Combinations & Permutations:

An Introduction to Counting Principles

Packet includes formulas, examples, explanations, and applications. (Also, a practice puzzle and math comics.)

Why is this not a combination lock?

Combinations vs. Permutations

Permutation: --- A rearrangement of the elements of a set.
--- An *ordered arrangement* of *n* different objects.
The number of permutations (i.e. the number of different possible ordered arrangements) is n!

Example 1: How many ways can you arrange 4 chair in a row? 4! 4 x 3 x 2 x 1 = 24 ways

1234	1243	1324	1342	1423	1432
2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

I listed all the possibilities (in numerical order to keep track).

Observe how we arrive at 4!

In position #1, you can choose from the 4 chairs: 4 choices..
Then, for position #2, you can choose from 3 remaining chairs..
Then, for position #3, you can choose from the 2 remaining chairs..
Finally, the last chair goes into position #4..
4 x 3 x 2 x 1 = 24 (or, 4!)

Combination: --- A *collection* of objects in a set. (*order does not matter..)

Example 2a: At an ice cream shop, there are 5 flavors. How many combinations of 3 flavors can we make?

C= Chocolate V=Vanilla S=Strawberry M=Mint Chip R=Rainbow Sherbet

CMRSV are the 5 flavors... The following are the combinations
(listed in alphabetical order to avoid confusion or 'double counting')

CMR CMS CMV CRS CRV CSV
MRS MRV MSV
RSV

Notice that order does not matter! Therefore, we have fewer combinations (than permutations).
EX: Since we used CMR, we can ignore all other combinations of C, M, and R..
Why? Because scoops of chocolate, rainbow, mint are the same as scoops of rainbow, mint, chocolate.

Example 2b: An obvious question: how many combinations of 5 flavors are there?

ONE!! Ordering all 5 flavors can be done in only one way...

Deriving Math Formulas

Let's start with permutations... Assume we have 8 dogs:

1: Astro
2: Buster
3: Chester
4: Dagwood
5: Emmy
6: Fritz
7: Gus
8: Homer

How many ways can we pick a Gold, Silver, and Bronze medal for "Man's best friend in the world"?

(**Note: This is different than Example 1 above.. Instead of arranging ALL 8 in order, we are arranging 3 out of 8)

1: Astro
2: Buster
3: Chester
4: Dagwood
5: Emmy
6: Fritz
7: Gus
8: Homer

Since each medal is specific (Gold, Silver, Bronze), the order we hand out these medals matters. Therefore, we must use permutations.

Here's how it breaks down:

*Gold Medal -- 8 choices: A B C D E F G H (clever how the names match up with letters!).
Let's assume A wins the gold.
*Silver Medal -- 7 choices: B C D E F G H
Now, suppose B wins the silver.
*Bronze Medal -- 6 choices remain: C D E F G H
Then, suppose C wins the bronze.

We picked A, B, and C in this example. But, the details don't matter. The number of possibilities will be 8 choices, then 7 choices, then 6 choices..
The total number of possibilities will be 8 x 7 x 6 = 336.

$${}_nP_k = \frac{n!}{(n-k)!}$$

where n is the number of total elements, and k is the number of elements selected.

or

$$P(n,k) = \frac{n!}{(n-k)!}$$

Remember, if we ordered all 8 dogs, it would be 8!
(8 choices, 7 choices, ... until we ran out of dogs..)

8 x 7 x 6 x 5 x 4 x 3 x 2 x 1

However, that does too much. We only want 8 x 7 x 6 (for 3 medals).
How can we "stop" the factorial at 5?

Notice how we want to get rid of 5 x 4 x 3 x 2 x 1. What's another expression for this? 5!

So, if we write $\frac{8!}{5!}$, we'll end up 8 x 7 x 6 $\quad \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = 336$

Reasoning: And, why did we use 5? Because, it was the number of dogs left over after we picked the 3 winners. In other words, the order of the 5 remaining didn't matter!

$$\frac{8!}{(8-3)!}$$

"Use the first k numbers of n"

$$\frac{n!}{(n-k)!}$$

Now, combinations.
Suppose instead of gold, silver, and bronze medals, we award milk bones to the top 3 dogs.

How many ways can I award 3 milk bones to 8 dogs?

In this case, order does not matter. You can mix up the top 3 finishers, and each still gets a milk bone.

Assume Astro, Buster, and Chester are the top 3 dogs again. If I give a milk bone to A, B, C, it's the same as giving a milk bone to B, A, and then C. 1st, 2nd, or 3rd: they all get the same prize.

This raises an interesting point: we have redundancies.

How many do we have?

And, how can we modify the permutation formula to eliminate them?

Let's figure out how many ways we can arrange Astro, Buster, and Chester:

ABC ACB BAC BCA CAB CBA $3! = 6$ A permutation!

So, if we have 3 milk bones to give away, there will be 6 variations for every three dogs we pick.

Therefore, if we want to figure out how many *combinations* we have, we must determine the number of permuations and divide it by all the redundancies.

In this case, we have 336 permutations, and we divide by 6 to eliminate all the extras.

$$\frac{336}{6} = 56$$

$$_{n}C_{k} = C(n,k) = \frac{P(n,k)}{(k!)}$$

which means "Find all the ways to pick k people from n, and divide by the k! variants"

$$_{n}C_{k} = \frac{n!}{(n-k)!\,k!}$$

Examples of combinations (order does not matter) and permutations (arrangements).

Picking a team of 3 people from a group of 10.

Combination: $C(10,3) = \frac{10!}{7!\,3!} = 120$

Selecting a President, VP and Waterboy from 10 applicants.

Permutation: $P(10,3) = \frac{10!}{7!} = 720$

Choosing 2 desserts from a menu of 8 choices.

Combination: $C(8,2) = \frac{8!}{6!\,2!} = 28$

Ranking your 4 favorite desserts from a menu of 8 choices.

Permutation: $P(8,4) = 8 \times 7 \times 6 \times 5 = 1680$

So, let's return to the original question: Why is this NOT a combination lock?

Because it's a *permutation* lock!

For this lock, order matters.

Example: If I told you the 3 numbers to open the lock are 14-22-24, you would need the order.
14-22-24 14-24-22 etc.. are all different arrangements.

Incidentally, there are 40 numbers on this lock. (Assuming a number may only be used once,)

\# of permutations = $P(40,3) = 40 \times 39 \times 38 = 59{,}280$

\# of combinations = $C(40,3) = 9880$

Summary notes:

"A permutation is an ordered combination"
Or, to help remember, think "Permutation ---> Position"
Rather than memorize the formulas, try to understand how and why they work.
And, remember, a set of numbers will have fewer combinations than permutations.

Variations: Repetition vs. Non-Repetition (Permutations/Combinations continued)

Suppose the lock company manufactured locks with "combinations" (permutations) that may have repeating numbers, such as 22-22-30? There would be more possibilities. How many more?

There are basically 2 types of permutation:

1) repetition is allowed: such as the example above '22' is used twice.
2) no repetition: for example, the first three people in a running race. You can't finish 1st and 2nd.

Permutations that allow repetition:

When you have n things to choose from, you'll have n choices *each* time.

When making r choices from n elements, the number of permutations is

$$n \cdot n \cdot \ldots \text{ (r times)} = n^r$$

There will be n possibilities for the first choice. THEN, again there will be n possibilities for the 2nd choice. And, so on..

Example: In the lock, there are 40 numbers to select (0, 1, 2, ... , 39). If you can set the order AND can repeat numbers, then the number of possibilities are:

$$40^3$$

40 x 40 x 40 = 64,000 permutations

Permutations WITHOUT Repetition:

$${}_nP_k = P(n,k) = \frac{n!}{(n-r)!}$$

Suppose the lock did not allow numbers to be repeated.

There would be 40 x 39 x 38 = 59,280 permutations

40 choices for the 1st number.
39 remaining choices for the 2nd number.
38 remaining choices for the 3rd number.

Note: 0! = 1 It may seem odd that multiplying no numbers together gets you 1, but it's accepted that 0! is 1. One reason: 0! is in the denominator, the equation would be undefined. Letting 0! = 1 fixes that. Also, consider this: how many ways can you arrange nothing? One.

Combinations without repetition:

We've learned combinations are permutations/redundancies.

$${}_nC_k = C(n,k) = \frac{n!}{(n-k)!\,k!}$$

Example: Assuming you have 16 pool balls (7 stripes, 7 solids, the 8 ball, and the cue ball), how many ways could you select 3 balls?

$$\frac{16!}{13!\,3!} = \frac{16 \times 15 \times 14}{3 \times 2 \times 1} = 560$$

Now, assuming you have 16 pool balls. How many ways could you select 13 balls?

$$\frac{16!}{3!\,13!} = 560$$

It's the same answer! (this makes sense: choosing 13 balls is the same as not picking 3 balls)

Combinations with repetition:

Let's use another ice cream example. This time, you may order 2 scoops of vanilla and 1 scoop of chocolate -- rather than having to choose 3 different flavors.

How many variations will there be?

Let's use letters to represent 5 flavors:

B Banana
C Chocolate
L Lemon
S Strawberry
V Vanilla

Selections may be {c, c, c} (3 scoops of chocolate)
{b, l, v} (one each of banana, lemon, and vanilla)
{b, v, v} (one banana, 2 vanilla)

So, there are n = flavors to choose from
r = 3 choices
flavors can be repeated
Order does not matter (combination) eg: {c, l, c} is the same as {l, c, c}

How do we count the possibilities?

Think about the ice cream being in a row of containers.
You could tell the ice cream man: "skip the 1st, then 3 scoops (chocolate), then skip the next 3 containers"
or,
"2 scoops (banana), skip, skip, 1 scoop (strawberry), skip"

Notice, each example has 7 "moves".

(skip, scoop, scoop, scoop, skip, skip, skip)
or,
(scoop, scoop, skip, skip, scoop, skip)

So, instead of worrying about different flavors, we have a simpler problem:

"how many different ways can you arrange 'skips and scoops'?"

There will aways be 3 scoops (of ice cream) and 4 skips (to get from the 1st container to the 5th)!!

In essence, there are $3 + (5 - 1)$ positions and we want to choose 3 of them.

$3 \to r$, $(5-1) \to (n-1)$, $3 \to r$

Then, since this is a combination, we must eliminate the redundancies: divide the permuation by

$(n + r - 1 - r)!$ ----> $(n - 1)!$

$$\frac{(n+r-1)!}{r!\,(n-1)!}$$

where n is the number of things to choose from, and you choose r of them. (repetition allowed, combination)

So, the number of ways to select 3 scoops from the ice cream shop are:

$$\frac{(5+3-1)!}{3!\,(5-1)!} = \frac{7!}{3!\,4!} = \frac{5040}{6 \times 24} = 35$$

Example: If there are 10 types of donuts on the menu and you want 3 donuts, how many choices can you make?

n = 10
r = 3

$$\frac{(10+3-1)!}{3!\,(10-1)!} = \frac{12!}{3!\,9!} = \frac{12 \times 11 \times 10}{3 \times 2 \times 1} = 220 \text{ choices}$$

"The Complement Principle" --- If A = # of outcomes that include A
$\overline{A}$ = # of outcomes that do not include A,

then $A + \overline{A}$ = total possible outcomes

This is a useful concept, because it can simplify counting strategies.

Example: How many numbers between 1 and 9999 contain the number '7'?

If we count them directly, we'd have to count numbers that include one '7', two '7's, three '7's, and 7777.

7, 17, 71, 700, 727, 1723, and on and on.... That is a multi-step, time consuming method.

OR, we can use "The Complement Principle"...

Total possible outcomes: 10,000 (there are 10,000 numbers between 1 and 9999)

Numbers that contain NO 7's: How many ways can we select a number without a seven?

View each digit as an individual choice.
First digit: 9 choices (any digit except 7)
Second digit: 9 choices
Third digit: 9 choices
Fourth digit: 9 choices
Total numbers that contain no 7's = 9 x 9 x 9 x 9 = 6561

Therefore, if there are 10,000 possible outcomes and 6561 do not contain a 7,
then 3439 contain at least one seven.

"Duplicate Elements are Combinations"

We know how to arrange different elements: n! Example: number of different 5-letter 'words' containing A B C D E is 5! = 120

But, suppose the set contains elements that are duplicates.

Example: How many ways can you arrange the letters A A B C D D D E ?

Since the first A and second A are identical, A_1A_2BCDDDE is the same as A_2A_1BCDDDE.

And, of course, all the D's could be switched without changing the appearance.

$AAD_1BC\ D_2D_3E$ is the same as $AAD_2\ BC\ D_3D_1E$

So, how do we eliminate the dupicates? Determine the number of repeats and divide them out of the permutations.

Total permutations: 8!
Number of A's that are repeats: 2!
Number of D's that are repeats: 3!

Total number of 'words' that can be made from the above letters is

$$\frac{8!}{2!\,3!} = 3360$$

Example: How many numbers contain three 1's, one 2, and one 3?

Total ways to arrange 5 numbers: 5!
Since each number will contain three 1's,
we'll need to divide by 3! to eliminate the redundancies.

$$\frac{5!}{3!\,1!\,1!} = 20$$

1's 2's 3's

11123 11132 11213 11231 11312 11321
12113 12131 12311 13112 13121 13211
21113 21131 21311 23111
31112 31121 31211 32111

Counting Questions

I. Determine whether the following are combinations or permutations:

1) phone number

2) 5 cards in a poker hand

3) numbers needed to open a combination lock

4) lottery numbers

5) social security number

6) license plate

II. Application

There are 9 members of a math club: five boys and 4 girls...
Three of them are going to represent the club at the national convention.
How many different delegations could be sent if

a) 3 members are randomly selected

b) 2 boys and 1 girl are selected

III. Probability

Using each letter A, B, C, D, and E once, what is the probability a 'word' has a *first letter A* and *last letter consonant*?

I. Determine whether the following are combinations or permutations: ANSWERS

Counting Questions

1) phone number — permutation (order matters: 555-6778 is a different phone number than 555-8776)

2) 5 cards in a poker hand — combination (It doesn't matter how the cards are held)

3) numbers needed to open a combination lock — permutation (if 21-4-8 is the code, then 4-21-8 will not open the lock!)

4) lottery numbers — combination

5) social security number — permutation

6) license plate — permutation

II. Application

There are 9 members of a math club: five boys and 4 girls...
Three of them are going to represent the club at the national convention.
How many different delegations could be sent if

a) 3 members are randomly selected

Since order does not matter, this is a combination...

$${}_9C_3 = \frac{9!}{6!\,3!} = \frac{9 \cdot 8 \cdot 7 \cdot 6!}{6! \cdot 3 \cdot 2 \cdot 1} =$$ **84 possible delegations**

b) 2 boys and 1 girl are selected

$${}_5C_2 \cdot {}_4C_1 = 40$$

$$\frac{5!}{3!\,2!} \cdot \frac{4!}{3!\,1!}$$

2 boys		1 girl
5	4	4
choices	remaining choices	choices

divided by two (because order doesn't matter, so we eliminate double counting) = **40**

III. Probability

Using each letter A, B, C, D, and E once, what is the probability a 'word' has a *first letter A* and *last letter consonant*?

Method 1: Using probability

P(1st letter A and last letter B,C,orD) =

$$\frac{1}{5} \cdot \frac{3}{4} = \frac{3}{20}$$

Method 2: Combinations/Permutations

$$\frac{\text{Number of 'words' that are correct}}{\text{Number of 5-letter words}} =$$

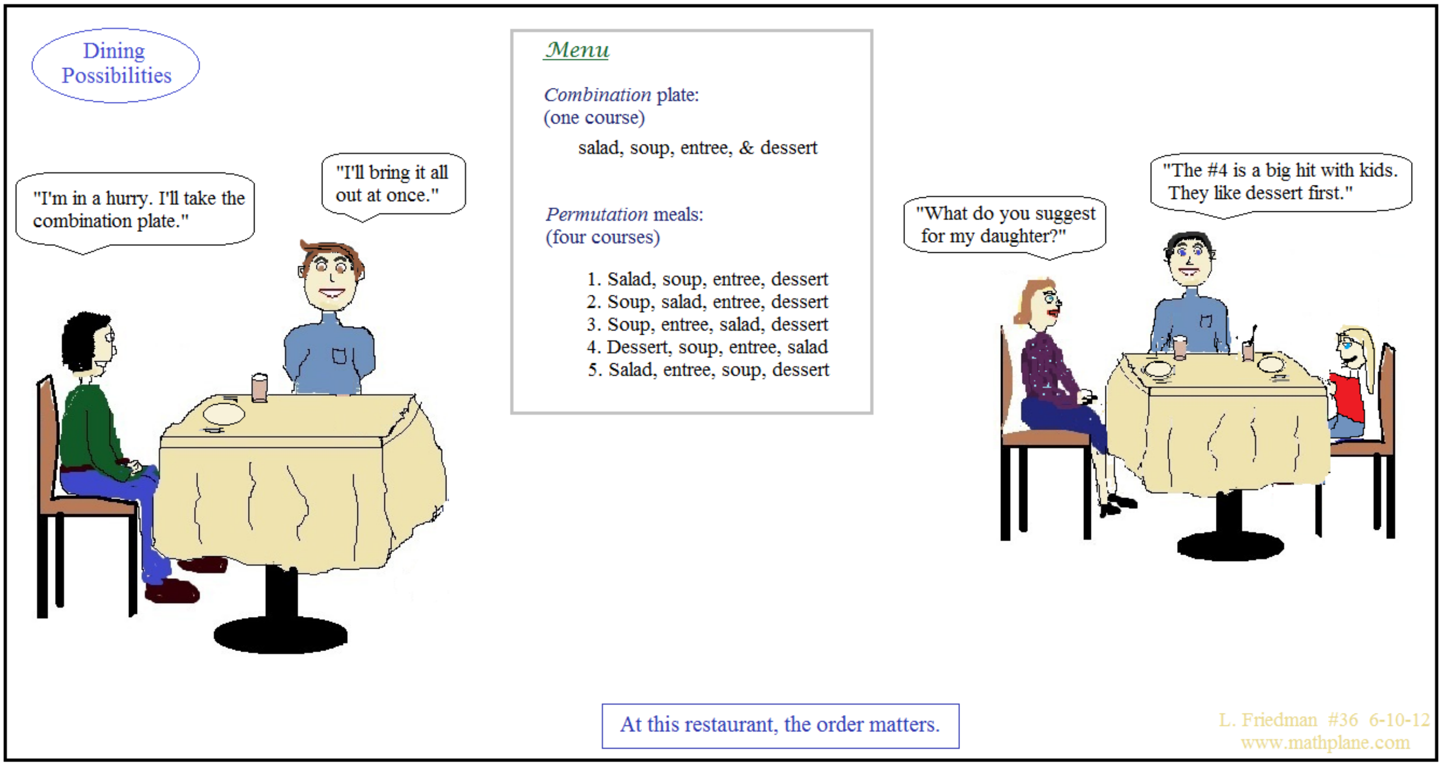
'A' 'consonant' the rest

$$\frac{{}_1C_1 \cdot {}_3C_1 \cdot 3!}{5!} = \frac{18}{120} = \frac{3}{20}$$

1 3 2 1 3

'A' one choice — 2nd letter (3 choices) — 3rd letter (2 remaining choices) — 'consonant' 3 choices (B, C, D)

1 x 3 x 2 x 1 x 3 = 18



Practice Puzzle on the next page -→

Treasure Chest & the Combination/Permutation Lock

Bad News: You're stranded on a desert island.

Good News: You found a treasure chest!!

Bad News: The treasure chest has a lock on it.

Good News: There is a note attached to the lock offering clues to the combination.

"Congratulations, you found the chest...
To get the treasure, here's my test..."

Solve the eight problems below. Then, add the answers together. The 6-digit sum will open the lock.

1) The number of ways to arrange 5 chairs in a row. ________

2) Your school locker has 4 digits on the lock. How many possibilities are there? (each digit can be 0-9) ________

3) How many different license plates can be made using the following format: letter, letter, number, number, number? EX: BX 263 ________

4) At the pizza parlor, there are 3 available toppings on the menu: sausage, pepperoni, and mushroom. How many different pizzas can they offer? ________

5) In your closet, you have 3 ties, 4 shirts, and 3 pairs of pants. How many different outfits could you make (consisting of 1 tie, 1 shirt, and 1 pair of pants)? ________

6) At the Kentucky Derby, there are 12 horses racing. You must try to pick the 3 horses that will finish in 1st, 2nd, and 3rd. How many choices (permutations) are there? ________

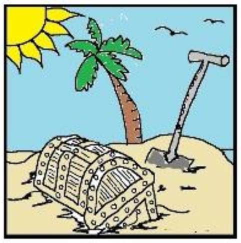
7) While playing scrabble, you draw X, M, N, 2 S's, and 2 T's.. Although you cannot make any real words (no vowels!), how many different 7-letter arrangements can you make? ________

8) There are 10 boys on the varsity basketball team. The coach must divide them into 2 teams for a scrimmage. How many different ways can the boys be grouped? ________

The "Combination" is

__ __ | __ __ | __ __

Treasure Chest & the Combination/Permutation Lock



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SOLUTIONS

"Congratulations, you found the chest...
To get the treasure, here's my test..."

Solve the eight problems below. Then, add the answers together. The 6-digit sum will open the lock.

1) The number of ways to arrange 5 chairs in a row.

5! 5 x 4 x 3 x 2 x 1 = 120

120

2) Your school locker has 4 digits on the lock.
How many possibilities are there? (each digit can be 0-9)

Each position (digit) has 10 possibilities: 0-9
So, 10 x 10 x 10 x 10 = 10,000

(also, consider every number from 0000 to 9999..)

10,000

3) How many different license plates can be made using the following format: letter, letter, number, number, number? EX: BX 263

26 x 26 x 10 x 10 x 10 = 676,000

Five "slots": 1st: 26 choices
2nd: 26 choices (26 letters in the alphabet)
3rd: 10 choices
4th: 10 choices
5th: 10 choices (numbers 0-9)

676,000

4) At the pizza parlor, there are 3 available toppings on the menu: sausage, pepperoni, and mushroom. How many different pizzas can they offer?

8 different pizzas

0 toppings: 1 way (plain cheese)
1 topping: 3 ways (S, P, M only)
2 toppings: 3 combinations (S/P, S/M, P/M)
3 toppings: 1 combination of all 3

8

5) In your closet, you have 3 ties, 4 shirts, and 3 pairs of pants. How many different outfits could you make (consisting of 1 tie, 1 shirt, and 1 pair of pants)?

3 x 4 x 3 = 36 possibilities

36

6) At the Kentucky Derby, there are 12 horses racing. You must try to pick the 3 horses that will finish in 1st, 2nd, and 3rd. How many choices (permutations) are there?

$${}_{12}P_3 = \frac{12!}{9!} = \frac{12 \times 11 \times 10 \times 9!}{9!} = 1320$$

Consider each place a "slot".. 1st slot: 12 choices. 2nd slot: select from 11 remaining horses. 3rd slot: choose one from 10 remaining horses.

12 x 11 x 10 = 1320

1320

7) While playing scrabble, you draw X, M, N, 2 S's, and 2 T's..
Although you cannot make any real words (no vowels!), how many different 7-letter arrangements can you make?

There are seven "slots" -- # of permutations = 7!
But, there are redundancies! 2 S's and 2 T's..
(ex: $S_1S_2T_1$XMN T_2 is the same as $S_2S_1T_1$XMN T_2)

We must eliminate the "double counting":

$$\frac{7!}{2!\,2!} = 1260$$

2! eliminates the S redundancies
2! eliminates the T redundancies

1260

8) There are 10 boys on the varsity basketball team. The coach must divide them into 2 teams for a scrimmage. How many different ways can the boys be grouped?

Reasoning: When dividing the 10 boys into 2 teams, we're merely selecting 5 boys (because, the remaining 5 are all put on the other team). So, selecting 5 boys: 10 x 9 x 8 x 7 x 6.. (the other 5 are irrelevant).
Then, regarding the 5 boys we chose: it's a combination (because the order of selection doesn't matter!)
Ex: Steve, John, Al, Kevin, Jack is the same as Steve, Al, John, Jack, Kevin...
So, to eliminate the repetition, divide by all the extra combinations of the five boys: 5!

$${}_{10}C_5 = \frac{10!}{5!\,5!} = 252$$

252

The "Combination" is

6 8 | 8 9 | 9 6

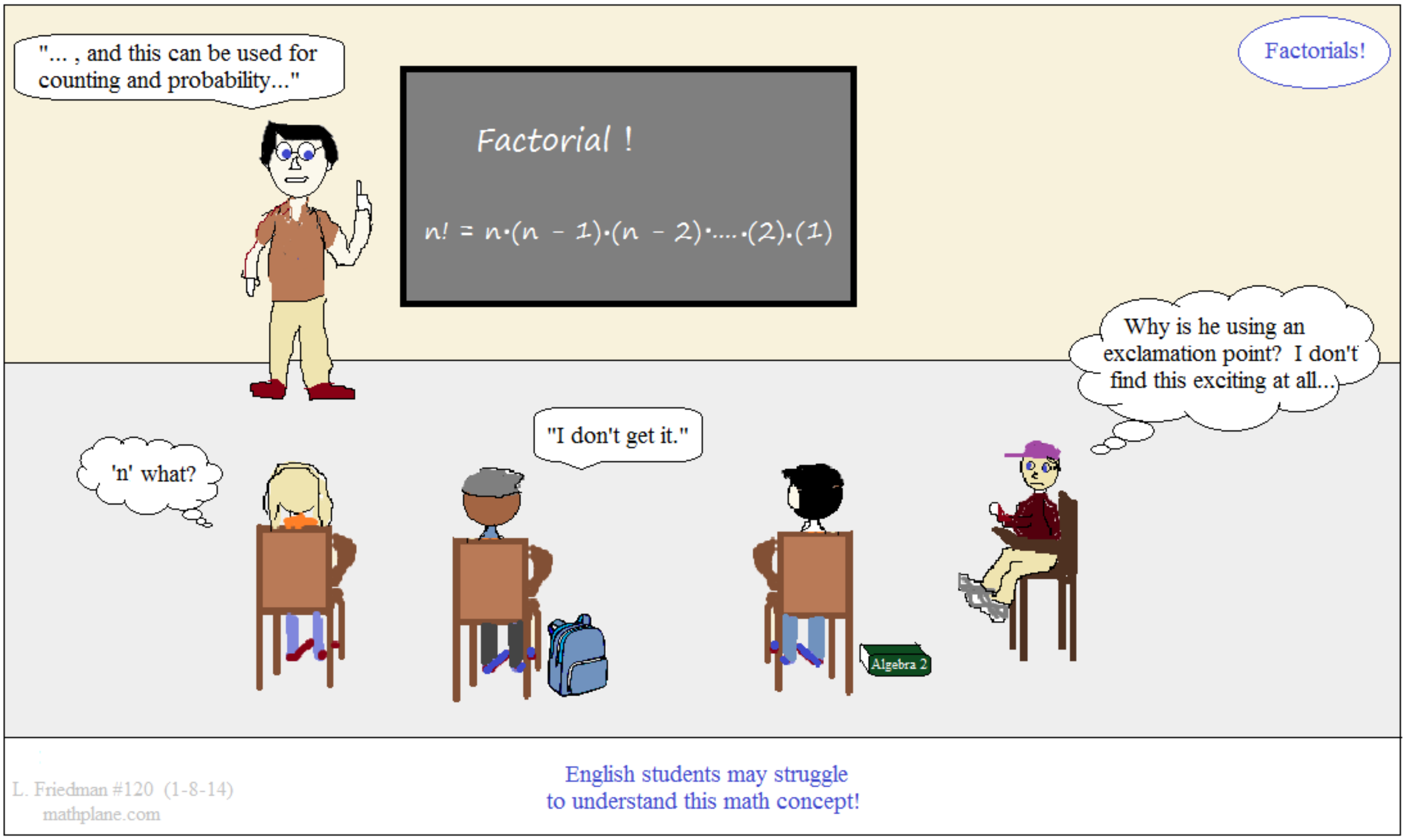
Add the 8 solutions:

120
10,000
676,000
8
36
1320
1260
+ 252
688996

Thanks for visiting. (Hope it helped!)

Find more resources at Mathplane.com

Good luck!



2 more counting questions:

1) If you listed all whole numbers between 1 and 100, how many 7's would appear in the list?

2) How many integers between 1 and 1000 contain at least one 7?

Solution on next page --→

7 7 7 7 7 7

Counting 7's

1) If you listed all whole numbers between 1 and 1000,
how many 7's would appear in the list?

Answer: 300

there will be 100 7's in the ones place
100 7's in the tens place
100 7's in the hundreds place

2) How many integers between 1 and 1000 contain
at least one 7?

Answer: 271

between 1 and 100: 19 (7, 17, 27, 37, 47, 57, 67, 70-79, 87, 97)
101 and 200: 19
201 and 300: 19
301 and 400: 19
401 and 500: 19
501 and 600: 19
601 and 699: 19
700 and 799: 100 (all of them have a seven)
800 and 900: 19
901 and 1000: 19