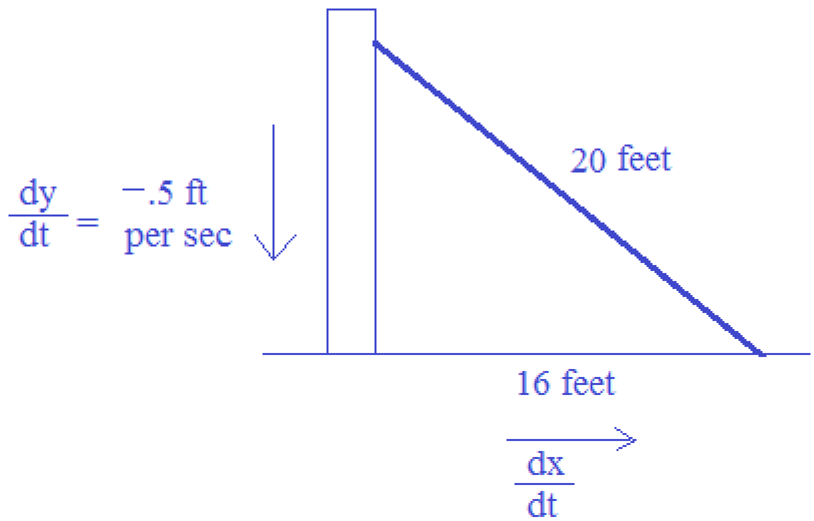
Calculus

Related Rates of Change

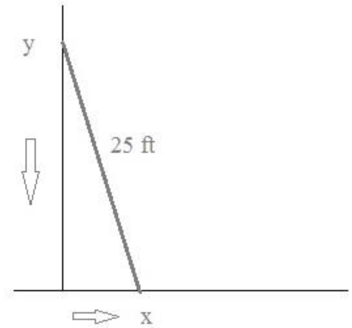


Includes notes, examples, and a practice quiz (with solutions)

Implicit Differentiation: Word Problem Examples

1) A 25-foot ladder is leaning against a wall. If the top of the ladder is slipping down the wall at a rate of 2 feet/second, how fast will the bottom be moving away from the wall when the top is 20 feet above the ground?

Step 1: Draw diagram, list variables and formulas



length to bottom of ladder = x
length to top of ladder = y

$x^2 + y^2 = 625\ ft^2$ (pythagorean theorem)

down the wall at a rate of 2 ft/sec $\quad \frac{dy}{dt} = -2$ ft/sec

(change of y *with respect to time*)

moving away from the wall $\quad \frac{dx}{dt} = ?$

(change of x *with respect to time*)

Important note: we're seeking dx/dt, (the change of x *with respect to time*)..
Simply taking the derivative of $y = \sqrt{625 - x^2}$

$1/2\ (625 - x^2)^{(-1/2)} \cdot (-2x) = \frac{-x}{\sqrt{(625 - x^2)}}$

shows us dy/dx, (the change in *y with respect to x*)

Step 2: Set up equation and use implicit differentiation.

$x^2 + y^2 = 625\ ft^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ derivative with respect to time

Substitute and solve:

$2x\frac{dx}{dt} + 2(20\ ft)(-2\ ft/sec) = 0$

$x^2 + y^2 = 625\ ft^2$
$(x)^2 + (20\ ft)^2 = 625\ ft^2$
When y = 20 ft, x = 15 feet

$2(15\ ft)\frac{dx}{dt} + (-80\ ft^2/sec) = 0$

$30\ ft\ \frac{dx}{dt} = 80\ ft^2/sec$

$\frac{dx}{dt} = \frac{80\ ft^2/sec}{30\ ft} =$ **2.67 ft/sec**

Using explicit differentiation & chain rule

$x = \sqrt{625 - y^2}$

$\frac{dx}{dy} = 1/2\ (625 - y^2)^{(-1/2)} \cdot (-2y) = \frac{-y}{\sqrt{(625 - y^2)}}$

$\frac{dx}{dt} = \frac{dy}{dt} \cdot \frac{dx}{dy}$

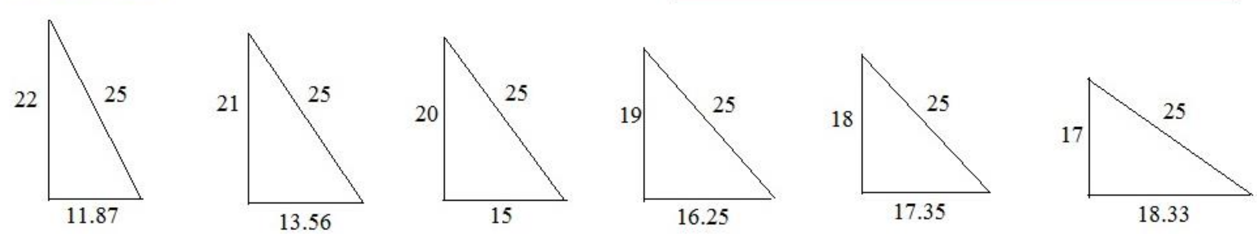
$-2\ \frac{feet}{sec} \bullet \frac{-y}{\sqrt{625\ ft^2 - y^2}}$

If y = 20 feet, then

$\frac{dx}{dt} = -2\ \frac{feet}{sec} \bullet \frac{-(20\ feet)}{\sqrt{625\ ft^2 - 400\ ft^2}}$

$= -2\ \frac{feet}{sec} \bullet \frac{-20\ feet}{15\ feet} =$ **2.67 ft/sec**

Step 3: Check answer



From 22 to 20 feet (one second), the ladder moved out 3.13 feet

From 21 to 19 feet (one second), the ladder moved out 2.69 feet...

From 20 to 18 feet (one second) the ladder moved 2.35 feet...

2.67 feet per second is a reasonable answer! ✓

Implicit Differentiation: Word Problem Examples (continued)

2) Oil erupts from a ruptured tanker, spreading in a circle whose area increases at a constant rate of 6 square miles per hour. How fast is the radius of the spill increasing when the area is 9π square miles?

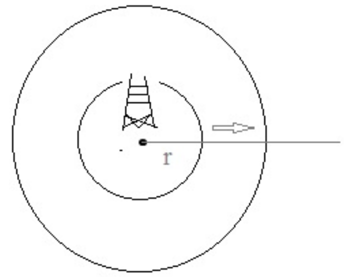
Step 1: Draw a diagram, list variables, and consider formulas

spill area $A = \pi r^2$

"area is increasing at a rate of 6 square miles per hour" $\quad \frac{dA}{dt} = 6 \frac{\text{miles}^2}{\text{hour}}$

"how fast is the radius of the spill increasing?" $\quad \frac{dr}{dt} = ?$

When area is 9π sq. miles, the radius is 3 miles.



Step 2: Implicit differentiation

Take derivative *with respect to t*

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Plug in values and solve

$$6 \frac{\text{miles}^2}{\text{hour}} = 2\pi (3 \text{ miles}) \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{6 \text{ miles}^2}{\text{hour} \cdot 2\pi (3 \text{ miles})} = \frac{1}{\pi} \text{ miles/hour}$$

(or, .318 miles/hour)

When area of spill is 9π square miles, the radius is increasing at .318 miles per hour.

9π sq. miles (28.27 sq miles) → (one hour) → 9π sq. miles + 6 sq. miles = 34.27 sq. miles

r = 3 miles → r = 3.30

radius changed .30 miles in one hour

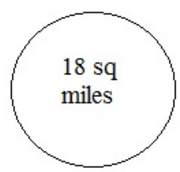
Step 3: Verify Answer

$A = (3.14)r^2$

$12 = (3.14)r^2$

r = 1.95 miles

(one hour later)



$18 = (3.14)r^2$

r = 2.39 miles

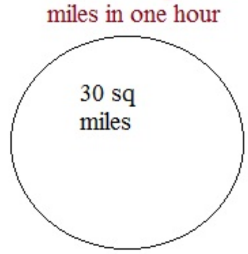
(one hour later)

$24 = (3.14)r^2$

r = 2.76 miles

radius changed .33 miles in one hour

(one hour later)



$30 = (3.14)r^2$

r = 3.09 miles

Calculus: Related Rates of Change

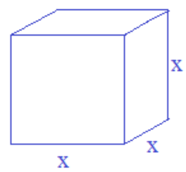
> Step 1: Draw a picture, list variables, write formulas.
>
> Step 2: Substitute given information and create and equation that will lead to the solution.
>
> Step 3: Solve and check for reasonableness.

Example 1: An ice cube melts *uniformly*, where the volume decreases by 3 cm^3/sec. How fast is the surface area decreasing when the cube's edge is 5 cm?

Step 1: Picture, variables, formulas

(volume of a cube) $V = x^3$

(surface area of a cube) $SA = 6x^2$



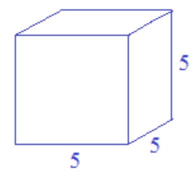
Step 2: Substitute given information

We need to find how fast the surface area is moving with respect to time. $\frac{dSA}{dt}$

$\frac{dV}{dt} = -3\ cm^3/sec$

$x = 5$ cm

$V = 125\ cm^3$



$x = \sqrt[3]{V}$ then, $SA = 6(\sqrt[3]{V}\,)^2$

Step 3: Solve

Find $\frac{dSA}{dt}$ $\quad SA = 6(V)^{\frac{2}{3}}$ Take derivative with respect to change in time (t)

$$\frac{dSA}{dt} = 4(V)^{\frac{-1}{3}}\frac{dV}{dt}$$

$$\frac{dSA}{dt} = 4(125)^{\frac{-1}{3}}(-3\ cm^3/sec)$$

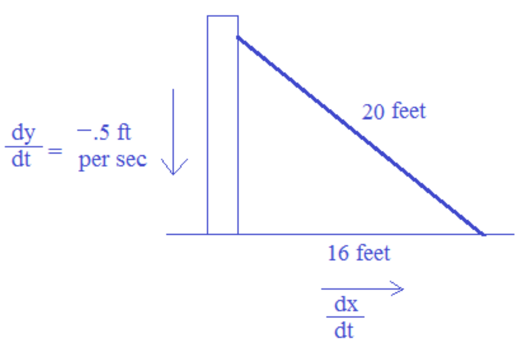
$$\frac{dSA}{dt} = \frac{-12}{5}\ cm^3/sec$$

	Volume	Side	Surface area	change
	131	5.08	154.76	
(3 sec)				2.37
	128	5.04	152.39	
(3 sec)				2.39
	125	5	150	
(3 sec)				2.41
	122	4.96	147.59	
(3 sec)				2.43
	119	4.92	145.16	

$\frac{12}{5} = 2.4$

Example 2: The *top* of a 20-foot ladder slides down the side of a house at the rate of 6 inches/second. When the *bottom* of the ladder is 16 feet from the house, how fast is the *bottom* of the ladder moving away from the house?

Step 1: Diagram and relevant formulas.



pythagorean theorem:

$$x^2 + y^2 = \text{hypotenuse}^2$$

Step 2: Create the equation that we need to solve.

$$x^2 + y^2 = 20^2$$

Find change of distance from house with respect to time...... $\frac{dx}{dt}$

$$x^2 + y^2 = 20^2$$

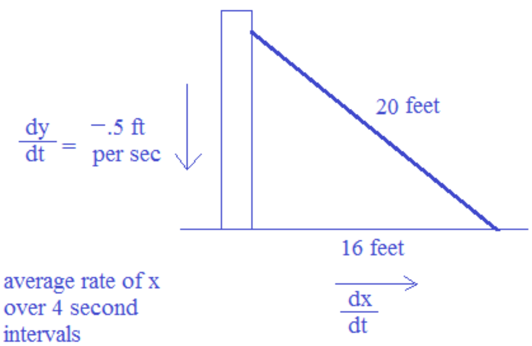
Use implicit differentiation to find the change with respect to time (t).

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

since $x^2 + y^2 = 20^2$

when $x = 16, \quad y = 12$

and, $\frac{dy}{dt} = -.5$



$$2(16)\frac{dx}{dt} + 2(12)(-.5) = 0$$

$$32\frac{dx}{dt} - 12 = 0$$

$$\frac{dx}{dt} = \frac{3}{8}\text{ feet/second}$$

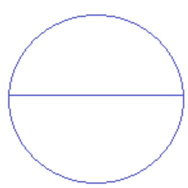
4.5 inches/second

elapsed time t	distance x	height y	average rate of x over 4 second intervals
0	12	16	.57 feet/sec
4	14.28	14	.43 feet/sec
8	16	12	.33 feet/sec
12	17.32	10	.25 feet/sec
16	18.33	8	

.375 feet/sec

Example 3: A (spherical) balloon inflates at a rate of 8 inches^3/second. How fast is the diameter increasing when the balloon's volume is 36π ?

Step 1: Write formulas and draw a picture.



$$V = \frac{4}{3}\pi r^3$$

Diameter = 2r

Want to find $\frac{dD}{dt}$

Step 2:

$$V = \frac{4}{3}\pi r^3$$

$$r = \frac{\text{Diameter}}{2}$$

$$36\pi = \frac{4}{3}\pi r^3$$

$$r^3 = 27$$

$$r = 3$$

$$D = 6$$

$$V = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3$$

$$\frac{dV}{dt} = 8\frac{\text{inches}^3}{\text{sec}}$$

$$V = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3$$

$$V = \frac{4}{24}\pi D^3$$

$$V = \frac{1}{6}\pi D^3$$

Find change related to time (t)

$$\frac{dV}{dt} = \frac{3}{6}\pi D^2\frac{dD}{dt}$$

$$8\frac{\text{inches}^3}{\text{sec}} = \frac{1}{2}\pi(6\text{ inches})^2\frac{dD}{dt}$$

$$\frac{16\text{ inches}^3}{\pi\text{ sec}} = 36\text{inches}^2\frac{dD}{dt}$$

$$\frac{dD}{dt} = \frac{4}{9\pi}\text{ inches/second}$$

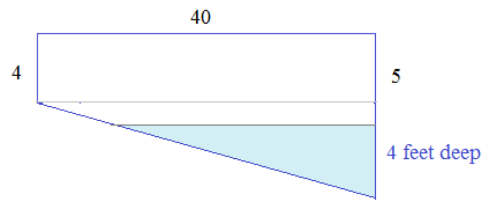
Example 4: A rectangular swimming pool is 10 feet by 40 feet.
The depth of the pool ranges from 4 feet from the shallow side to 9 feet on the deep side.
(The depth is a constant decline from one side of pool to the other)

If the deep end of the pool is 4 feet deep, and the pool is filling at a rate of 10 cubic feet/minute,

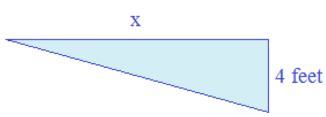
a) what is the volume of water currently in the pool?

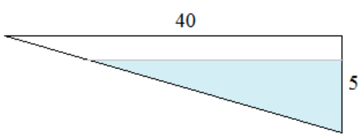
b) how fast is the water level rising?

side view of pool



a) recognizing similar triangles:





***the ratio of the triangles is 5:4
Therefore, the ratio of their areas will be 25:16...

$$\frac{25}{16} = \frac{100}{x}$$

If the area of the large triangle is 100, then the area of the small triangle is 64

Since the area of the small triangle is 64,
the volume of water in the pool is 64 x 10 = **640 cubic feet**

b) How fast is the water level rising?

What is $\frac{dh}{dt}$?

We know $\frac{dV}{dt} = 10\ ft^3/minute$

$V = 640\ ft^3$

$h = 4 \qquad depth = 10$

Volume of the water (i.e. volume of a triangular prism) is

$$V = \frac{1}{2}(base)(height)(depth)$$

$$\frac{40'}{5'} = \frac{base}{height}$$

$$base = 8 \cdot height$$

$$V = \frac{1}{2}(10')(base)(height)$$

$$V = 5(8h)(h)$$

$$V = 40h^2$$

$$\frac{dV}{dt} = 80h\frac{dh}{dt}$$

$$10 = 80(4)\frac{dh}{dt}$$

$$\boxed{\frac{dh}{dt} = 1/32}$$

Steps and Strategies for Calculus Related Rates of Change

Example:

The length of a rectangle is decreasing at a rate of 2 feet/minute.
The width of a rectangle is increasing at a rate of 2 feet/minute.

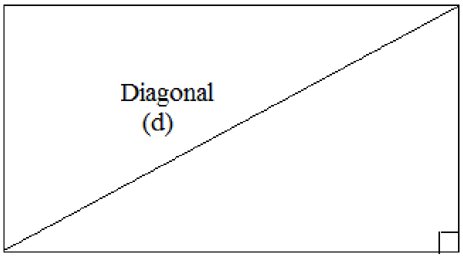
If the length is 12 feet and the width is 5 feet find the rates of the change of the:

a) Area

b) Perimeter

c) Diagonal Length

Step 1: Draw a picture and label given values



width (w) = 5

⇧ 2 feet/minute (increasing)

length (l) = 12 ⇦ 2 feet/minute (decreasing)

rates of change (with respect time (t))

$$\frac{dw}{dt} = 2 \text{ ft/min}$$

$$\frac{dl}{dt} = -2 \text{ ft/min}$$

Step 2: Write equations (that show how the variables relate to each other)

Area = length x width

Perimeter = 2(length) + 2(width)

$$\text{Diagonal} = \sqrt{(\text{length})^2 + (\text{width})^2} \quad \text{(Pythagorean Theorem)}$$

Step 3: Solve using (implicit) differentiation

a) To find the change of area with respect to time,

$$\frac{dA}{dt} = \frac{dl}{dt}w + \frac{dw}{dt}l \quad \text{(product rule)}$$

$$\frac{dA}{dt} = -2 \text{ ft/min } (5) + 2 \text{ ft/min } (12) \quad \text{(substitution)}$$

$$\boxed{\frac{dA}{dt} = 14 \text{ feet/minute}}$$

b) To find the change in perimeter with respect to time,

$$\frac{dP}{dt} = 2\frac{dl}{dt} + 2\frac{dw}{dt}$$

$$\boxed{\frac{dP}{dt} = 2(-2 \text{ ft/min}) + 2(2 \text{ ft/min}) = 0 \text{ feet/minute}}$$

c) To find the change in each diagonal with respect to time,

$$\frac{dD}{dt} = \frac{1}{2}(l^2 + w^2)^{\frac{-1}{2}}\left(2l\frac{dl}{dt} + 2w\frac{dw}{dt}\right) \quad \text{(power rule/chain rule)}$$

$$\frac{dD}{dt} = \frac{1}{2}(144 + 25)^{\frac{-1}{2}}(-48 + 20)$$

$$\boxed{\frac{dD}{dt} = \frac{-28}{2(13)} = -\frac{14}{13} \text{ feet/minute}}$$

approx. -1.08

Step 4: Check for reasonableness

one minute ago: length = 14, width = 3, area = 42

△ area = 18

now: length = 12, width = 5, area = 60

14 is in between!

△ area = 10

one minute later: length = 10, width = 7, area = 70

That makes sense... As the lengths increase 2 feet each, the widths decrease 2 feet each. Although the shape is changing, the perimeter does not change.

one minute ago: length = 14, width = 3, diagonal ≈ 14.3

△ diagonal = -1.3

now: length = 12, width = 5, diagonal = 13

-1.08 is in between!

one minute later: length = 10, width = 7, diagonal ≈ 12.2

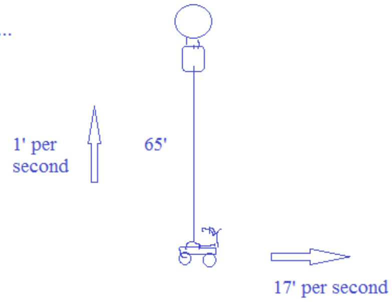
△ diagonal = -.8

Example: A balloon is 65' above the ground and rising at a rate of 1' per second.
Meanwhile, a bicycle rider is directly under the balloon, traveling on a flat road at a rate of 17' per second.

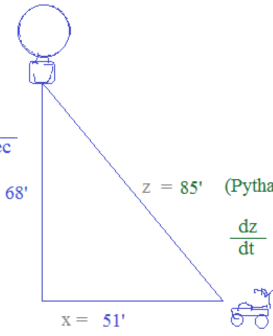
How fast is the distance between the balloon and the bicycle rider increasing *3 seconds later*?

Step 1: Draw diagram and establish variables

Now...



3 seconds later..



$\frac{dy}{dt} = 1 \frac{ft}{sec}$

$y = 68'$

$z = 85'$ (Pythagorean Theorem)

$\frac{dz}{dt} = ?$

$x = 51'$

$\frac{dx}{dt} = 17 \frac{feet}{second}$

Step 2: Write the basic equation (showing the relationship between variables)

$x^2 + y^2 = z^2$

Since we know x, y, z, and we know the dx/dt and dy/dt, we can find the related rates.. (i.e. the change in distance related to time)

Step 3: Find related rate (e.g. implicit differentiation)

The rates of change *with respect to time (t)* $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

Then, substitute... (after the units cancel,) $2(51)(17) + 2(68)(1) = 2(85)(\frac{dz}{dt})$

$1734 + 136 = 170\ (\frac{dz}{dt})$

$11 = \frac{dz}{dt}$

After 3 seconds, the distance between the bicycle and the balloon is increasing at a rate of 11 feet/second

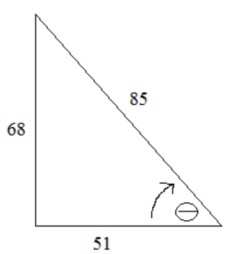
Step 4: Check for reasonableness...

Snapshots:

now	1 second later	2 seconds later	3 seconds later	4 seconds later
distance apart: 65'	distance apart: 68.15	distance apart: 75.13	distance apart 85	distance apart: 96.88
65' high 0' along	66' high 17' along	67' high 34' along	68' high 51' along	69' high 68' along

average rate between 0 and 1: 3.15
average rate between 1 and 2: 6.98
average rate between 2 and 3: 9.87
IROC at 3 seconds: 11
average rate between 3 and 4: 11.88

One more question: How fast is the angle between the ground and the line connecting the balloon and bicycle changing (after 3 seconds)?



One equation that relates the angle to the sides: $\tan\theta = \frac{y}{x}$ (opposite) (adjacent)

Then, the derivative that relates the rates to time:
(using implicit diff. and the quotient rule)

$\sec^2\theta \frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$

$\sec\theta = \frac{85}{51}$

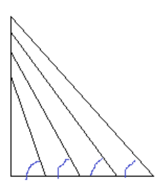
$\left(\frac{85}{51}\right)^2 \frac{d\theta}{dt} = \frac{51(1) - 68(17)}{51^2}$

$\frac{d\theta}{dt} = -.1529$ radians or -8.76 degrees/second

quick check:

Now: angle is undefined
1 second later: 75.5 degrees
2 seconds later: 63.1 degrees
3 seconds later: 53.1 degrees
4 seconds later: 45.4 degrees

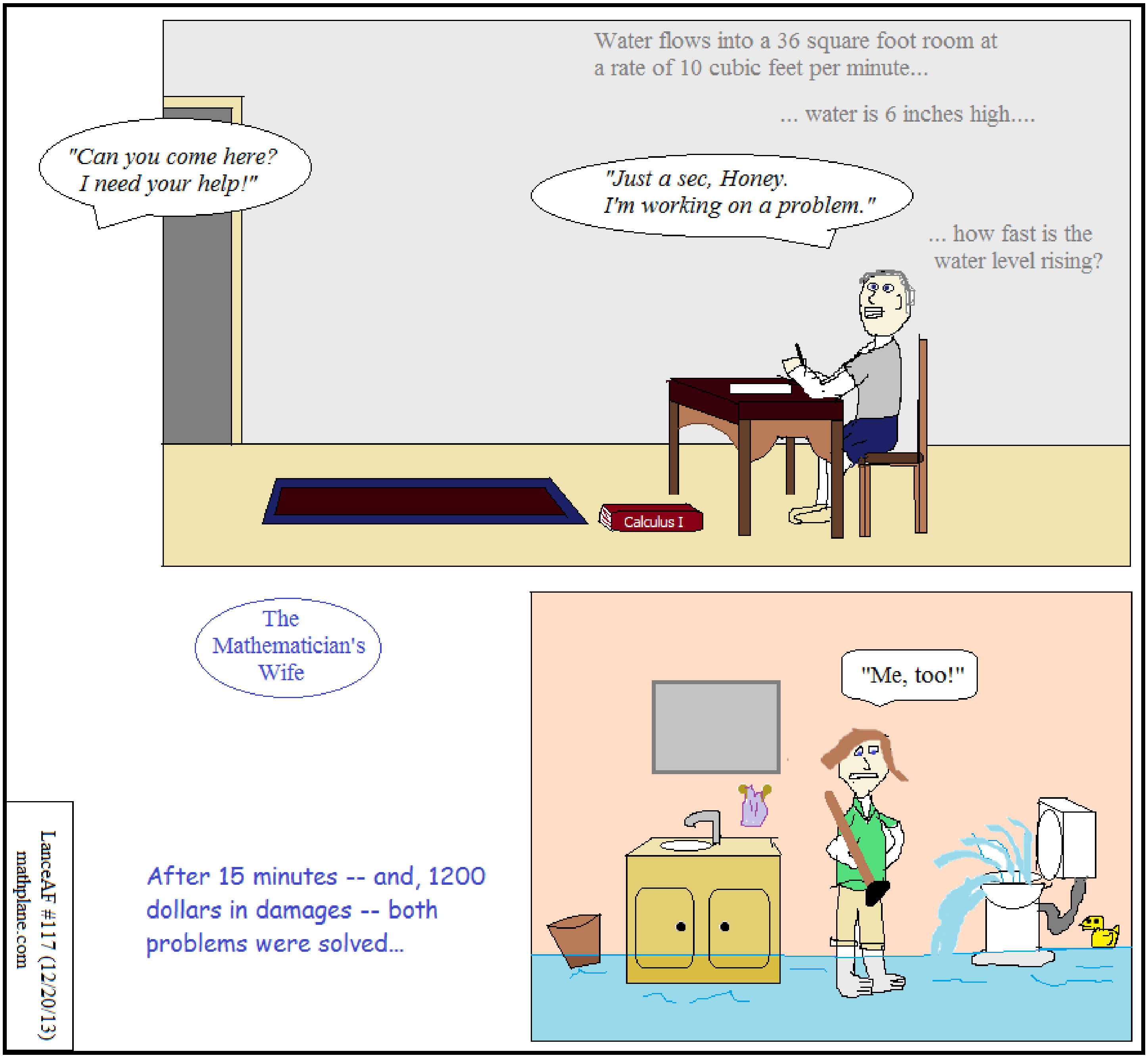
AROC (1 - 2) -12.4
AROC (2 - 3) -10
IROC -8.76 degrees
AROC (3 - 4) -7.7



Note: You may use another trig function such as $\sin\theta = \frac{y}{z}$

$\cos\theta \frac{d\theta}{dt} = \frac{z\frac{dy}{dt} - y\frac{dz}{dt}}{z^2}$ $\quad \frac{51}{85}\frac{d\theta}{dt} = \frac{85(1) - 68(11)}{85^2}$

$\frac{d\theta}{dt} = -.1529$ radians or -8.76 degrees/second ✓



RELATED RATES OF CHANGE TEST

(w/ SOLUTIONS)

Calculus: Related Rates of Change Test

1) A boat is pulled by means of a winch on the dock 12 feet above the deck of the boat.
If the winch pulls in rope at the rate of 4 ft/second, determine the speed of the boat when there is 20 feet of rope out.
What happens to the speed of the boat as it nears the deck?

2) At a sand and gravel plant, the sand is pouring off the conveyer belt onto a conical pile at a rate of 10 ft^3/minute.
The diameter of the base of the cone is 3 times the altitude.
At what rate is the height of the pile changing when it is 15 feet high?

3) A man 6 feet tall walks at a rate of 5 ft/second away from a light that is 15 feet above the ground. When the man is 10 feet from the base of the light,
a) at what rate is the tip of his shadow moving?
b) at what rate is the length of his shadow growing?

4) A 17-foot ladder leans against a wall.
The base of the ladder slides away from the wall at 5 ft/second.
When the base is 15 feet from the wall, what rate is the <u>angle</u> (between the floor and the ladder) changing at that moment?

5) A point moves along $y = (x - 3)^2$ such that the x-coordinate is moving at 2 units per minute.
At $x = 1$,
a) how fast is the y-coordinate moving?
b) how fast is the distance from the origin changing?

6) Two ships leave port, traveling straight paths that differ by 55 degrees.
Ship A is traveling at 30 miles per hour.
Ship B is traveling at 38 miles per hour.
After three hours, what rate is the distance between the two ships increasing?

Calculus: Related Rates of Change Test

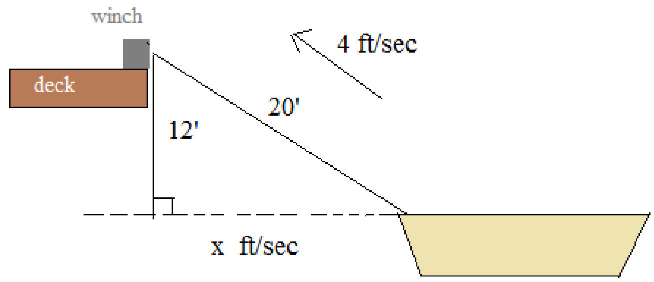
SOLUTIONS

1) A boat is pulled by means of a winch on the dock 12 feet above the deck of the boat. If the winch pulls in rope at the rate of 4 ft/second, determine the speed of the boat when there is 20 feet of rope out.
What happens to the speed of the boat as it nears the deck?

Step 1: Draw a diagram

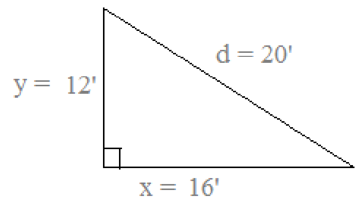
Step 2: Label the given parts

Step 3: Identity and establish equations



Pythagorean Theorem: $x^2 + y^2 = d^2$

We are looking for the speed of the boat: $\frac{dx}{dt}$ (change of x with respect to time (t))



We know $\frac{dd}{dt} = 4$ ft/sec

and $\frac{dy}{dt} = 0$ (the winch doesn't move)

Step 4: Find related rate of change

We are looking for the change in lengths *with respect to time (t)*

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2d\frac{dd}{dt}$$

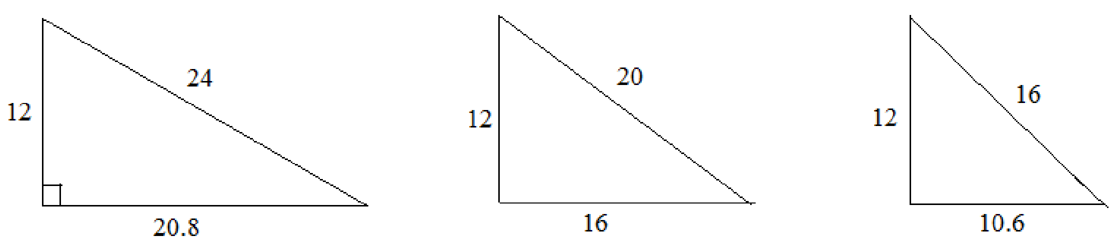
Step 5: Input values to solve

$$2(16\text{ ft})\frac{dx}{dt} + 2(12\text{ ft})(0\text{ ft/sec}) = 2(20\text{ ft})(4\text{ ft/sec})$$

$$(32\text{ ft})\frac{dx}{dt} = (160\text{ ft}^2\text{/sec})$$

$$\frac{dx}{dt} = 5\text{ ft/sec}$$

Quick Check:



one second → boat moves 4.8 feet one second → boat moves 5.4 feet

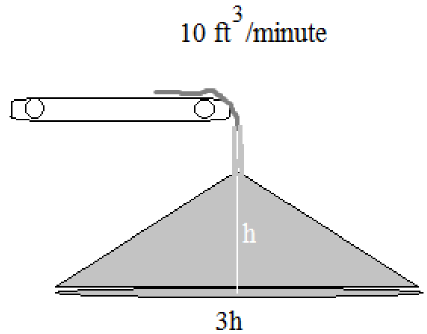
**boat speeds up as it nears the deck!

Calculus: Related Rates of Change Test

SOLUTIONS

2) At a sand and gravel plant, the sand is pouring off the conveyer belt onto a conical pile at a rate of 10 ft^3/minute.
The diameter of the base of the cone is 3 times the altitude.
At what rate is the height of the pile changing when it is 15 feet high?

Step 1: Draw a diagram and label parts



Step 2: Identify and establish equations

$$\frac{dV}{dt} = 10 \text{ ft}^3\text{/minute}$$

$V = 1/3 \; r^2 h$ (volume of cone)

radius = 3h/2 (radius = 1/2(diameter))

**we are looking for $\frac{dh}{dt}$ (rate the height of pile is changing)

Since we know dV/dt and we're looking for dh/dt,
we want to have an equation with V and h...

So, using substitution, $V = 1/3 \pi r^2 h$ $\quad r = 3h/2$

$$V = \frac{\pi}{3}\left(\frac{3h}{2}\right)^2 h = \frac{9h^2 \cdot h \,\pi}{3 \cdot 4} = \frac{3}{4} h^3 \pi$$

Step 3: Use implicit differentiation to find the rates of change with respect to t

$$V = \frac{3}{4} h^3 \pi$$

$$\frac{dV}{dt} = \frac{9}{4} \pi h^2 \frac{dh}{dt}$$

substitute values: $10 \text{ ft}^3\text{/minute} = \frac{9}{4} \pi (15 \text{ feet})^2 \frac{dh}{dt}$

$$10 \text{ ft}^3\text{/minute} = 506.25 \pi \text{ feet}^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{10}{506.25 \pi} \text{ ft/minute}$$

$$= \frac{8}{405 \pi} \text{ ft/minute}$$

(approx. .0062876 ft/min)

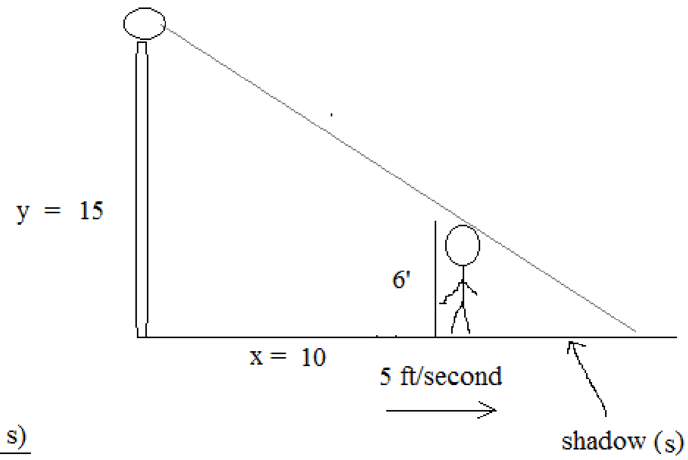
3) A man 6 feet tall walks at a rate of 5 ft/second away from a light that is 15 feet above the ground. When the man is 10 feet from the base of the light,

a) at what rate is the tip of his shadow moving?
b) at what rate is the length of his shadow growing?

Calculus: Related Rates of Change Test

SOLUTIONS

Step 1: draw a diagram and label



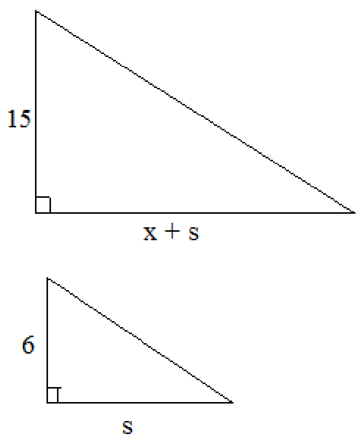
$$\frac{15}{6} = \frac{10+s}{s}$$

$9s = 60$ $\quad s = 20/3$

now

Step 2: Identify variables, equations, and relationships

There are 2 similar triangles



$$\frac{15}{6} = \frac{(x+s)}{s}$$

$15s = 6x + 6s$

$9s = 6x$

$x = (3/2)s$

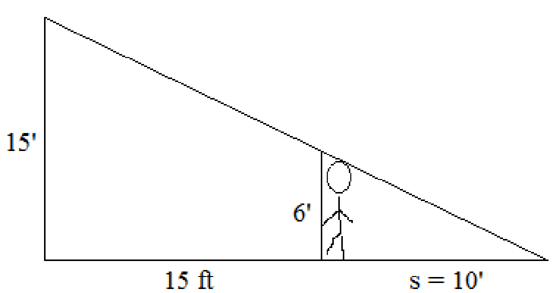
Step 3: Find related rates of change

b) $$1\,\frac{dx}{dt} = 3/2\,\frac{ds}{dt}$$

$$5 \text{ ft/sec} = 3/2\,\frac{ds}{dt}$$

$$\frac{ds}{dt} = \frac{10}{3} \text{ ft/sec}$$

The length of the shadow (s) is increasing at 10/3 feet per second



one second later

a) $$\frac{y}{6} = \frac{x+s}{s}$$

This ratio is constant

$$\frac{15}{6} = \frac{10+x+s}{s}$$

$15s = 60 + 6x + 6s$

$9s = 60 + 6x$

$$9\frac{ds}{dt} = 0 + 6\frac{dx}{dt}$$

$$9\frac{ds}{dt} = 30 \text{ ft/sec}$$

$$\frac{ds}{dt} = 10/3 \text{ ft/sec}$$

The tip of the shadow is moving at 25/3 ft/sec

so, x is increasing at 5 ft/sec and s is increasing at 10/3 ft/sec

so, the entire side is increasing at 25/3 ft/sec!

4) A 17-foot ladder leans against a wall.
The base of the ladder slides away from the wall at 5 ft/second.
When the base is 15 feet from the wall, what rate is the angle
(between the floor and the ladder) changing at that moment?

Calculus: Related Rates of Change Test

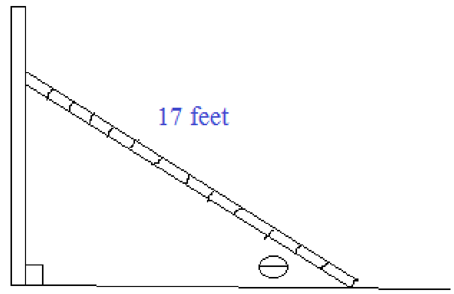
SOLUTIONS

Step 1: Draw a picture and label parts

Ladder, wall and floor form right triangle..

Pythagorean theorem: $15^2 + 8^2 = 17^2$

The angle between the floor and ladder is θ



Step 2: Identify variables and equations

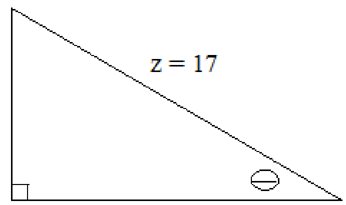
$x = 15$ feet $\qquad \frac{dx}{dt} = 5$ ft/second

$y = 8$ feet $\qquad \frac{dy}{dt} = ?$

$z = 17$ feet $\qquad \frac{dz}{dt} = 0$ (**the distance of the ladder never changes!)

Since we are looking for the rate of change of the angle, we'll use θ.
And, because we know the rate of change for x and z, we'll use x and z.
(and, ignore y)

$$\cos\theta = \frac{x}{z}$$



Step 3: Solve (i.e. find the related rates of change)

derivative with respect to time (t)

use quotient rule ---->

$$-\sin\theta \frac{d\theta}{dt} = \frac{(1)\frac{dx}{dt}(z) - (1)\frac{dz}{dt}(x)}{z^2}$$

substitution

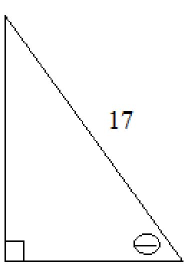
$$-\left(\frac{8}{17}\right)\frac{d\theta}{dt} = \frac{5 \text{ ft/sec} \cdot (17 \text{ ft}) - 0 \text{ ft/sec} \cdot (15 \text{ ft})}{17^2 \text{ ft}^2}$$

$$-\left(\frac{8}{17}\right)\frac{d\theta}{dt} = \frac{5}{17 \text{ sec}}$$

$$\boxed{\frac{d\theta}{dt} = \frac{-5}{8} \text{ per second}}$$

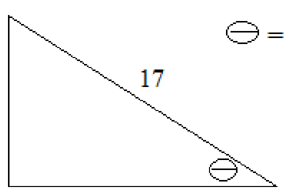
Step 4: Check/Estimate

The speed of the ladder is increasing..
one second ago, it moved at .38 rad/sec
currently, it moves at .625 rad/sec



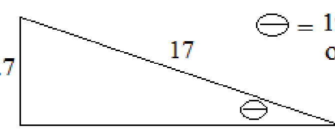
$\theta = 54^\circ$ or .94 radians

(one second ago)



$\theta = 32.2^\circ$ or .56 radians

(now)



$\theta = 19.7^\circ$ or .34 radians

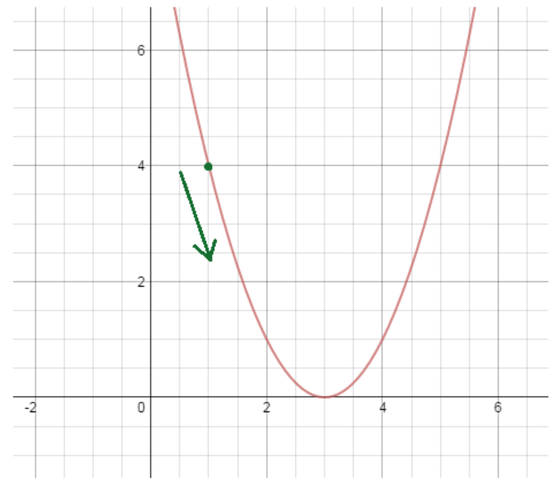
(.2 sec later)

5) A point moves along $y = (x - 3)^2$ such that the x-coordinate is moving at 2 units per minute.

At x = 1,

a) how fast is the y-coordinate moving?

b) how fast is the distance from the origin changing?



Step 1: Identify variables and equations

Since x = 1,

$y = (1 - 3)^2 = 4$

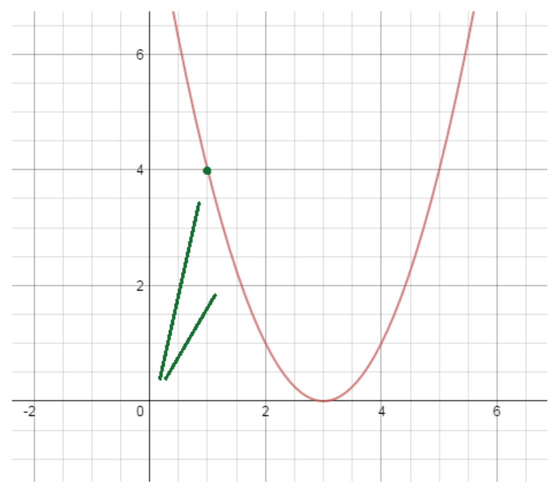
$\frac{dx}{dt} = 2$ units/minute

$\frac{dy}{dt} = ?$

$y = (x - 3)^2$

distance from (0, 0): $d = \sqrt{x^2 + y^2}$

Step 2: Find the rates of change



$y = (x - 3)^2$

$\frac{dy}{dt} = 2(x - 3)\frac{dx}{dt}$

$= 2(1 - 3)(2)$

$= -8$ units per minute

at x = 1, the y-coordinate is moving at -8 units/minute

$d = \sqrt{x^2 + y^2}$

$\frac{dd}{dt} = \frac{1}{2}(x^2 + y^2)^{\frac{-1}{2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)$

$\frac{1}{2}(1 + 16)^{\frac{-1}{2}}(2(1)(2) + 2(4)(-8))$

$\frac{1}{2}\frac{(-60)}{\sqrt{17}} = -7.28$ units/minute

at x = 1, the distance from the origin is decreasing at an instantaneous rate of -7.28 units/minute

Quick check: ("snapshots and AROC") $d = \sqrt{x^2 + y^2}$

1 minute earlier: (-1, 16)
distance = 16.03

y change: -12   d change: -11.91

now: (1, 4)   -8 is reasonable   -7.28 is reasonable
distance = 4.12

y change: -4   d change: -1.12

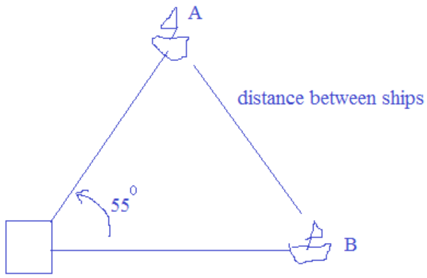
1 minute later: (3, 0)
distance = 3

6) Two ships leave port, traveling straight paths that differ by 55 degrees.

Ship A is traveling at 30 miles per hour.
Ship B is traveling at 38 miles per hour.

After three hours, what rate is the distance between the two ships increasing?

Step 1: Draw Diagram



Step 2: Identify the relevant equations and parts

Let a = distance ship A traveled
b = distance ship B traveled
c = distance between ships

Angle C = 55 degrees

law of cosines:

$c^2 = a^2 + b^2 - 2abCos(C)$

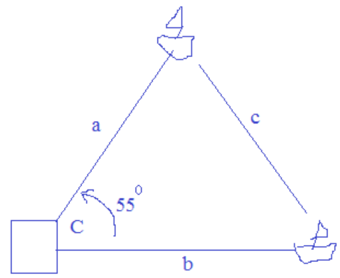
After 3 hours:

a = 90 miles
b = 114 miles

$c^2 = 90^2 + 114^2 - 2(90)(114)Cos(55^\circ)$

$c^2 = 21096 - 20520(.5736)$

c = 96.57 miles



and, the rates of change:

$\frac{da}{dt} = 30$ miles/hour

$\frac{db}{dt} = 38$ miles/hour

$\frac{dC}{dt} = 0$ (the angle never changes!)

$\frac{dc}{dt} = ?$

Step 3: Find the related rate of change

law of cosines:

$c^2 = a^2 + b^2 - 2abCos(C)$

$$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} - 2\left[\frac{da}{dt}bCos(C) + a\frac{db}{dt}Cos(C) + ab(-SinC)\frac{dC}{dt}\right]$$

$$193.1\frac{dc}{dt} = 180(30) + 228(38) - 2[30(114)(.5736) + 90(38)(.5736) + (90)(114)(-.8191)(0)]$$

$$193.1\frac{dc}{dt} = 5400 + 8664 - 2[1961.7 + 1961.7 + 0]$$

$$193.1\frac{dc}{dt} = 6217$$

$\frac{dc}{dt} = 32.2$ miles per hour..

Check: using 'snapshots'

hour 1: 30, 38 $c^2 = 900 + 1444 - 2(30)(38)Cos(55)$ distance c = 32.19

hour 2: 60, 76 $c^2 = 3600 + 5776 - 2(60)(76)Cos(55)$ distance c = 64.38

hour 3: 90, 114 distance c = 96.57

hour 4: 120, 152 $c^2 = 14400 + 23104 - 2(120)(152)Cos(55)$

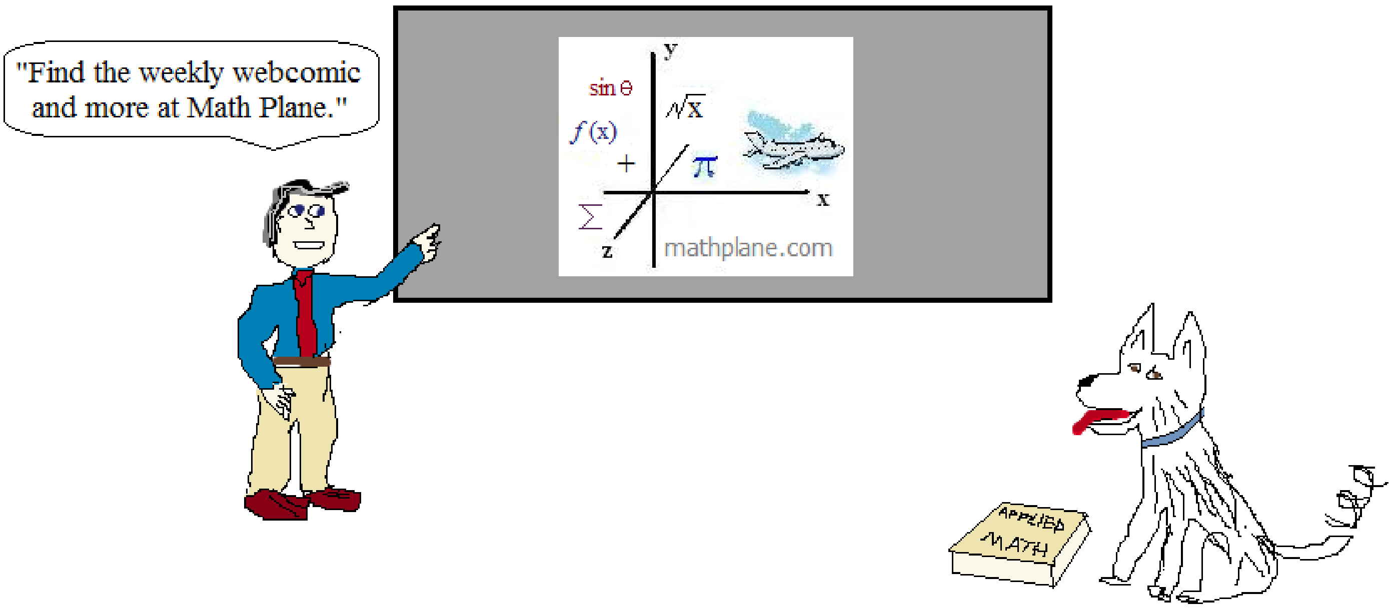
distance c = 128.76

The rate of change of c is 32.19...

Thanks for visiting. (Hope it helps!)

If you have questions, suggestions, or requests, let us know.

Enjoy.



Also, at Facebook, Google+, Pinterest, TES, and TeachersPayTeachers

Mathplane *Express* for mobile at Mathplane.org

One more question:

A police car is parked .3 miles away from the highway. The speed limit is 70 miles per hour.
A red mustang cruises past a mile marker that is .4 miles down the road.
If the radar gun reads 60 miles per hour, is the mustang going over the speed limit?

SOLUTION-→

Related Rates Application: Speeding car

A police car is parked .3 miles away from the highway. The speed limit is 70 miles per hour.
A red mustang cruises past a mile marker that is .4 miles down the road.
If the radar gun reads 60 miles per hour, is the mustang going over the speed limit?

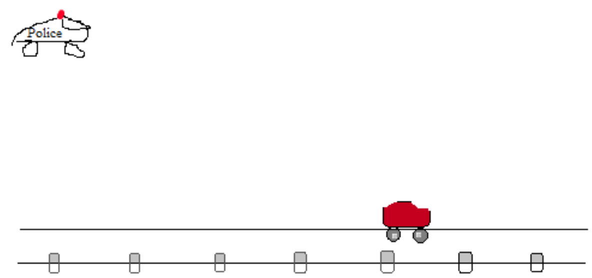
Step 1: Set up the variables

$y = .3$ miles (vertical) distance of police car to the road

$\frac{dy}{dt} = 0$ (since the police car and road don't move, the change of y with respect to time is 0)

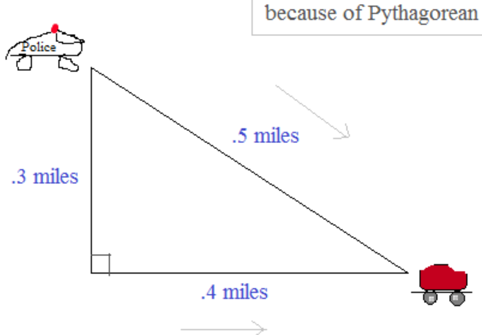
$x = .4$ miles (horizontal) distance of mustang down the road

$\frac{dx}{dt} = ?$ (that's the speed of the mustang with respect to time, which is what we're looking for!)



Step 2: Identify the relationship of given information

distance is .5 miles between cars, because of Pythagorean Theorem



d = distance between mustang and police radar = .5 miles

$\frac{dd}{dt} = 60$mph (rate that the distance increases between radar and mustang)

The relationship of the moving variables is $x^2 + y^2 = d^2$

Step 3: Find the related rates of change

Take the derivative *with respect to time (t)*

$$d^2 = x^2 + y^2$$

$$2d\frac{dd}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$2(.5 \text{ miles})(60 \text{m/hr}) = 2(.4 \text{ miles})\frac{dx}{dt} + 2(.3 \text{ miles})(0)$$

$$(1 \text{ mile})(60 \text{ m/hr}) = (.8 \text{ miles})\frac{dx}{dt}$$

$$\frac{dx}{dt} = 75 \text{ miles per hour}$$

Step 4: Answer the question (and check for reasonableness)

The mustang is driving 75 miles per hour (over the speed limit)