrm{i}$.

Determine the possible values of the real constant λ if

$$\left| \frac{z}{w} + \lambda \right| = \sqrt{\lambda + 2}$$

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48. The complex number z satisfies the equation

$$z^2 = 3 + 4i$$

a) Find the possible values of …

i. … z.

ii. … z^3.

b) Hence, by showing detailed workings, find a solution of the equation

$$w^6 - 4w^3 + 125 = 0, w \in \mathbb{C},$$

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49. The following complex numbers are given

$$z = \frac{1 + \mathrm{i}}{1 - \mathrm{i}} \text{ and } w = \frac{\sqrt{2}}{1 - \mathrm{i}}$$

a) Calculate the modulus of z and the modulus of w.

b) Find the argument of z and the argument of w.

In a standard Argand diagram, the points A, B and C represent the numbers z, $z + w$ and w respectively. The origin of the Argand diagram is denoted by O.

c) By considering the quadrilateral $OABC$ and the argument of $z + w$, show that

$$\tan\left(\frac{3\pi}{8}\right) = 1 + \sqrt{2}$$

50. The complex conjugate of z is denoted by $\bar{z}$.

Solve the equation

$$z + 2\bar{z} = |z+2|, z \in \mathbb{C}.$$

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51. It is given that

$$z = \cos\theta + i\sin\theta, 0 \leq z < 2\pi$$

Show clearly that

$$\frac{2}{1+z} = 1 - \mathrm{i}\tan\left(\frac{\theta}{2}\right)$$

52.

$$\frac{(3+4\mathrm{i})(1+2\mathrm{i})}{1+3\mathrm{i}} = q(1+\mathrm{i}), \quad q \in \mathbb{R}$$

a) Find the value of q.

b) Hence simplify

$$\arctan\frac{4}{3} + \arctan 2 - \arctan 3$$

giving the answer in terms of π.

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53. Sketch on a standard Argand diagram the locus of the points $z = \sqrt{2}(1+\mathrm{i}), \quad w = \sqrt{3} - \mathrm{i}$ and $z + w$, and use geometry to prove that

$$\tan\left(\frac{\pi}{24}\right) = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2$$

You must justify all the steps in this proof.

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54. It is given that

$$z = \frac{1+8\text{i}}{1-2\text{i}}$$

a) Express z in the form $x + \text{i}y$, where x and y are real numbers.

b) Find the modulus and argument of z.

c) Show clearly that

$$\arctan 8 + \arctan 2 + \arctan \frac{2}{3} = \pi.$$

55.

$$z = (2+3\text{i})^{4n+2} + (3-2\text{i})^{4n+2}, n \in \mathbb{N}$$

Show clearly that $z = 0$ for all $n \in \mathbb{N}$.

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56. Two distinct complex numbers z_1 and z_2 are such so that

$|z_1| = |z_2| = r \neq 0$.

Show clearly that $\frac{z_1+z_2}{z_1-z_2}$ is purely imaginary.

You may find the result $z\bar{z} = |z|^2 = r^2$ useful.

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2 De moivre's theorem Drill Questions

57.

$$z_1 = \sqrt{2}\left(\cos\frac{\pi}{3} - \cos\frac{5\pi}{6}\mathrm{i}\right)$$
$$\text{and } z_2 = 2\left(\sin\frac{5\pi}{6} - \sin\frac{\pi}{3}\mathrm{i}\right)$$
$$\text{find } \frac{z_1^3}{z_2^5}$$

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58. Find the Cartesian form of $\left(\frac{\sin\theta + i\cos\theta}{\cos\theta - i\sin\theta}\right)^{2019}$.

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59. geometric sequence of complex numbers is given: $2+i, 1+3i, -2+4i, \ldots$

a) Find the common ratio.

b) What is the ninth term of the sequence?

c) Find the sum of the first nine terms of the sequence.

60.

$$z^4 = -16, z \in \mathbb{C}$$

a) Determine the solutions of the above equation, giving the answers in the form $a+b\mathrm{i}$, where a and b are real numbers.

b) Plot the roots of the equation as points in an Argand diagram.

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61. Given that the complex number $\omega \neq 1$ is a solution of $z^3 = 1$, show that:

a) $\omega^2 + \omega + 1 = 0$

b) $(\omega^*)^2 + \omega^* + 1 = 0$

c) Hence, calculate $\omega^{2019} + \omega^{2020} + \omega^{2021} + \omega^{2022}$.

62.

Solve the equation $z^4 = -8 + 8\sqrt{3}i$. Draw the solutions on the Argand diagram.

63. The complex number $z = -4\sqrt{2} + 4\sqrt{2}i$ is given.

a) Write the number in Euler's form.

b) Show that the real part of the complex number $\sqrt[6]{z}$ is $\frac{\sqrt{2\sqrt{2}+4}}{2}$.

c) Find the exact form of the imaginary part of the complex number $\sqrt[6]{z}$.

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64. Let $z = \text{cis}\,\theta = \cos\theta + i\sin\theta$.

a) Use the binomial theorem to find the real and imaginary part of z^3.

b) Use De Moivre's theorem to find the formulae for:

i) $\cos 3\theta$ ii) $\sin 3\theta$ iii) $\tan 3\theta$

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65. Let the number $z = \text{cis}\,\theta$.

a) Use the binomial theorem to expand $\left(z - \frac{1}{z}\right)^4$.

b) Use the formulae $z^n + \frac{1}{z^n} = 2\cos n\theta$ and $z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta$ to find $\sin^4\theta$.

c) Hence, find $\int \sin^4 x \, \mathrm{d}x$.

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66. The complex number is defined as $z = \cos\theta + \mathrm{i}\sin\theta, -\pi < \theta \leq \pi$.

a) Show clearly that ...

i. $\ldots z^n + \frac{1}{z^n} = 2\cos\theta$.

ii. $\ldots z^n - \frac{1}{z^n} = 2i\sin\theta$.

iii. $\ldots 8\sin^4\theta = \cos 4\theta - 4\cos 2\theta + 3$.

b) Hence solve the equation

$$8\sin^4\theta + 5\cos 2\theta = 3, -\pi < \theta \leq \pi.$$

67*. It is given that

$$\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$$

a) Use de Moivre's theorem to prove the validity of the above trigonometric identity.

It is further given that

$$\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta.$$

b) Solve the equation

$$\sin 5\theta = 5\sin 3\theta \text{ for } 0 \leq \theta < \pi,$$

giving the solutions correct to 3 decimal places.

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68. The complex number $z = -9\mathrm{i}$ is given.

a) Determine the fourth roots of z, giving the answers in the form $re^{\mathrm{i}\theta}$, where $r > 0$ and $0 \leq \theta < 2\pi$.

b) Plot the points represented by these roots in Argand diagram, and join them in order of increasing argument, labelled as A, B, C and D.

The midpoints of the sides of the quadrilateral $ABCD$ represent the 4^{th} roots of another complex number w.

c) Find w, giving the answer in the form $x + \mathrm{i}y$, where $x \in \mathbb{R}, y \in \mathbb{R}$.

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69. The complex number z is given in polar form as

$$\cos\left(\frac{2}{5}\pi\right) + i\sin\left(\frac{2}{5}\pi\right)$$

a) Write z^2, z^3 and z^4 in polar form, each with argument θ, so that $0 \leq \theta < 2\pi$.

In an Argand diagram the points A, B, C, D and E represent, in respective order, the complex numbers

$1, \quad 1+z, \quad 1+z+z^2, \quad 1+z+z^2+z^3, \quad 1+z+z^2+z^3+z^4.$

b) Sketch these points, in the sequential order given, in a standard Argand diagram.

c) State the exact argument of

$$1+z+z^2.$$

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70. Let $z = e^{i\theta}, \theta \in \mathbb{R}$.

Show that $\frac{1}{1+z} = \frac{1}{2}\left(1 - i\tan\frac{\theta}{2}\right)$.

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71. The equation $z^5 - 1 = 0$ is given.

a) Find all the solutions in polar form.

b) Hence, show that

$$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} + \cos\frac{6\pi}{5} + \cos\frac{8\pi}{5} = -1$$
