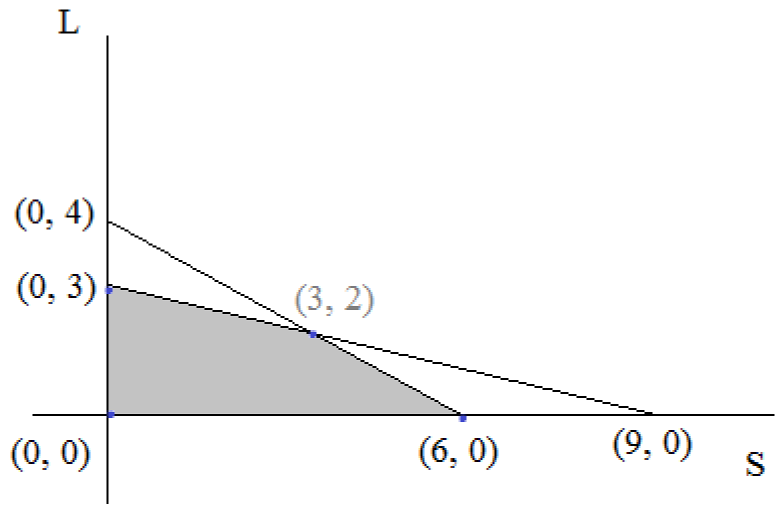
Algebra: Linear Programming (Optimization)



lesson, word problem examples, and exercises (w/ solutions)

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Algebra: Linear Programming Notes and Examples

I. Introduction, terms and illustrations

Linear programming is a method of determining a way to achieve the best outcome in a given mathematical model.

It's a useful way to discover how to allocate a fixed amount of resources (constraints) in a manner that optimizes productivity.

The *objective function* is the function that is to be minimized or maximized.

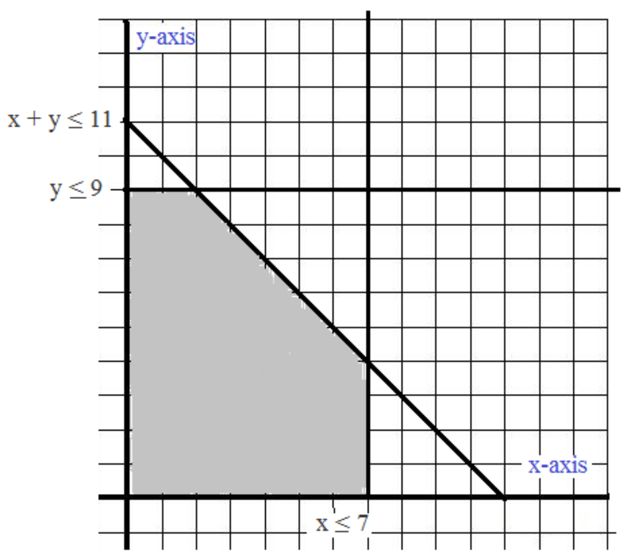
(The objective function is often referred to as the *optimization equation.*)

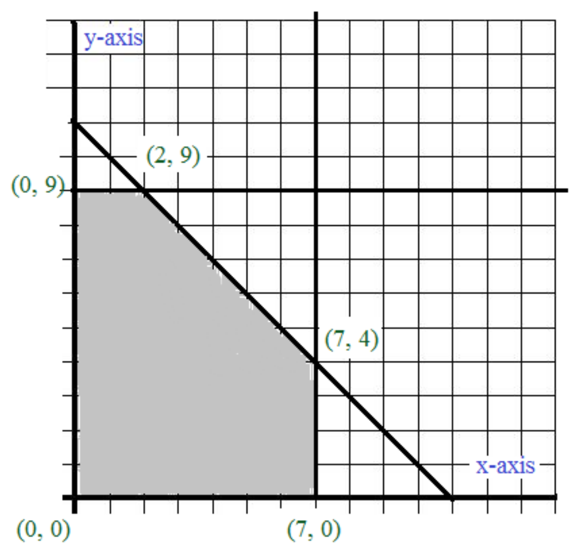
The *feasibility region* contains all the solutions that are within all the constraints.

The *maximum* or *minimum value* of the objective function will be one of the corner points (i.e. vertices) of the feasibility region.

Example: Constraints: $x + y \leq 11$
$x \leq 7$
$y \leq 9$

The shaded area is the feasibility region.
Any point in the shaded area is a value that satisfies all the constraints!





(Note: in many cases, it is understood that x and y must be non-negative numbers. For example, if x and y represent hours of work, it is implied that x and y cannot be negative hours)

The 5 corners are possible optimization points.
(the values that lead to maximum or minimum values in the objective function)

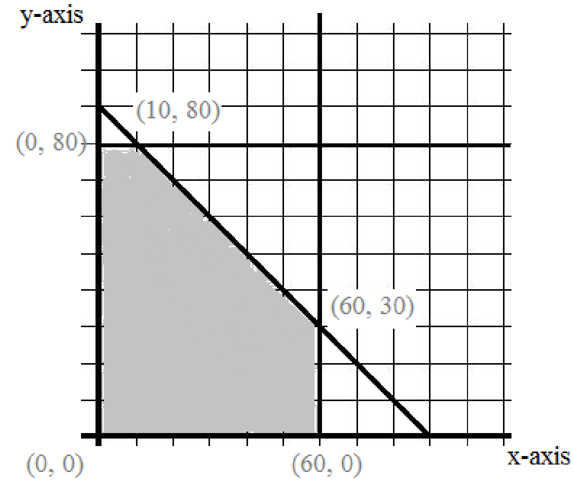
II. Finding the optimization points

Example: Objective function: $.5x + .2y = P$

Constraints:

$x + y \leq 90$
$x \leq 60$
$y \leq 80$
$x \geq 0$
$y \geq 0$

What combination of x and y -- subject to the constraints -- would maximize P?



Test each corner point in the objective function:

$(0, 0): \ .5(0) + .2(0) = 0$

$(0, 80): \ .5(0) + .2(80) = 16$

$(10, 80): \ .5(10) + .2(80) = 21$

$(60, 30): \ .5(60) + .2(30) = \boxed{36}$

$(60, 0): \ .5(60) + .2(0) = 30$

Example: Objective function: $40A + 25B = C$

Constraints:

$A \geq 10$

$B \geq A + 5$

$-A + 5B \leq 175$

What is the minimum cost *subject to the constraints*?

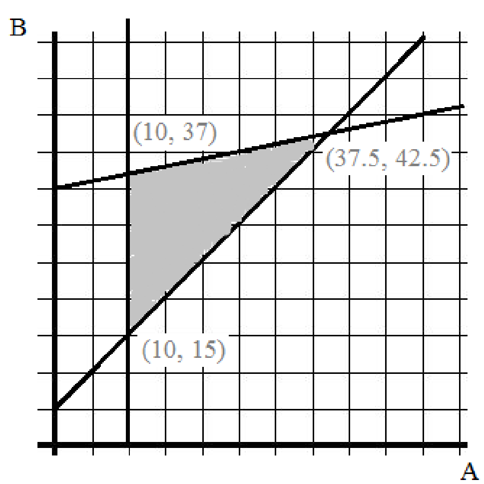
Find corner points:

$A = 10$
$B = A + 5$
$(10, 15)$

$A = 10$
$-A + 5B = 175$
$-10 + 5B = 175$
$5B = 185$
$(10, 37)$

$B = A + 5$
$-A + 5B = 175$
(use substitution)
$-A + 5(A + 5) = 175$
$4A = 150$
$A = 37.5$
$B = 42.5$

$(37.5, 42.5)$



Test each corner point:

$(10, 15): \ 40(10) + 25(15) = \boxed{775}$

$(10, 37): \ 40(10) + 25(37) = 1325$

$(37.5, 42.5): \ 40(37.5) + 25(42.5) = 2562.5$

III. Solving optimization word problems

A procedure to solve linear programming word problems is illustrated below.

Note how each phrase and number is translated into linear equations and inequalities.

Then, the inequalities are graphed to show the feasibility region.

And, finally, each corner point is tested in the objective function to determine which variables achieve the best outcome.

> 4 basic steps:
>
> 1) Identify and label *variables*
>
> 2) Determine the *objective function*
>
> 3) List and Graph the *constraints*
>
> 4) Test *corner points* of feasibility region

A math test consists of number problems and graphing problems. Number problems are worth 6 points each, and graphing problems are worth 10 points each. You can accurately solve a number problem in 2 minutes and a graphing problem in 4 minutes. Assuming you have 40 minutes and may choose no more than 12 problems to answer, how many of each type should you solve to get the highest score?

1) Identify and label variables: N = # of number problems G = # of graphing problems

2) Determine the objective function: "how many to get highest score?"

$$6N + 10G = \text{Score}$$

3) List and graph the constraints: (time) $2N + 4G \leq 40$

(problems) $N + G \leq 12$

4) Test the corner points of the feasibility region

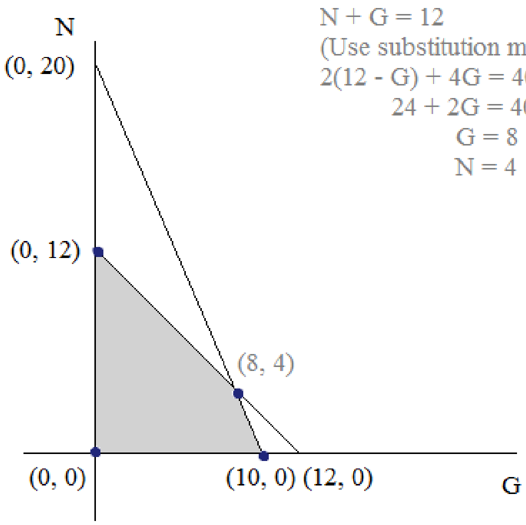
(0, 0) : $6(0) + 10(0) = 0$

(0, 12) : $6(12) + 10(0) = 72$

(8, 4) : $6(4) + 10(8) = \boxed{104}$

(10, 0) : $6(0) + 10(10) = 100$

Under the test constraints, answering 8 graphing problems and 4 number problems would get the best score!



2N + 4G = 40
N + G = 12
(Use substitution method)
2(12 - G) + 4G = 40
24 + 2G = 40
G = 8
N = 4

Linear Optimization Examples

Example: "Open Feasibility Region"

Find the minimum for $C = x + 3y$ where $\begin{cases} x + 2y \geq 8 \\ x \geq 2 \\ y \geq 0 \end{cases}$

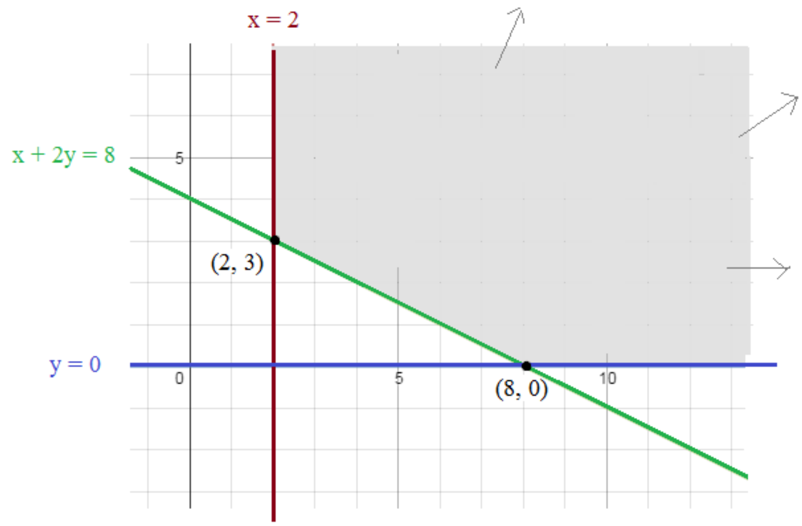
Although there are 3 constraints, the feasibility region isn't necessarily a triangle....

There are only 2 vertices to test:

at (2, 3): $C = 2 + 3(3) = 11$

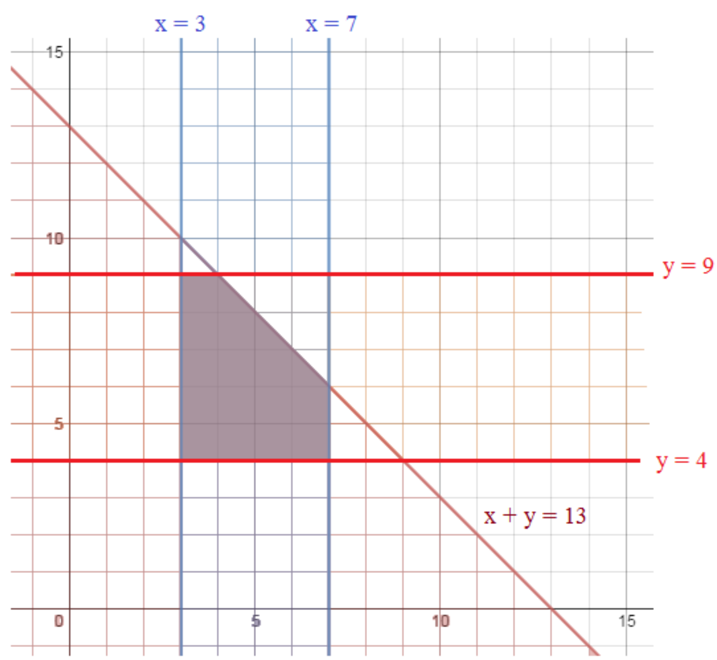
at (8, 0): $C = 8 + 3(0) = 8$

The minimum value 8 occurs at (8, 0)



Example: Graphing Compound Inequalities

Find the maximum for $P = 3x + 2y$ where $\begin{cases} 3 \leq x \leq 7 \\ 4 \leq y \leq 9 \\ x + y \leq 13 \end{cases}$



5 vertices to test:

at (3, 9): $P = 3(3) + 2(9) = 27$

at (3, 4): we would expect this to be < (3, 9)

$P = 3(3) + 2(4) = 17$

at (7, 4): $P = 3(7) + 2(4) = 29$

at (7, 6): we would expect this to be > (7, 4)

$P = 3(7) + 2(6) = 33$

at (4, 9): $P = 3(4) + 2(9) = 30$

The maximum value 33 occurs at (7, 6)

Example: Ye Olde Furniture Shopped manufactures desks and bookcases.
It makes $75 per desk and $40 per bookcase. A desk requires 10 hours of woodworking and 3 hours of finishing. And, a bookcase needs 5 hours for woodworking and 4 hours for finishing. The staff of woodworkers are available for 600 hours per week. And, the finishers work for 240 hours per week.
How many desks should the Furniture Shoppe produce each week to maximize profits?

Step 1: Read question and select variables

Since we're working with desks and bookcases, we'll

let D = the number of desks
B = the number of bookcases

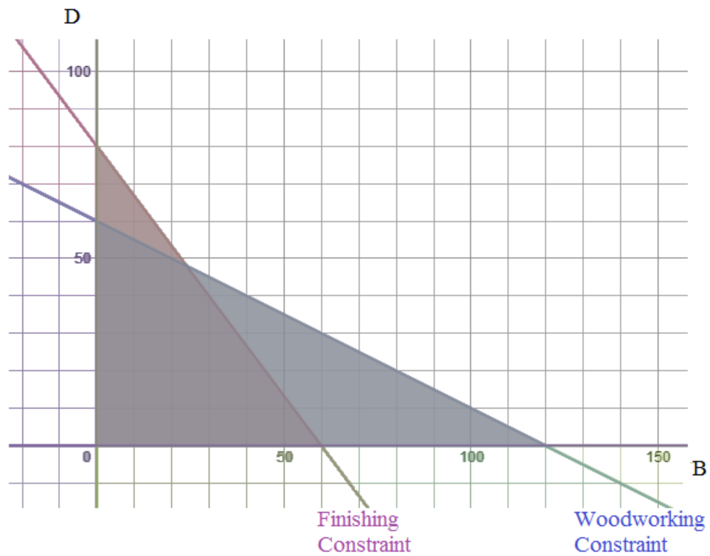
Step 2: Identify the objective function

"produce each week to *maximize profits*"

Profits = Desk profit + Bookcase profit

$P = 75D + 40B$

(75 dollars per desk and 40 dollars per bookcase)



Step 3: List constraints

After reading the question, we can identify 2 constraints:

Woodworking (labor) $10\text{hrs}(D) + 5\text{hrs}(B) \leq 600\text{hrs}$

Finishing (labor) $3\text{hrs}(D) + 4\text{hrs}(B) \leq 240\text{hrs}$

And, there are 2 'natural' constraints:

Bookcases produced $B \geq 0$

Desks produced $D \geq 0$

(there are no "negative desks" or "negative bookcases"!)

Step 4: Find Feasibility Region

After graphing the constraints, we see a shaded feasiblity region...
Then, we identify the vertices by finding where the constraints intersect...

$10D + 5B = 600$
$3D + 4B = 240$
(elimination method)

$30D + 15B = 1800$
$-30D - 40B = -2400$

$-25B = -600$

$B = 24$
$D = 48$

(24, 48)

$B = 0$
$10D + 5B = 600$
(substitution)

$10D + 5(0) = 600$
$D = 60$

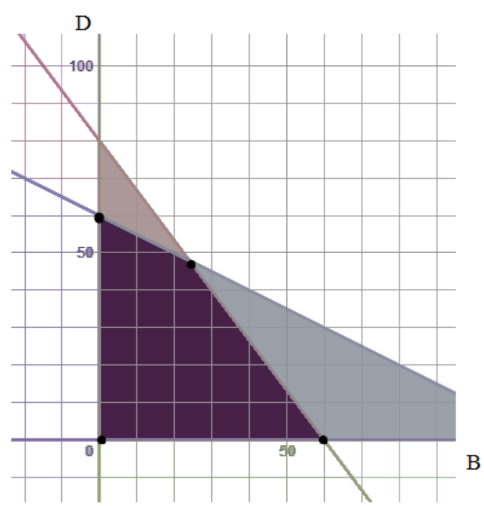
(0, 60)

$D = 0$
$3D + 4B = 240$
(substitution)

$3(0) + 4B = 240$
$B = 60$

(60, 0)

The graph illustrates the intersection points
(vertices of the feasibility region)



Step 5: Apply vertices to objective function

(0, 0): $40(0) + $75(0) = 0

(0, 60): $40(0) + $75(60) = $4500

(24, 48): $40(24) + $75(48) = $4560

(60, 0): $40(60) + $75(0) = $2400

The Furniture Shoppe should produce
24 bookcases and 48 desks
to maximize profits

Linear optimization discussion problem: "Applying mathematical results to real world"

Example: After the summer, you have 210 tomatoes and 20 onions remaining in your garden. So, you decide to make cans of tomato sauce and jars of salsa and sell them. Each can of sauce requires 10 tomatoes and 1 onion. And, each jar of salsa needs 7 tomatoes and 1/2 an onion. There is a $2 profit for each can of sauce. And, a $1.50 profit per jar of salsa. Because of demand, you need to produce at least 2 1/2 cans of sauce as 1 jar of salsa. What is the optimal amount of each you should make to maximize profit?

This is a linear optimization problem..

Objective Function: $P = \$1.5(J) + \$2(C)$

C = # of cans of sauce
J = # of jars of salsa

Constraints:

(Tomatoes): $10(C) + 7(J) \leq 210$

(Onions): $1(C) + .5(J) \leq 20$

(Demand): $1C \geq 2.5J$

Natural constraints: $C \geq 0$ $J \geq 0$

(Salsa and Sauce cannot be negative)

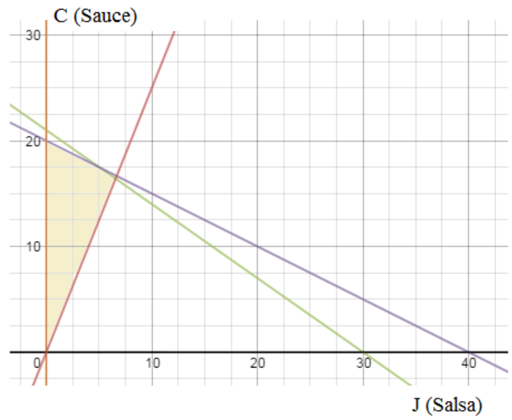
Vertices:

(0, 20): $40
(0, 0): $0
(5, 17.5): $42.50
(6.56, 16.4): $42.65

$$\begin{array}{r} 10C + 7J = 210 \\ -10C - 5J = -200 \\ \hline 2J = 10 \\ J = 5 \\ C = 17.5 \end{array}$$

$$\begin{array}{r} 1C = 2.5J \\ 10C + 7J = 210 \\ 32J = 210 \end{array}$$

$J = 6.56$ $C = 16.40$

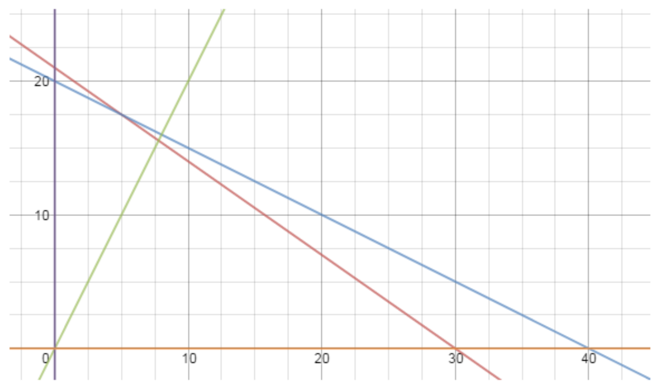


***Since you cannot have partial jars, we'll round down... Therefore, the maximum profit comes from 5J and 17C

$41.50
(real world max profit)

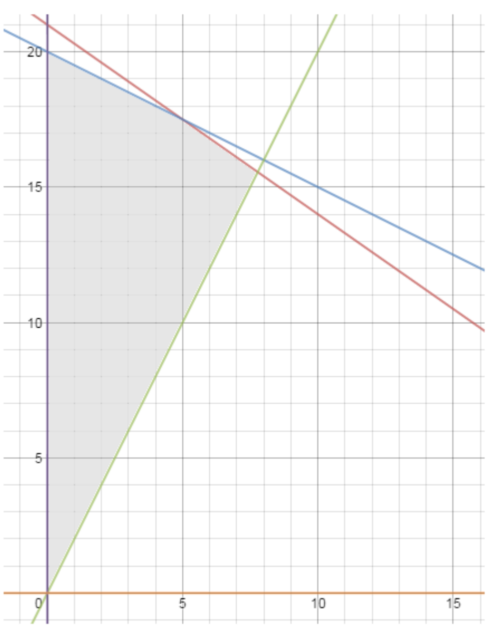
Example: Change the above question to "at least 2 cans of sauce as 1 jar of salsa" What is the *feasibility region*?

A graph of the linear system of constraints

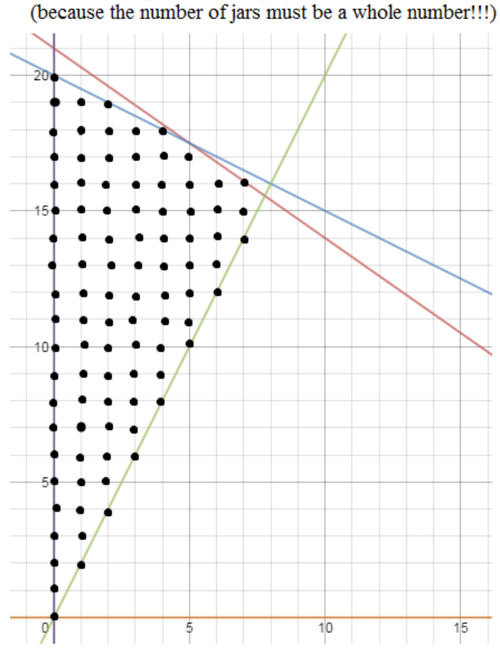


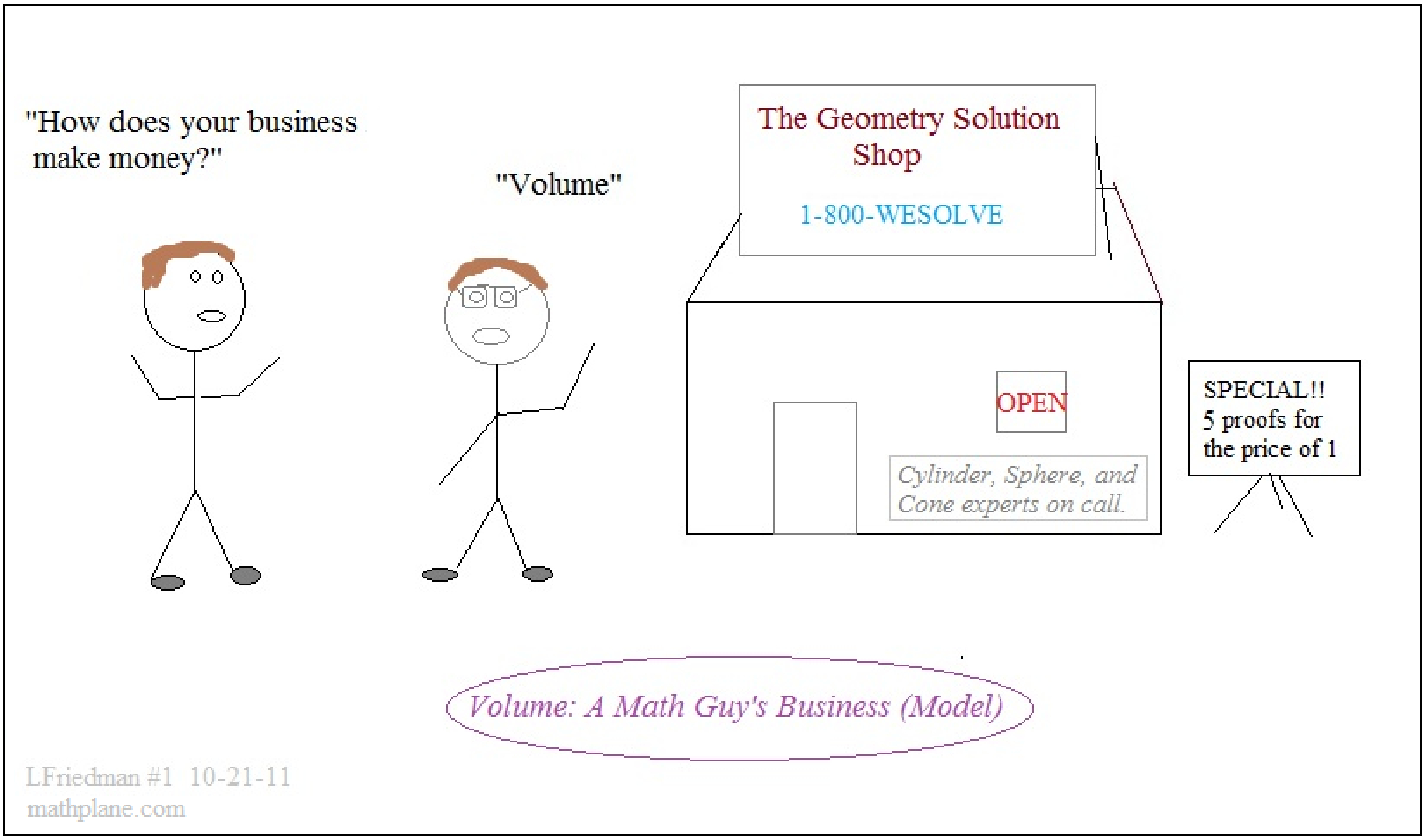
?

The feasiblity region is the gray shaded area?



Or, the set of points in this graph?
(because the number of jars must be a whole number!!!)





Warm-up Exercises and Solutions-→

Linear Programming - Warm-up

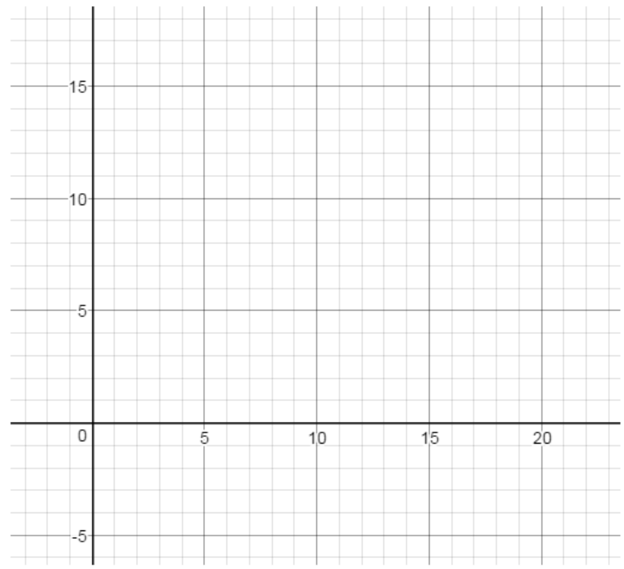
1) Sketch the system of inequalites:

$x \geq 0$
$y \geq 0$

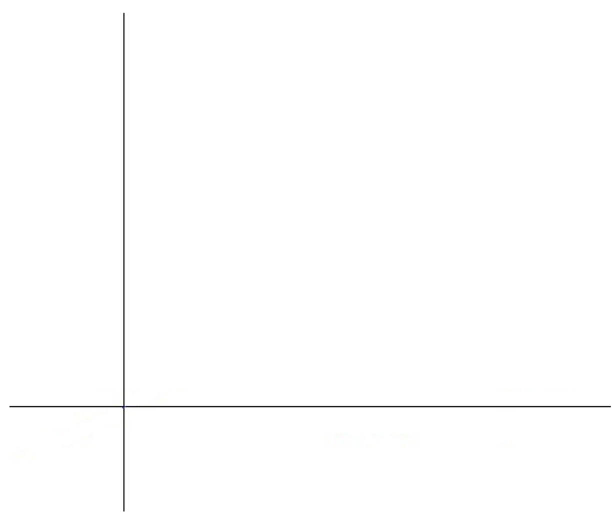
$-x + y \geq 0$

$2x + y \geq 4$
$2x + y \leq 13$

Identify the 'feasibility region' and all possible points where a minimum or maximum output could be found.



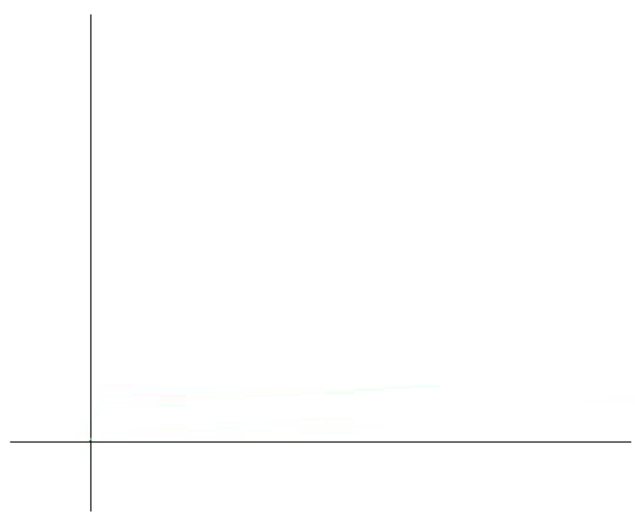
2) An electronics store sells 2 brands of smartphones. Due to customer demand, the store must stock at least twice as many units of brand A as of brand B. It is also necessary to stock at least 10 units of brand B. The store shelves have room for not more than 100 boxes of smartphones.
Find and graph a system of inequalities that shows all the possibilities for stocking the smartphones.



Linear Programming - Warm-up

3) A company manufactures radios. Its profit for the deluxe model is $45 per radio and for the standard model is $30 per radio. The company must produce at least 40 deluxe and 80 standard radios per day. If the company is able to make up to 200 radios per day, how many of each should they produce to maximize profit?

4) A young family has $15,000 to invest. They decide to invest at least $2,000 in high-risk technology stocks, and at least three times that amount in low-risk dividend stocks. Find and graph the feasibility region describing all the possible investment allocations the family can make.



1) Sketch the system of inequalites:

SOLUTIONS

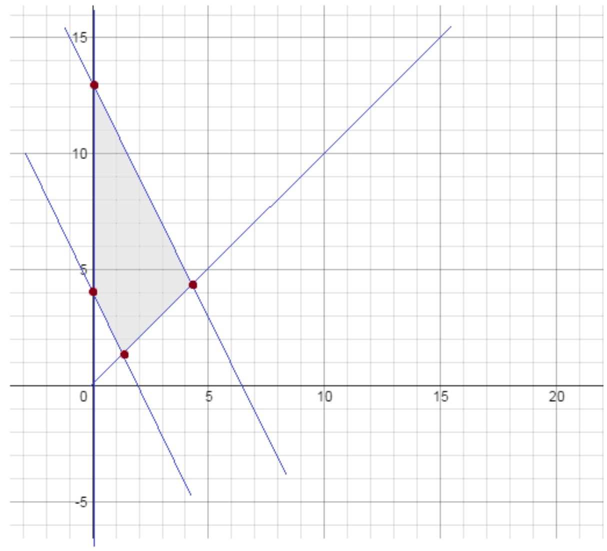
Linear Programming - Warm-up

$x \geq 0$
$y \geq 0$

$-x + y \geq 0$

$2x + y \geq 4$
$2x + y \leq 13$

Identify the 'feasibility region' and all possible points where a minimum or maximum output could be found.



The shaded area is the feasibility region...

For any given 'objective function', the red points are *possible* maximum or minimum outputs (subject to the constraints).

2) An electronics store sells 2 brands of smartphones. Due to customer demand, the store must stock at least twice as many units of brand A as of brand B. It is also necessary to stock at least 10 units of brand B. The store shelves have room for not more than 100 boxes of smartphones.
Find and graph a system of inequalities that shows all the possibilities for stocking the smartphones.

Let A = # of brand A smartphones
B = # of brand B smartphones

Constraints: 1) "at least twice as many brand A as brand B" -----> $A \geq 2B$

EX: If there are 5 B's, there must be at least 10 A's...

2) "at least 10 sets of B" ------> $B \geq 10$

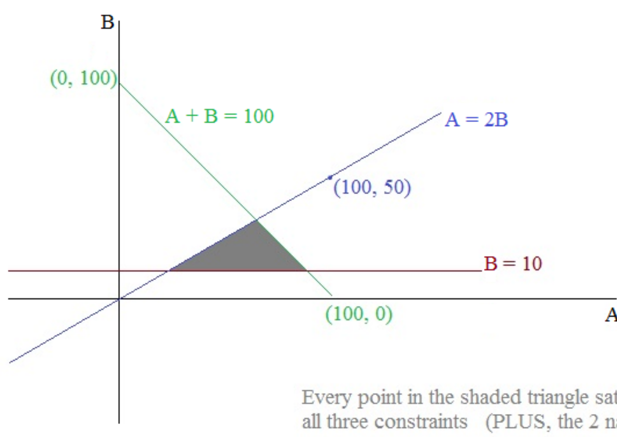
3) "room for not more than 100 sets" ------> $A + B \leq 100$

Also, there are 2 natural constraints:

4) $A \geq 0$

5) $B \geq 0$

(I assume you can't have "negative tv sets"!!)



Every point in the shaded triangle satisfies all three constraints (PLUS, the 2 natural constraints)

3) A company manufactures radios. Its profit for the deluxe model is \$45 per radio and for the standard model is \$30 per radio. The company must produce at least 40 deluxe and 80 standard radios per day. If the company is able to make up to 200 radios per day, how many of each should they produce to maximize profit?

D = number of deluxe radios
S = number of standard radios

Constraints

$D \geq 0$
$S \geq 0$ (it's impossible to have negative radios)

$D \geq 40$ "at least 40 deluxe"
$S \geq 80$ "at least 80 standard radios"

$D + S \leq 200$ "... up to 200 radios per day"

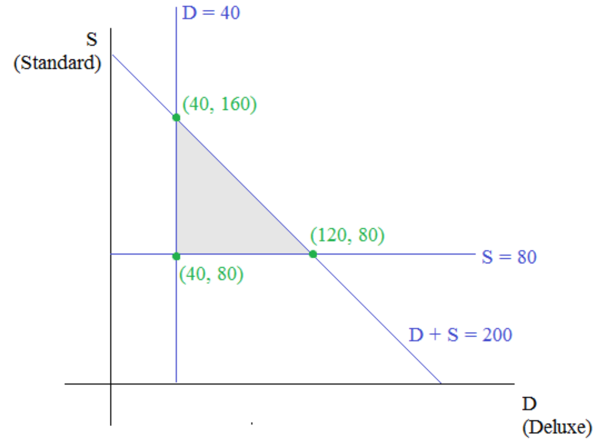
Objective Function: $P = 45D + 30S$

Three possible points:

(40, 160): $P = 45(40) + 30(160) = \$6600$

(40, 80) : $P = 45(40) + 30(80) = \$4200$

(120, 80) : $P = 45(120) + 30(80) = \$7800$



4) A young family has \$15,000 to invest. They decide to invest at least \$2,000 in high-risk technology stocks, and at least three times that amount in low-risk dividend stocks. Find and graph the feasibility region describing all the possible investment allocations the family can make.

Let H = high-risk investment
L = low-risk investment

Constraints:
1) "\$15,000 to invest" -----> $H + L \leq 15{,}000$
2) "at least \$2000 in high risk" ------> $H \geq 2000$
3) "at least 3 times that amount" ------> $L \geq 3H$

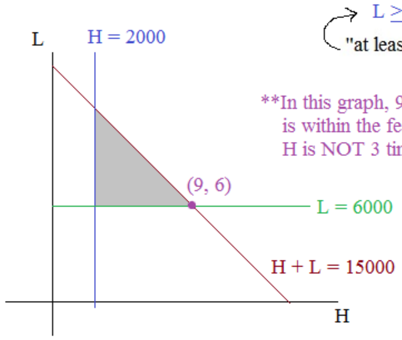
EX: If the family invests 3000 in H, then they must invest at least 9000 in L...

And, there are natural constraints:

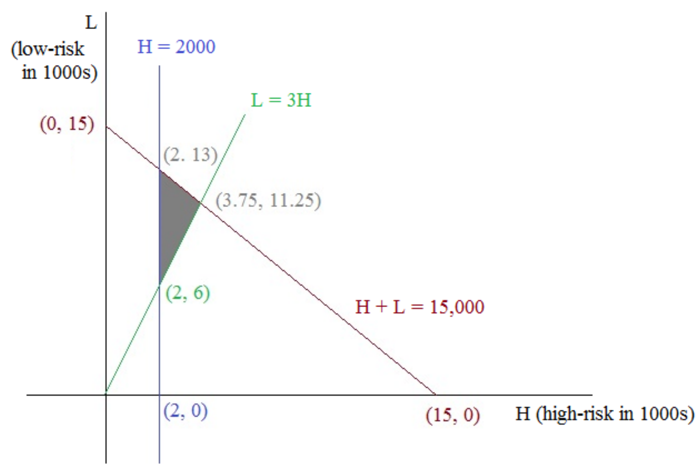
4) $H \geq 0$
5) $L \geq 0$

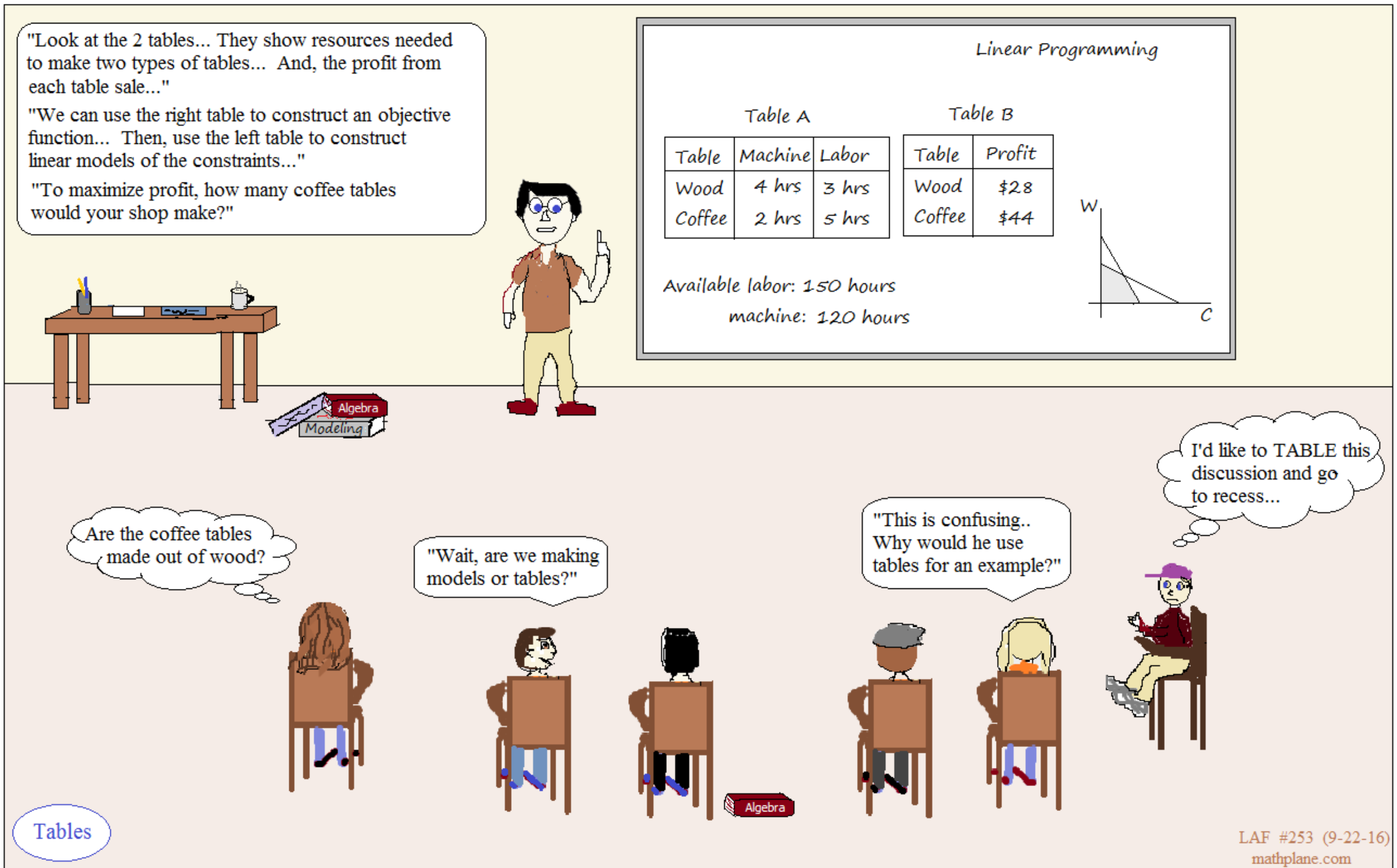
INCORRECT interpretation...

$H + L \leq 15{,}000$
$H \geq 2000$
$L \geq 6000$
"at least 3 times that amount"



**In this graph, 9000 H and 6000 L is within the feasibility region.. BUT, H is NOT 3 times the amount of L!!





Apparently, the teacher's lesson about linear programming was not optimized!

Linear Programming Exercises (and solutions)-→

Linear Programming Practice Exercise

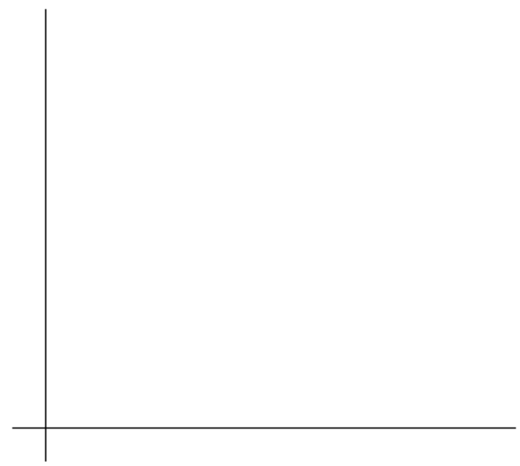
A) The area of a parking lot is 600 square yards. A car requires 6 square yards. And, a bus needs 30 square yards. The attendant can oversee only 60 vehicles. If a car is charged $2.50 and a bus is charged $7.50, how many of each should be accepted to maximize income?

B) Spot builds dog houses. He needs 10 wooden boards and 15 nails to build a small dog house; and, he uses 15 boards and 45 nails for a large dog house. Spot makes a $40 profit on every small dog house and $52 profit on every large dog house. If he has 60 wooden boards and 135 nails, how many of each type of dog house should he make to maximize his profit? What will be Spot's maximum profit?

C) A furniture store makes two types of chairs: rockers and swivels. Machines A an B are required to make each type of chair. Machine A can be run no longer than 20 hours in a day. Machine B is limited to 15 hours per day.
The following Table shows the time needed to produce each chair and the profit.

Chair	Machine A	Machine B	Profit
Rocker	2 hours	3 hours	$12
Swivel	4 hours	1 hour	$10

How many of each chair should be made each day to maximize profits?



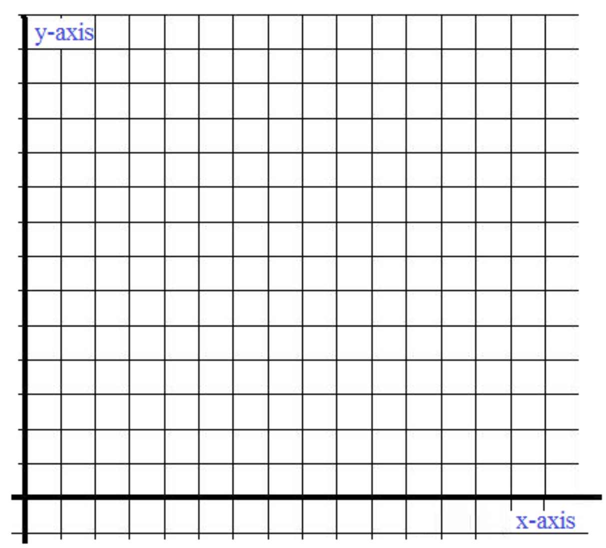
D) Given the following constraints, determine the maximum and minimum values of $Z = .2x - 3.3y$

$x \geq 0 \quad y \geq 2$

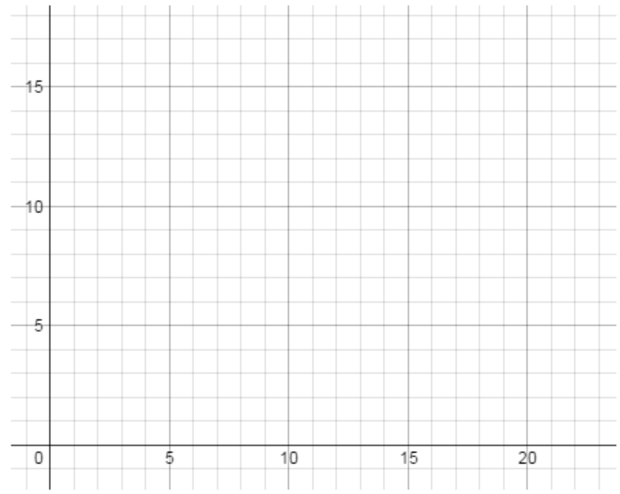
$y \leq 3x + 4$

$x + 2y \leq 15$

$x - \frac{1}{3}y \leq 5$



E) A math baker produces bran and corn muffins. A batch of corn muffins requires 3 cups of milk and 3 cups of flour. And, a batch of bran muffins needs 4 cups of flour and 2 cups of milk. In the bakery, he has 48 cups of flour and 30 cups of milk. If the math baker nets \$3 for each batch of bran muffins and \$4 for each batch of corn muffins, what is the maximum profit possible?



F) On a typical Saturday, a pizza place sells between 75 & 95 small pizzas and between 110 & 150 large pizzas. Due to workers and oven availability, it can prepare up to 225 pizzas per day. If the pizza place earns \$1.50 for each small it sells, and it earns \$2.25 for each large pizza it sells, then what's the maximum profit it can earn?

Linear Programming (Solutions)

A) The area of a parking lot is 600 square yards. A car requires 6 square yards. And, a bus needs 30 square yards. The attendant can oversee only 60 vehicles. If a car is charged \$2.50 and a bus is charged \$7.50, how many of each should be accepted to maximize income?

1) Identify and label the variables: Let C = # of cars   B = # of buses

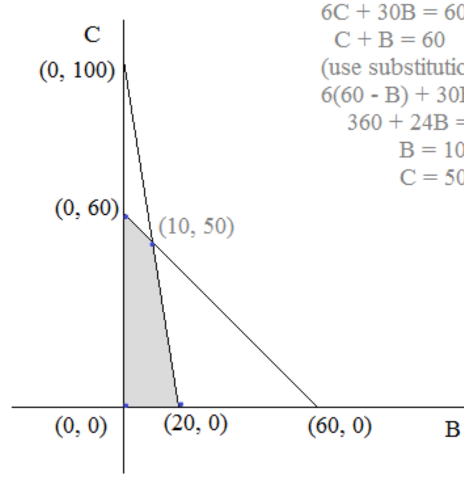
2) Determine the objective function: "How many to maximize income"

$$2.5C + 7.5B = \text{Income}$$

3) List and graph the constraints: (space) $6C + 30B \le 600$

(attendant) $C + B \le 60$

$6C + 30B = 600$
$C + B = 60$
(use substitution)
$6(60 - B) + 30B = 600$
$360 + 24B = 600$
$B = 10$
$C = 50$



4) Test the corner points of the feasibility region.

(0, 0) : $2.5(0) + 7.5(0) = 0$

(0, 60) : $2.5(60) + 7.5(0) = 150$

(10, 50) : $2.5(50) + 7.5(10) = \boxed{200}$

(20, 0) : $2.5(0) + 7.5(20) = 150$

To maximize income -- under the given constraints -- the parking lot should have 10 buses and 50 cars!

B) Spot builds dog houses. He needs 10 wooden boards and 15 nails to build a small dog house; and, he uses 15 boards and 45 nails for a large dog house. Spot makes a \$40 profit on every small dog house and \$52 profit on every large dog house. If he has 60 wooden boards and 135 nails, how many of each type of dog house should he make to maximize his profit? What will be Spot's maximum profit?

1) Identify the variables:

let S = # of small dog houses
L = # of large dog houses

2) Determine the objective function:

(i.e. optimization equation)

$$40S + 52L = \text{Profit}$$

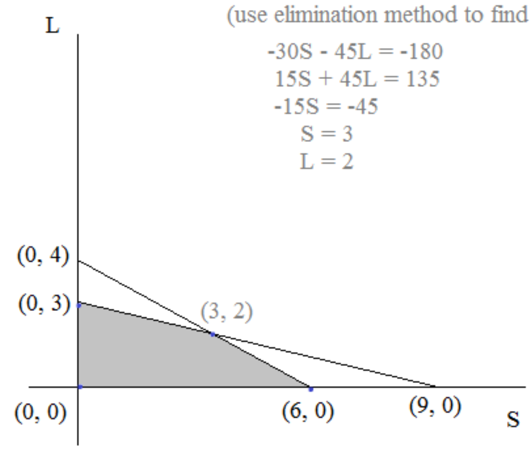
3) List and graph the constraints:

$10S + 15L \le 60$ (available boards)

$15S + 45L \le 135$ (available nails)

$S \ge 0$
$L \ge 0$ (# of doghouses cannot be less than zero)

$10S + 15L = 60$
$15S + 45L = 135$
(use elimination method to find solution point)
$-30S - 45L = -180$
$15S + 45L = 135$
$-15S = -45$
$S = 3$
$L = 2$



4) Test the corner points of the feasibility region:

(0, 0) : $40(0) + 52(0) = 0$

(0, 3) : $40(0) + 52(3) = 156$

(3, 2) : $40(3) + 52(2) = 224$

(6, 0) : $40(6) + 52(0) = \boxed{240}$

Spot should only construct (6) small doghouses.
He would maximize his profit (\$240).

Linear Programming (Solutions)

C) A furniture store makes two types of chairs: rockers and swivels. Machines A an B are required to make each type of chair. Machine A can be run no longer than 20 hours in a day. Machine B is limited to 15 hours per day.
The following Table shows the time needed to produce each chair and the profit.

Chair	Machine A	Machine B	Profit
Rocker	2 hours	3 hours	\$12
Swivel	4 hours	1 hour	\$10

How many of each chair should be made each day to maximize profits?

1) Identify and label variables: R = # of rockers S = # of swivels

2) Determine the objective function: "how many to maximize profits?"

$$12R + 10S = \text{Profit}$$

3) List and graph the constraints: (time of machine use)

Machine A: $2R + 4S \leq 20$

Machine B: $3R + 1S \leq 15$

4) Test the corner points of the feasibility region.

(0, 0) : 12(0) + 10(0) = 0

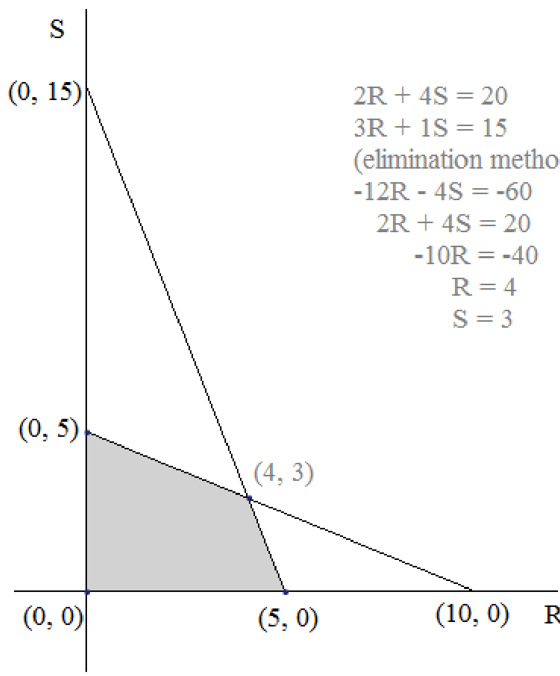
(0, 5) : 12(0) + 10(5) = 50

(4, 3) : 12(4) + 10(3) = **78**

(5, 0) : 12(5) + 10(0) = 60

The furniture store should produce 4 rockers and 3 swivel chairs each day to maximize profits!

$2R + 4S = 20$
$3R + 1S = 15$
(elimination method)
$-12R - 4S = -60$
$2R + 4S = 20$
$-10R = -40$
$R = 4$
$S = 3$



Linear Programming (solutions)

D) Given the following constraints, determine the maximum and minimum values of $Z = .2x - 3.3y$

$x \geq 0 \qquad y \geq 2$

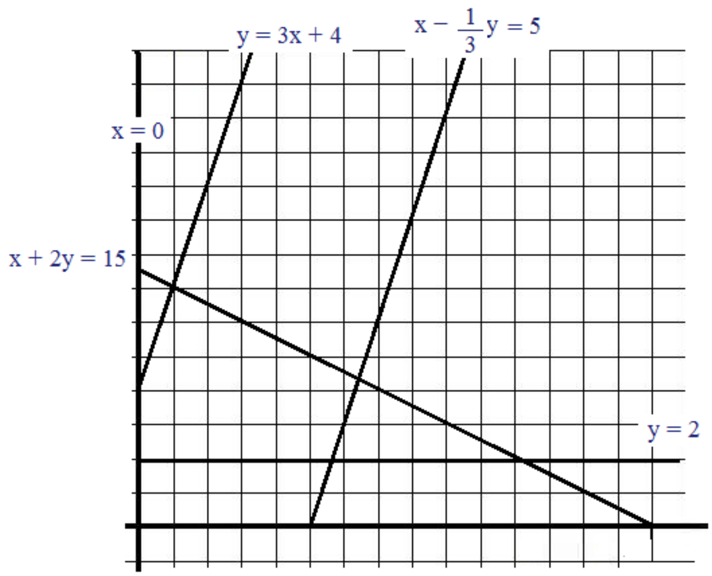
$y \leq 3x + 4$

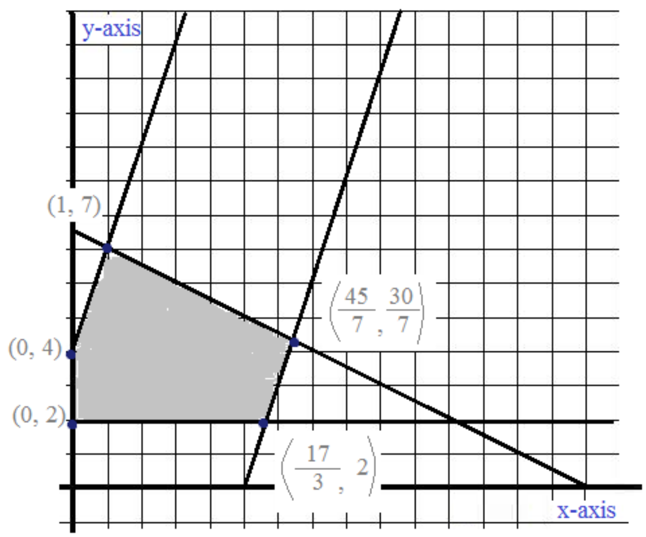
$x + 2y \leq 15$

$x - \frac{1}{3}y \leq 5$

Graph each inequality....

.... then, establish the feasibility region...





To verify the corner points (vertices), you can algebraically determine where the inequalities intersect.

$x = 0$
$y = 2$ (0, 2)

$x = 0$
$y = 3x + 4$ (0, 4)
(substitution)
$y = 3(0) + 4 = 4$

$x + 2y = 15$
$y = 3x + 4$
(substitution)
$x + 2(3x + 4) = 15$ (1, 7)
$7x + 8 = 15$
$x = 1$
$y = 7$

$x + 2y = 15$
$x - 1/3y = 5$ (45/7, 30/7)
(elimination/subtraction)
$7/3\ y = 10$
$y = 30/7$
$x = 45/7$

$x - 1/3y = 5$
$y = 2$
(substitution) (17/3, 2)
$x - 1/3(2) = 5$
$x - 2/3 = 5$
$x = 5\ 2/3$

Finally, test each corner point in the objective function to determine the optimal values:

(0, 2): $.2(0) - 3.3(2) = -6.6$

(0, 4): $.2(0) - 3.3(4) = -13.2$

(1, 7): $.2(1) - 3.3(7) = -22.9$

(45/7, 30/7): $.2(6.4) - 3.3(4.3) = -12.9$

(17/3, 2): $.2(5.67) - 3.3(2) = -5.5$

"Maximum value": -5.5 where x = 17/3 and y = 2
"Minimum value": -22.9 where x = 1 and y = 7

E) A math baker produces bran and corn muffins. A batch of corn muffins requires 3 cups of milk and 3 cups of flour. And, a batch of bran muffins needs 4 cups of flour and 2 cups of milk. In the bakery, he has 48 cups of flour and 30 cups of milk. If the math baker nets \$3 for each batch of bran muffins and \$4 for each batch of corn muffins, what is the maximum profit possible?

SOLUTIONS

1) Identify the 2 variables:

Let B = # of batches of bran muffins
C = # of batches of corn muffins

2) "maximum profit possible" -- this is our objective!

$P = \$4(C) + \$3(B)$

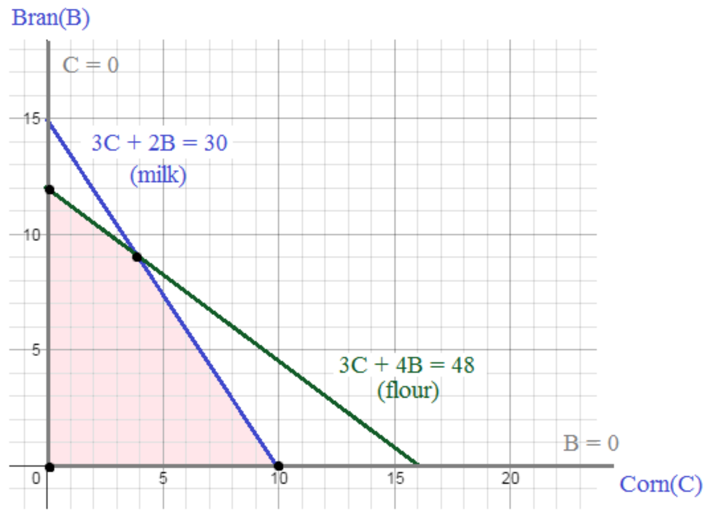
3) The constraints will describe the baker's limited resources: milk and flour

$3\text{cups}(C) + 2\text{ cups}(B) \le 30\text{ cups}$ (milk)

$3\text{cups}(C) + 4\text{cups}(B) \le 48\text{ cups}$ (flour)

4) And, the obvious, natural constraints

$B \ge 0$
$C \ge 0$ (there are no 'negative' muffins!)



5) The vertices of the feasibility region are:

(0, 0), (10, 0), (0, 12) and (4, 9)

$3C + 2B = 30$
$-\ 3C + 4B = 48$
$-2B = -18$
$B = 9$

if B = 9, then $3C + 2(9) = 30$
$C = 4$

6) Find profit at each corner:

(0, 0): \$0
(10, 0): \$40
(0, 12): \$36
(4, 9): \$43

The math baker can earn \$43 if he makes 4 batches of corn and 9 batches of bran muffins

F) On a typical Saturday, a pizza place sells between 75 & 95 small pizzas and between 110 & 150 large pizzas. Due to workers and oven availability, it can prepare up to 225 pizzas per day. If the pizza place earns \$1.50 for each small it sells, and it earns \$2.25 for each large pizza it sells, then what's the maximum profit it can earn?

Let S = # of small pizzas
L = # of large pizzas

Objective function: "maximum profit"

$P = \$1.50S + \$2.25L$

Restraints: "between 75 & 95 small"

$75 \le S \le 95$

"between 110 & 150 large"

$110 \le L \le 150$

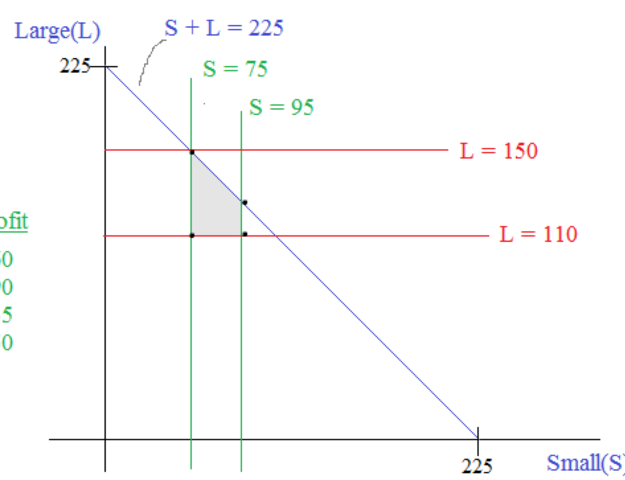
"can prepare up to 225 pizzas"

$S + L \le 225$

Vertices of trapezoid (feasibility region)

Vertex	Profit
(75, 110)	\$360
(95, 110)	\$390
(95, 130)	\$435
(75, 150)	\$450

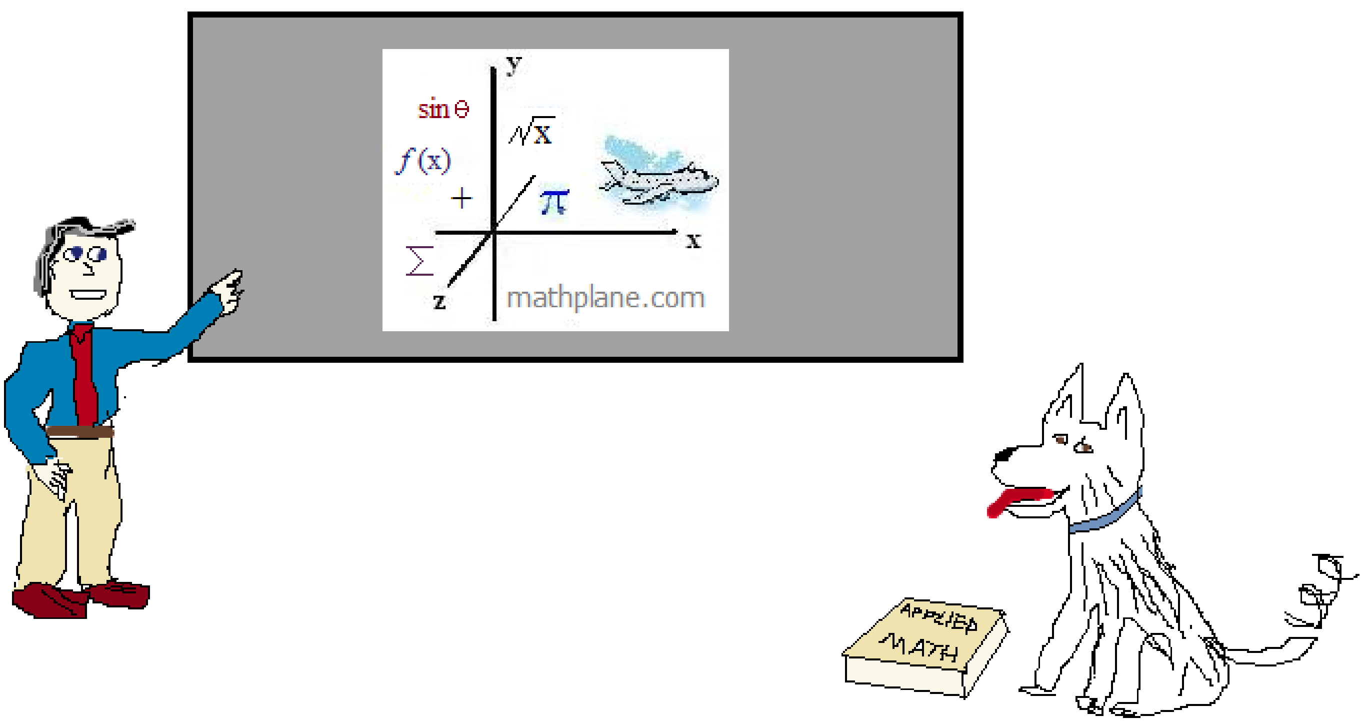
If the pizza shop sells 75 small and 150 large pizzas, it'll make \$450



Thanks for visiting the site. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers.



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