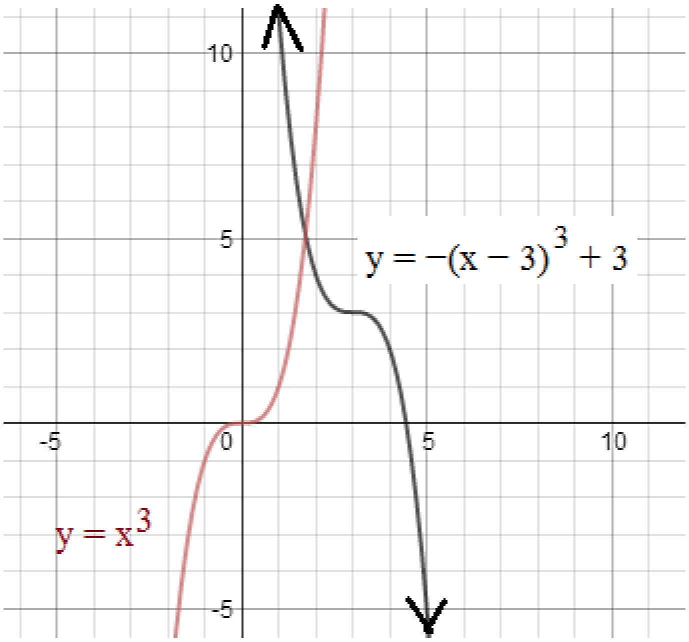
Transformations of Functions (Advanced)

Notes, Examples, and Practice Questions (with solutions)



Topics include shifts, stretches, reflections, graphing, odd/even, domain/range, and more.

Identifying Properties and Transformations of Functions

Example: If the point (2, 7) is on the EVEN function $f(x)$, identify another point.

$(-2, 7)$ If a function is even, then for every point, there is another point reflected over the y-axis (the function's line of symmetry is the y-axis)

Definition of 'even function' : $f(-x) = f(x)$

Since $f(2) = 7$ and $f(-2) = f(2)$

then $f(-2) = 7$

Suppose $h(x) = \frac{1}{2} f(3 - x) + 5$

Determine 2 points in the function $h(x)$

Approach 1: Finding the 2 points and solving

Since we know $f(2)$ and $f(-2)$, we'll select these points for $h(x)$.

In other words, where does $f(3 - x) = f(2)$?

$$3 - x = 2$$

$$x = 1$$

So, we'll use 1: $h(1) = \frac{1}{2} f(3 - 1) + 5$

$= \frac{1}{2} f(2) + 5$ and, we know $f(2) = 7$

$= \frac{1}{2} \cdot 7 + 5 = 17/2$

$(1, \frac{17}{2})$

And, where does $f(3 - x) = f(-2)$?

$$3 - x = -2$$

$$x = 5$$

Then, we'll use 5: $h(5) = \frac{1}{2} f(3 - 5) + 5$

$= \frac{1}{2} f(-2) + 5$ and, we know from above that $f(-2) = 7$

$= \frac{1}{2} \cdot 7 + 5 = 17/2$

$(5, \frac{17}{2})$

Approach 2: Recognizing translations/transformations

$$h(x) = \frac{1}{2} f(3 - x) + 5$$

If we rewrite the equation: $\frac{1}{2} f(-x + 3) + 5$

$$\frac{1}{2} f(-x + 3) + 5$$

$$\frac{1}{2} f\Big(-(x - 3)\Big) + 5$$

$$\frac{1}{2} f\Big(-(x - 3)\Big) + 5$$

a (-) b c d

		(2, 7)	(-2, 7)
(b)	horizontal expansion is 1 (none)	(2, 7)	(-2, 7)
(-)	horizontal reflection over y-axis	(-2, 7)	(2, 7)
(c)	horizontal shift of 3 to the right	(1, 7)	(5, 7)
(a)	vertical shrink (x 1/2)	(1, 7/2)	(5, 7/2)
(d)	vertical shift of up 5	$\boxed{(1, 17/2)}$	$\boxed{(5, 17/2)}$

Observation: Because of the vertical shift, the function $h(x)$ is not an 'even' function any more

Graphing and identifying transformations

Example: If the point (-3, 5) is on the ODD function $f(x)$, identify another point.

(3, -5) If a function is odd, then for every point, there is *another point reflected over the origin.*

Definition of 'odd function' : $f(-x) = -f(x)$

Since $f(-3) = 5$, then $f(-(-3)) = -f(-3)$

$f(3) = -5$

Suppose $g(x) = -4f(\frac{1}{5}x + 2) - 1$

Determine 2 points in function $g(x)$

Approach 1: Finding the 2 inputs and solving

Since we know the outputs for $f(3)$ and $f(-3)$, we'll choose those points for $g(x)$

where does $f(\frac{1}{5}x + 2) = f(3)$?

$\frac{1}{5}x + 2 = 3$

$x = 5$

So, we'll use 5: $g(5) = -4f(\frac{1}{5}(5) + 2) - 1$

$= -4f(3) - 1$

$= -4(-5) - 1 = 19$

(5, 19)

Note: if we used another number, such as 10, what happens?

$g(10) = -4f(\frac{1}{5}(10) + 2) - 1$

$g(10) = -4f(2 + 2) - 1$

$= -4f(4) - 1$

Since we don't know what $f(4)$ equals, we can't determine that point!

Then, where does $f(\frac{1}{5}x + 2) = f(-3)$?

$\frac{1}{5}x + 2 = -3$

$x = -25$

So, we'll use -25: $g(-25) = -4f(\frac{1}{5}(-25) + 2) - 1$

$= -4f(-3) - 1$

$= -4(5) - 1 = -21$

(-25, -21)

Approach 2: Using transformations and translations

$g(x) = -4f(\frac{1}{5}x + 2) - 1$

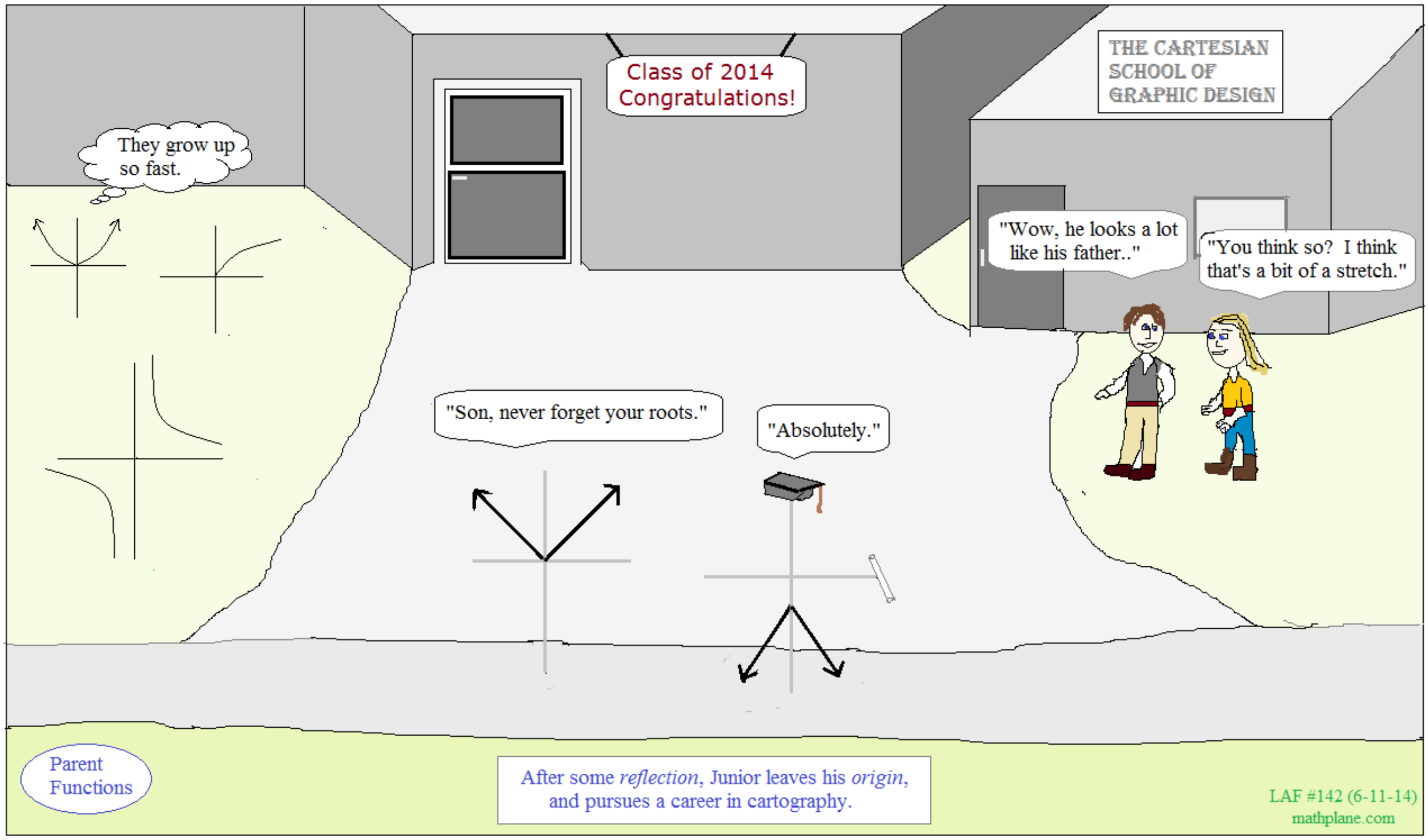
$-4f\left(\frac{1}{5}(x + 10)\right) - 1$ $\longrightarrow$ $-4f\frac{1}{5}(x + 10) - 1$

(-) a b c d

	Taking a point in $f(x)$,	(-3, 5)	(3, -5)
(-)	reflect over the x-axis	(-3, -5)	(3, 5)
a)	vertical stretch of 4	(-3, -20)	(3, 20)
b)	horizontal expansion by 5	(-15, -20)	(15, 20)
c)	horizontal shift of 10 to the left	(-25, -20)	(5, 20)
d)	vertical shift of 1 unit down	$\boxed{(-25, -21)}$	$\boxed{(5, 19)}$

Note: The order that each point is translated and transformed matters! Be careful..
(the shifts are last)

			(-3, 5)	(3, -5)
change in x	(b)	expansion (inside function)	(-15, 5)	(15, -5)
	(c)	horizontal shift (inside)	(-25, 5)	(5, -5)
change in y	(a)	vertical stretch (outside)	(-25, 20)	(5, -20)
	(-)	reflection (outside)	(-25, -20)	(5, 20)
	(d)	vertical shift (outside)	$\boxed{(-25, -21)}$	$\boxed{(5, 19)}$

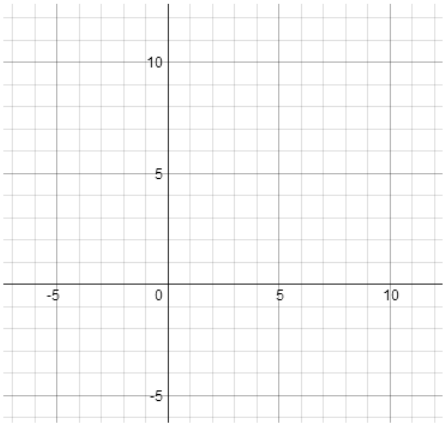


Practice Exercises-→

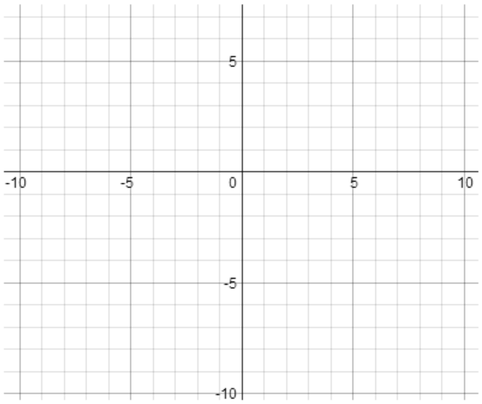
Graphing and Parent Functions Quiz

In the following, a) identify the parent function
b) describe any translations and transformations
c) sketch the functions
d) (optional) determine the domain and range

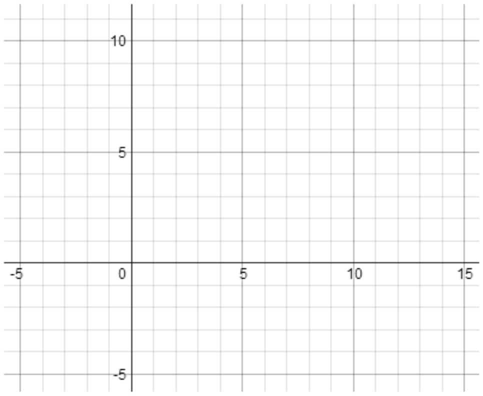
1) $y = |x - 2| + 4$



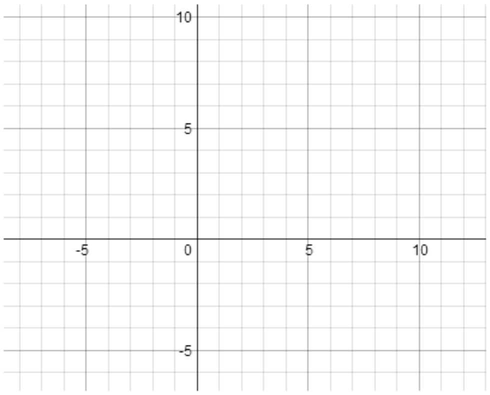
2) $f(x) = -\frac{1}{2}(x + 3)^2$



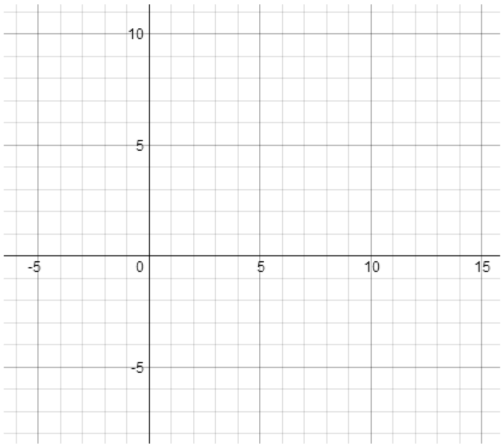
3) $y = 2\sqrt{x - 1} + 3$



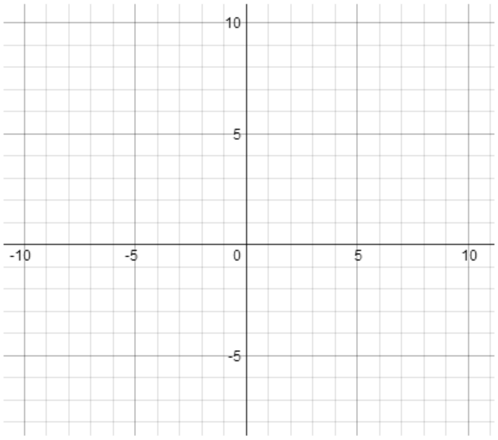
4) $y = -|3x - 3| + 5$



5) $y = -(x - 3)^3 + 3$



6) $g(x) = \frac{(x + 4)^2}{2}$



Transformations (Advanced)

1) What is the equation of a circle with diameter 20 and its center translated 8 units to the left and 11 units up from the origin?

2) Find the domain and range of the following:

a) $x^2 + y^2 - 4x = 21$

b) $5x^2 + 5y^2 - 20x + 25y = 100$

Transformations (Advanced)

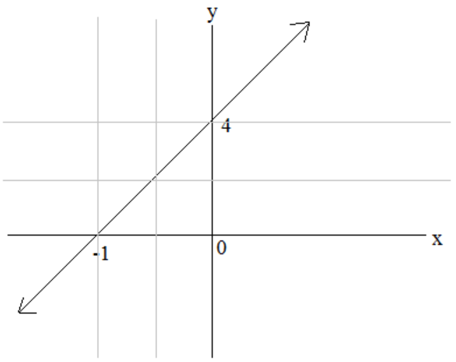
1) Which is the equation of the line?

a) $y = -x + 4$

b) $y = 8x + 4$

c) $y = x + 4$

d) $y = 4x + 4$



2) If you shifted $y = 3x + 6$ five units to the right, what would the new linear equation be?

a) $y = 3x + 11$
b) $y = 8x + 6$
c) $y = 3x + 1$
d) $y = 3x - 9$
e) $y = 8x + 11$

3) The function $f(x) = x$ is linear. Express each transformation in slope intercept form ($y = mx + b$).

a) $-f(x + 7) + 4 =$

b) $f(3x + 9) - 5 =$

c) $3\,f(4 - x) + 6 =$

4) Write the equation of a line that bisects quadrants II and IV.

5) Find the missing term:

x	y
-12	17
-2	-3
-1	-5
0	☐
6	-19

6) What is the equation of a line that is perpendicular to the y-axis and passes through the (-4, 5)?

Transformations (Advanced)

Part I: determine the following from the graph:

1) Interval(s) where the function is

a) increasing

b) decreasing

2) Relative

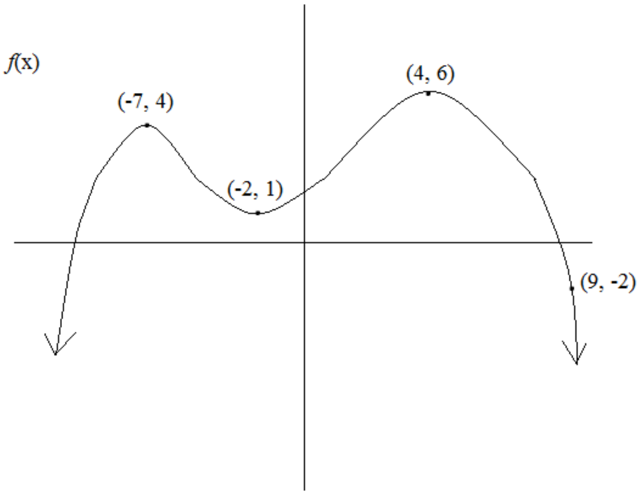
a) maximum(s)

b) minimum(s)

3) Absolute

a) maximum

b) minimum

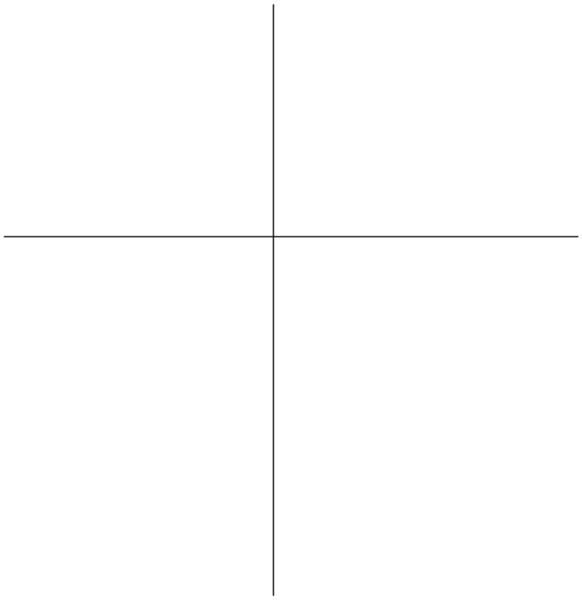


Part II: $g(x) = -2f(x + 3) - 4$

Find the intervals where $g(x)$ is increasing and decreasing

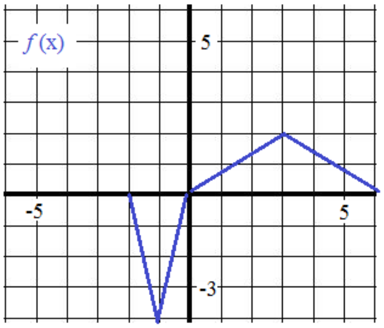
Determine the absolute and relative maximum(s) and minimum(s)

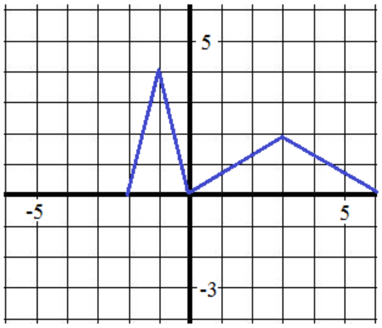
Sketch $g(x)$



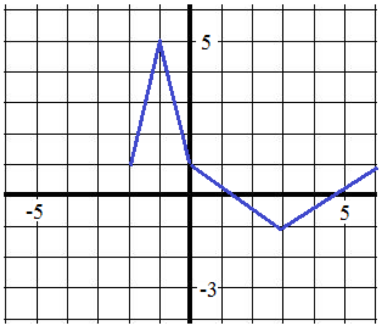
Transformations (Advanced)

Identify the transformed functions in the graphs:



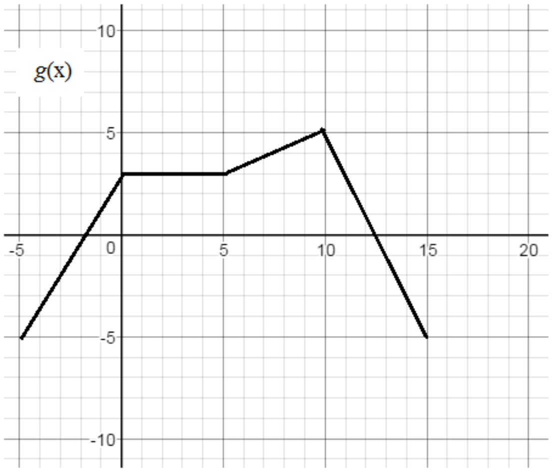


a) ____________________

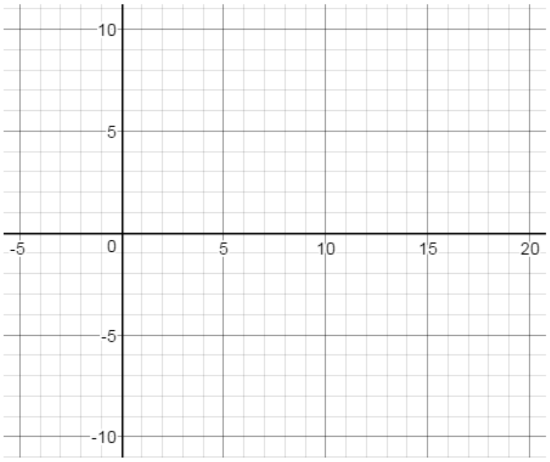


b) ____________________

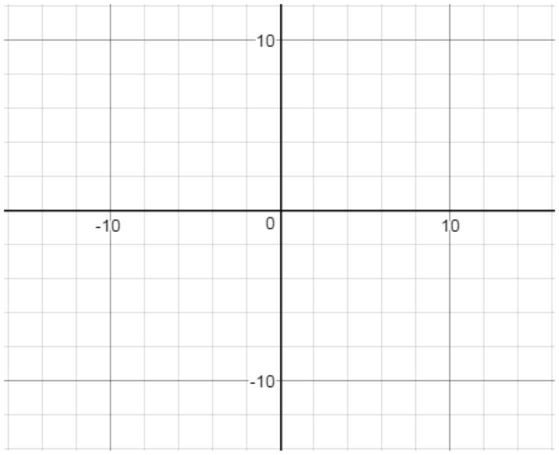
Sketch the following transformations:



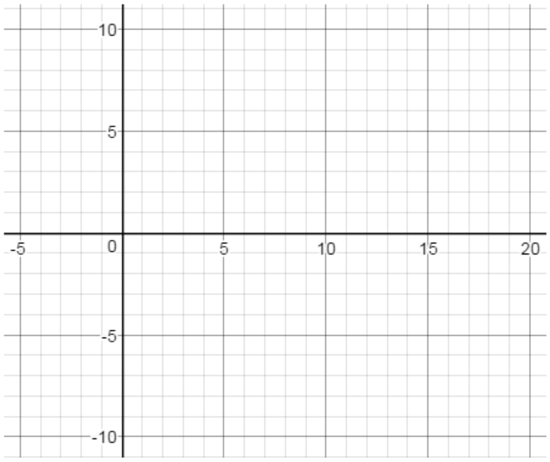
|g(x)|



g(|x|)



2g(2x) − 2



Transformations (Advanced)

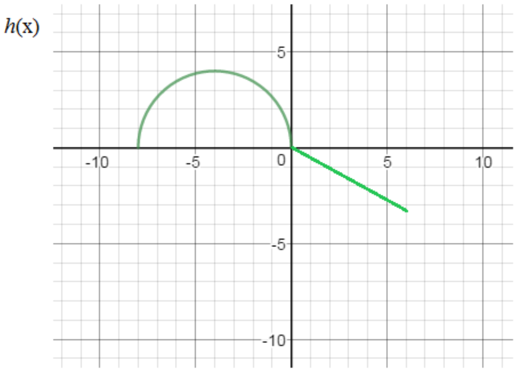
$f(x)$ has a range $[5, \infty)$ and domain $(-4, 11]$

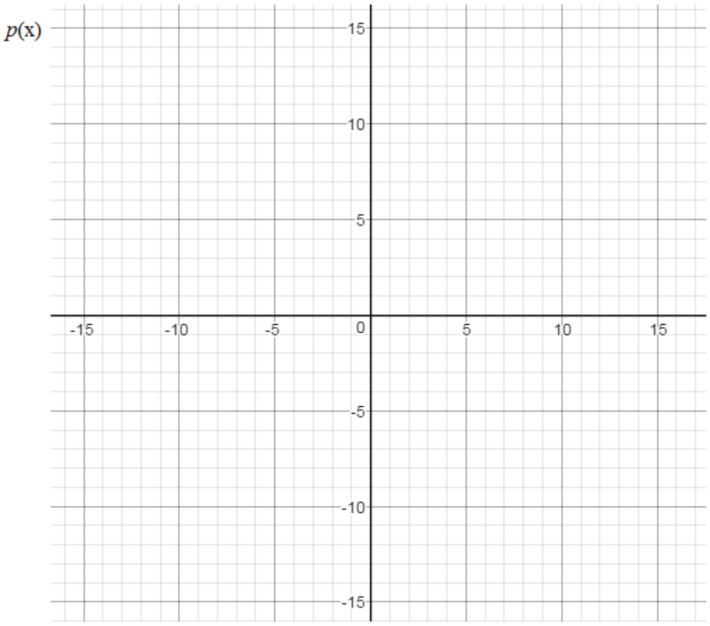
$g(x) = 3f(-2x + 4) - 5$

The range of $g(x)$?

The domain of $g(x)$?

$p(x) = -3h(-2x + 4) - 1$

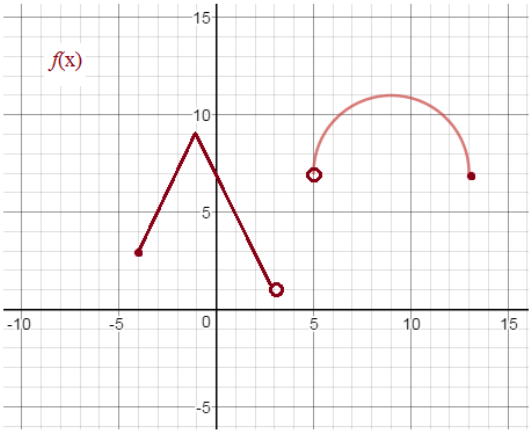




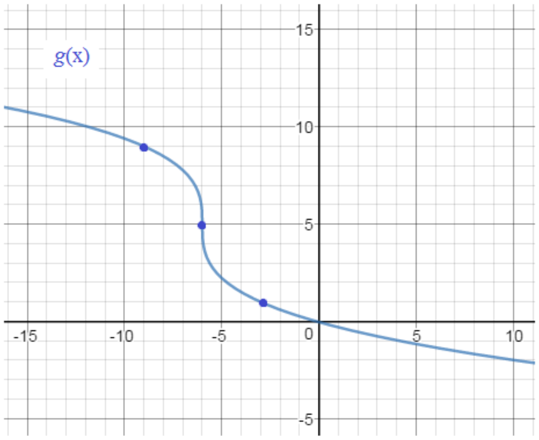
Write the equation of the functions in each graph.

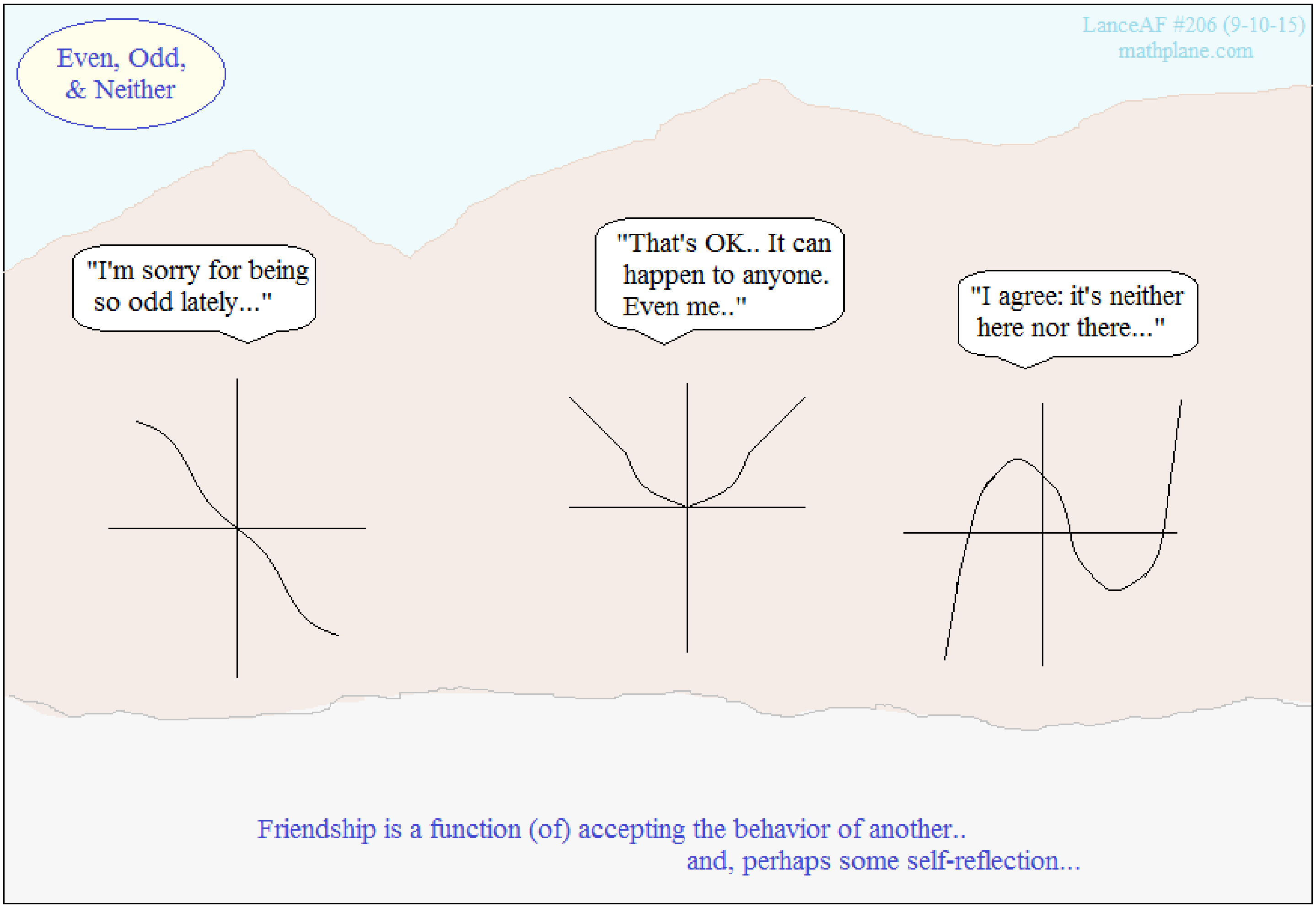
Transformations (Advanced)

$f(x)$ =



$g(x)$ =




Even, Odd, & Neither
LanceAF #206 (9-10-15)
mathplane.com
"I'm sorry for being so odd lately..."
"That's OK.. It can happen to anyone. Even me.."
"I agree: it's neither here nor there..."
Friendship is a function (of) accepting the behavior of another..
and, perhaps some self-reflection...

Solutions-→

Graphing and Parent Functions Quiz

SOLUTIONS

In the following, a) identify the parent function
b) describe any translations and transformations
c) sketch the functions
d) (optional) determine the domain and range

If $f(x)$ is the parent function,

$a\,f(b(x - c)) + d$ is the transformed function where

a is the "stretch"
b is the "compression"
c is the "horizontal shift"
d is the "vertical shift"

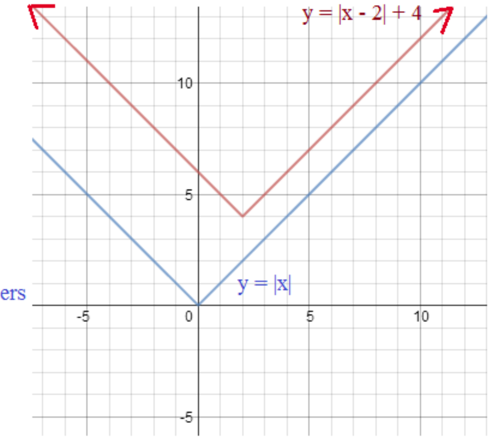
1) $y = |x - 2| + 4$

parent function:
$y = |x|$

horizontal shift (c):
2 units to the right

vertical shift (d):
4 units up

domain: all real numbers
range: $y \geq 4$



2) $f(x) = -\frac{1}{2}(x + 3)^2$

parent function:
$f(x) = x^2$

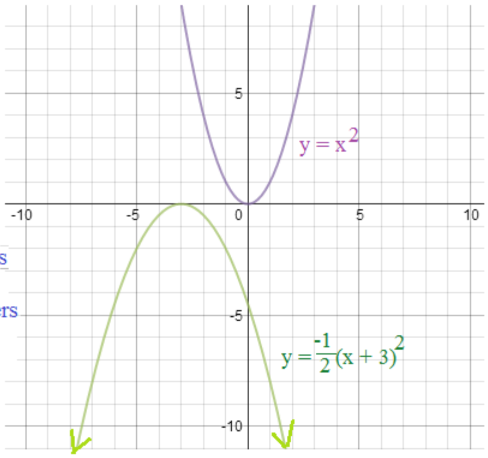
horizontal shift (c):
3 units to the left

amplitude (a): 1/2
(shrink by 2)

reflection over the x-axis

domain: all real numbers
$(-\infty, \infty)$

range: $f(x) \leq 0$
$(-\infty, 0]$



3) $y = 2\sqrt{x - 1} + 3$

parent function:
$y = \sqrt{x}$

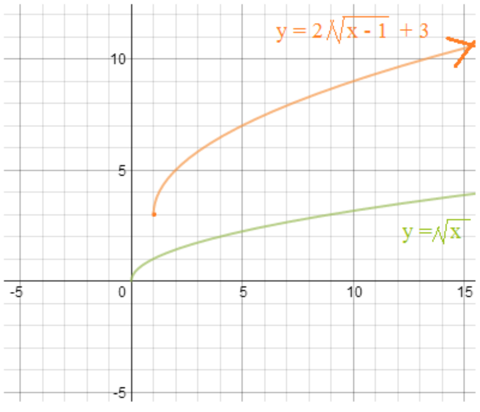
horizontal shift (c):
1 unit to the right

vertical shift (d):
3 units up

amplitude (a):
vertical stretch by 2

domain: $x \geq 1$
(term under radical must be non-negative)

range: $y \geq 3$



4) $y = -|3x - 3| + 5$

**first, rewrite the equation

$\boxed{y = -|3(x - 1)| + 5}$

horizontal shift (c):
1 unit to the right

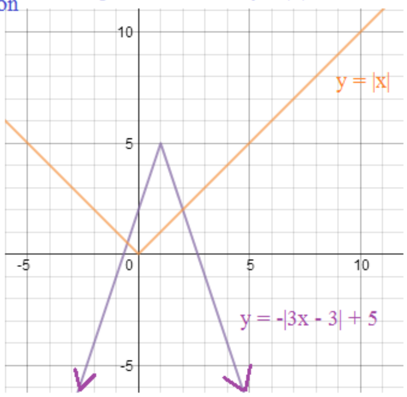
vertical shift (d):
5 units up

reflected over the x-axis

"compression" (b):
1/3 of the width

domain: all real numbers
range: $y \leq 5$

parent function: $y = |x|$



5) $y = -(x - 3)^3 + 3$

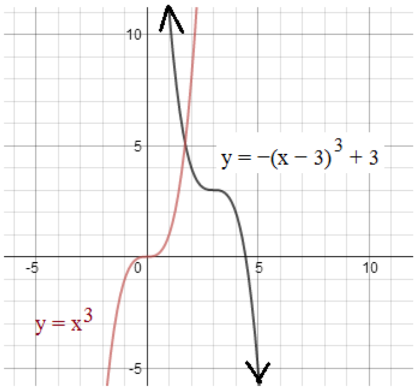
parent function:
$y = x^3$ (cubic)

horizontal shift (c):
3 units to the right

vertical shift (d):
up 3 units

reflected over the x-axis

domain: all real numbers
range: all real numbers



6) $g(x) = \frac{(x + 4)^2}{2}$

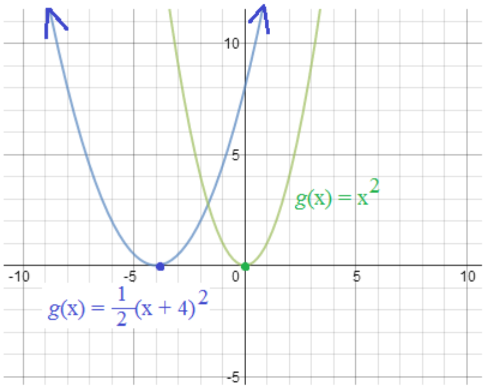
$\boxed{g(x) = \frac{1}{2}(x + 4)^2}$

parent function:
$y = x^2$

horizontal shift (c):
4 units to the left

amplitude (a):
1/2, so it shrinks

domain: all real numbers
range: $g(x) \geq 0$



Transformations (Advanced)

SOLUTIONS

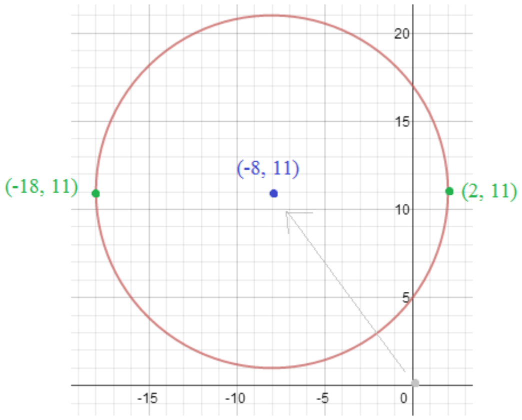
1) What is the equation of a circle with diameter 20 and its center translated 8 units to the left and 11 units up from the origin?

since diameter is 20, radius is 10...

origin shifted 8 units to the left: (-8
origin shifted 11 units up: (-8, 11)

$$(x-h)^2 + (y-k)^2 = r^2$$

$$\boxed{(x+8)^2 + (y-11)^2 = 100}$$



2) Find the domain and range of the following:

a) $x^2 + y^2 - 4x = 21$

The coefficients of x^2 and y^2 are the same, so we know it's a circle.

$$x^2 + y^2 - 4x = 21$$

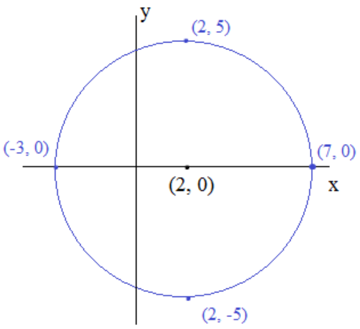
To find the center, we complete the square. Then, express the equation in standard form.

$$x^2 + y^2 - 4x = 21$$
$$x^2 - 4x + y^2 = 21$$
$$x^2 - 4x + 4 + y^2 = 21 + 4$$
$$(x-2)(x-2) + y^2 = 25$$
$$\boxed{(x-2)^2 + (y-0)^2 = 25}$$

$h = 2$
$k = 0$
The center is (2, 0)
The radius is 5

Domain: [-3, 7]

Range: [-5, 5]



To check your solutions, plug points into original equation:

(2, 5): $(2)^2 + (5)^2 - 4(2) = 21$

$4 + 25 - 8 = 21$ ✓

(-3, 0): $(-3)^2 + (0)^2 - 4(-3) = 21$

$9 + 0 + 12 = 21$ ✓

b) $5x^2 + 5y^2 - 20x + 25y = 100$

First, divide entire equation by 5

$$x^2 + y^2 - 4x + 5y = 20$$

Then, complete the square (to convert to standard form)

$$x^2 - 4x + 4 + y^2 + 5y + \frac{25}{4} = 20 + 4 + \frac{25}{4}$$

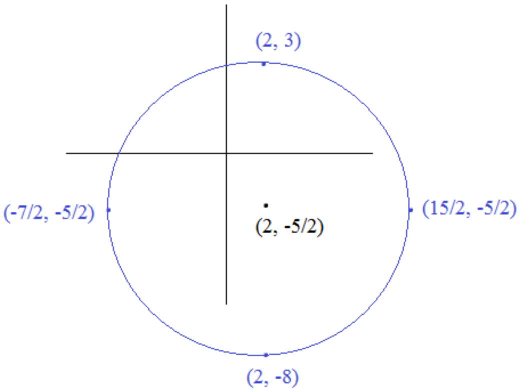
$$\boxed{(x-2)^2 + \left(y+\frac{5}{2}\right)^2 = \frac{121}{4}}$$

$h = 2 \quad k = -\frac{5}{2}$ center: (2, -5/2)

radius: 11/2

Domain: [-7/2, 15/2]

Range: [-8, 3]



SOLUTIONS

Transformations (Advanced)

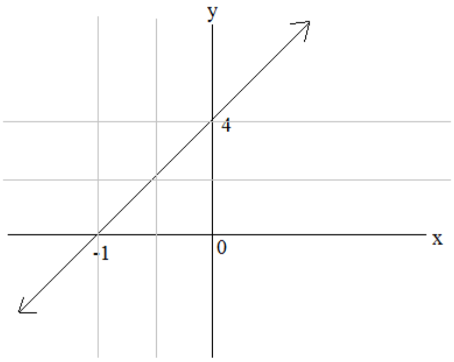
1) Which is the equation of the line?

a) $y = -x + 4$

b) $y = 8x + 4$

c) $y = x + 4$

$\boxed{\text{d) } y = 4x + 4}$



The y-intercept is (0, 4)

The slope is "rise"/"run"

$4/1 = 4$

$\boxed{y = 4x + 4}$

2) If you shifted $y = 3x + 6$ five units to the right, what would the new linear equation be?

a) $y = 3x + 11$

b) $y = 8x + 6$

c) $y = 3x + 1$

$\boxed{\text{d) } y = 3x - 9}$

e) $y = 8x + 11$

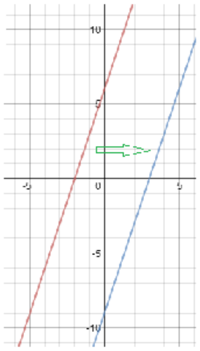
Since the entire line is shifted, the slope is the SAME...

slope is 3

If the line is shifted 5 units to the right, then presumably, the x-intercept would move 5 units to the right...

original x-intercept is (-2, 0)... Then, new x-intercept is (3, 0)

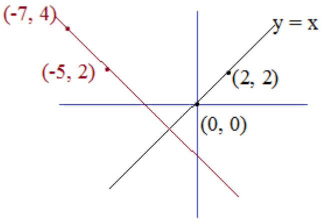
therefore, equation is $y - 0 = 3(x - 3)$ or $y = 3x - 9$



3) The function $f(x) = x$ is linear. Express each transformation in slope intercept form ($y = mx + b$).

a) $-f(x + 7) + 4 =$

$\boxed{y = -x - 3}$



b) $f(3x + 9) - 5 =$

(0, 0) ---> (-3, -5)

(3, 3) ---> (-2, -2)

Equation of a line going through (-3, -5) and (-2, -2)

$y + 2 = 3(x + 2)$

$\boxed{y = 3x + 4}$

c) $3\,f(4 - x) + 6 =$

(0, 0) ---> (4, 6)

(1, 1) ---> (3, 9)

Equation of a line going through (4, 6) and (3, 9)

$y - 6 = -3(x - 4)$

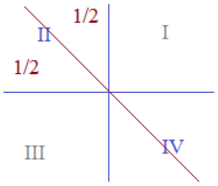
$\boxed{y = -3x + 18}$

$3\,f(-(x - 4)) + 6$

each point on the line is
reflected over the y-axis..
shifted to the right by 4
vertically stretched by 3 (slope)
and shifted up 6

4) Write the equation of a line that bisects quadrants II and IV.

$\boxed{\text{Answer: } y = -x}$



5) Find the missing term:

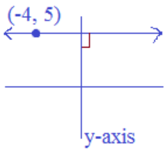
x	y
-12	17
-2	-3
-1	-5
0	☐
6	-19

$\boxed{\text{Answer: } -7}$

(slope/rate of change is -2)

6) What is the equation of a line that is perpendicular to the y-axis and passes through the (-4, 5)?

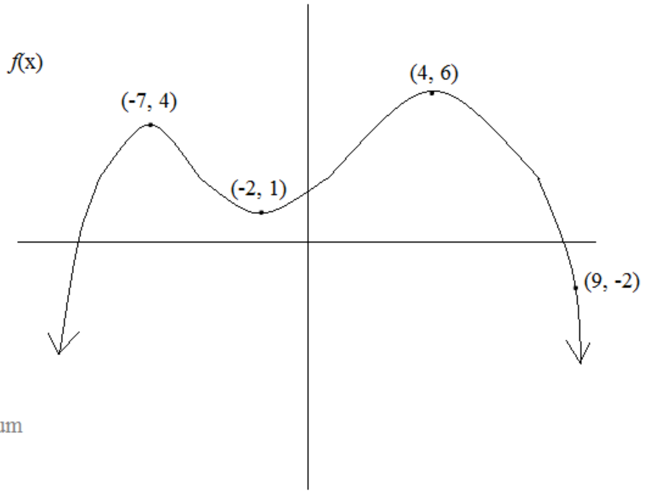
$\boxed{y = 5}$



Part I: determine the following from the graph:

SOLUTIONS

Transformations (Advanced)



1) Interval(s) where the function is

a) increasing $(-\infty, -7) \cup (-2, 4)$

b) decreasing $(-7, -2) \cup (4, \infty)$

Note: Increasing and decreasing intervals
do NOT include the maxima and minima

2) Relative

a) maximum(s) $(-7, 4)$ and $(4, 6)$

b) minimum(s) $(-2, 1)$

3) Absolute

a) maximum (4, 6)

Note: The highest 'relative' maximum (4, 6)
is also the 'absolute' (or global) maximum

b) minimum None (function goes to negative infinity)

Part II: $g(x) = -2f(x + 3) - 4$

Find the intervals where g(x) is increasing and decreasing

Increasing intervals: $(-10, -5) \cup (1, \infty)$

Decreasing intervals: $(-\infty, -10) \cup (-5, 1)$

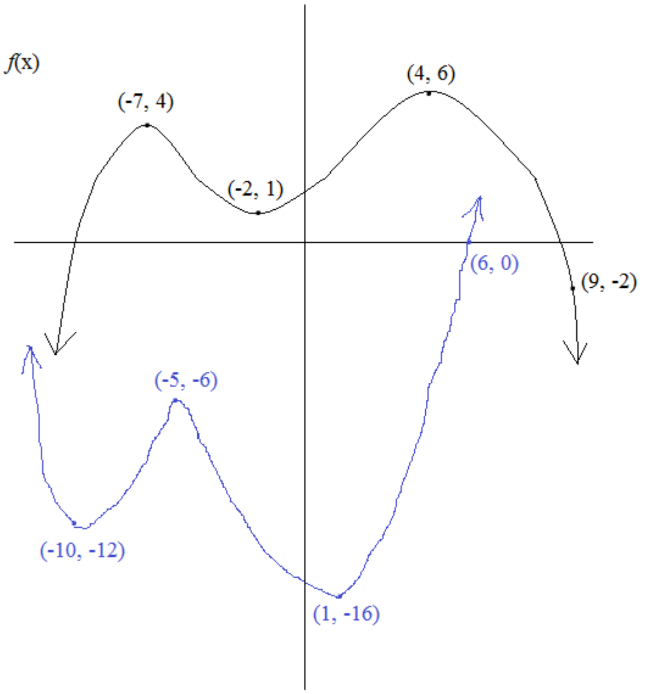
Determine the absolute and relative maximum(s) and minimum(s)

Relative max: (-5, -6)
Relative min: (-10, -12) and (1, -16)

Absolute max: none
Absolute min: (1, -16)

Note: absolute min of g(x) corresponds
to absolute max of f(x)

Sketch g(x)



Let's transform each of the 4 labeled points in f(x)

(-7, 4): shift left 3 units (-10, 4)
reflect over x-axis (-10, -4)
vertical stretch by factor of 2 (-10, -8)
vertical shift down 4 units (-10, -12)

(-2, 1): left 3 units (-5, 1)
reflect over x-axis, vertical stretch by 2 (-5, -2)
vertical shift (-5, -6)

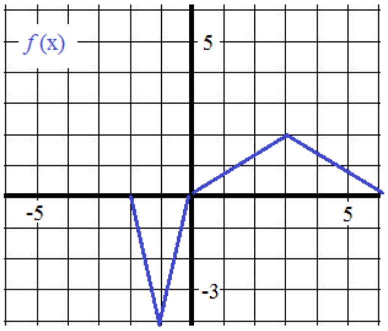
(9, -2) ---> (6, 0)

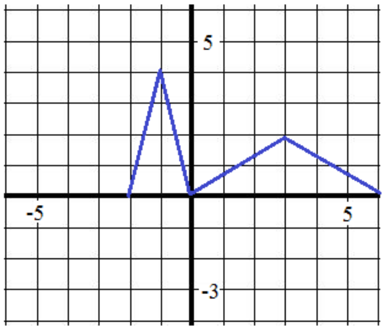
(4, 6) ---> (1, -16)

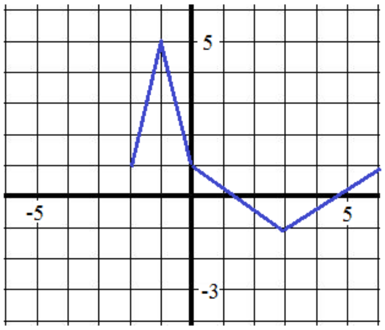
Transformations (Advanced)

SOLUTIONS

Identify the transformed functions in the graphs:







a) $|f(x)|$

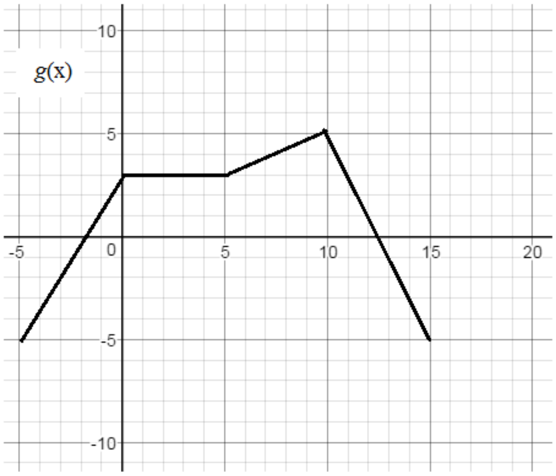
(The negative outputs only are reflected over the x-axis)

b) $-f(x) + 1$

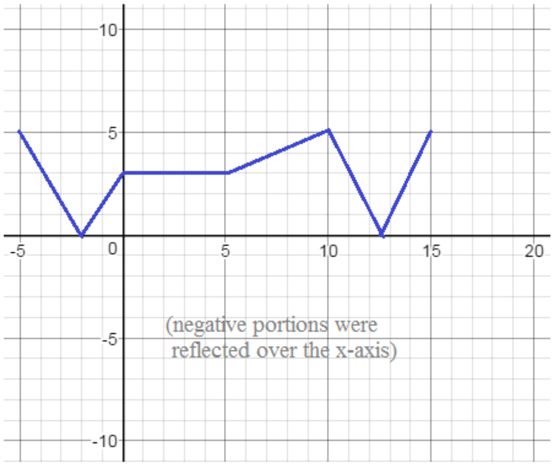
(reflected over x-axis; shifted up 1)

Sketch the following transformations:

g(x)

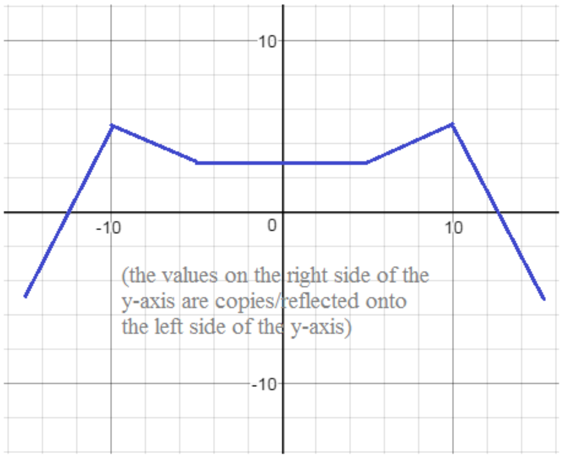


|g(x)|



(negative portions were reflected over the x-axis)

g(|x|)



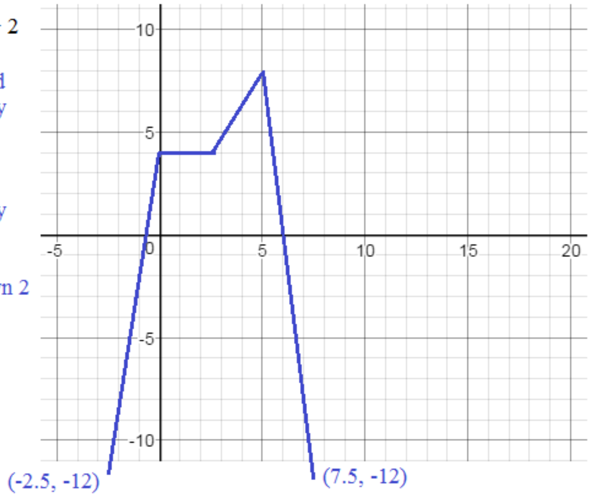
(the values on the right side of the y-axis are copies/reflected onto the left side of the y-axis)

$2g(2x) - 2$

compressed (horizontally by 1/2)

vertically stretched by factor of 2

shifted down 2



Transformations (Advanced)

SOLUTIONS

$f(x)$ has a range $[5, \infty)$ and domain $(-4, 11]$

$g(x) = 3f(-2x + 4) - 5$

The range of $g(x)$? $\boxed{[10, \infty)}$

The domain of $g(x)$? $\boxed{[-7/2, 4)}$

where does (-2x + 4) = 11?

at x = -7/2

where does (-2x + 4) = -4?

at x = 4

Find the transformations of the boundaries...

The 'outers' affect the range:

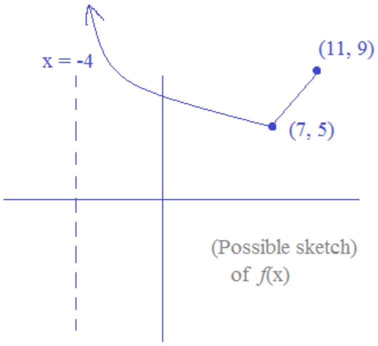
vertical stretch: 3
vertical shift: down 5

The 'inners' of the function affect the domain:

$f(-2(x - 2))$

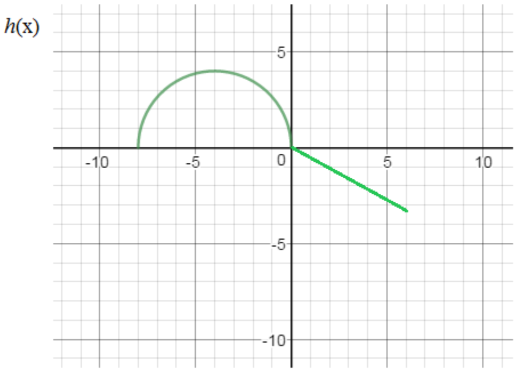
horizontal reflection: over the y-axis
horizontal compression: cuts domain in 1/2
horizontal shift: right 2

(Note: reflection, compression, shift
must be in that order!!)



(Possible sketch)
of $f(x)$

$p(x) = -3h(-2x + 4) - 1$



outer affects the output (y)

(-8, 0) 0 ---> 0 x -3 = 0

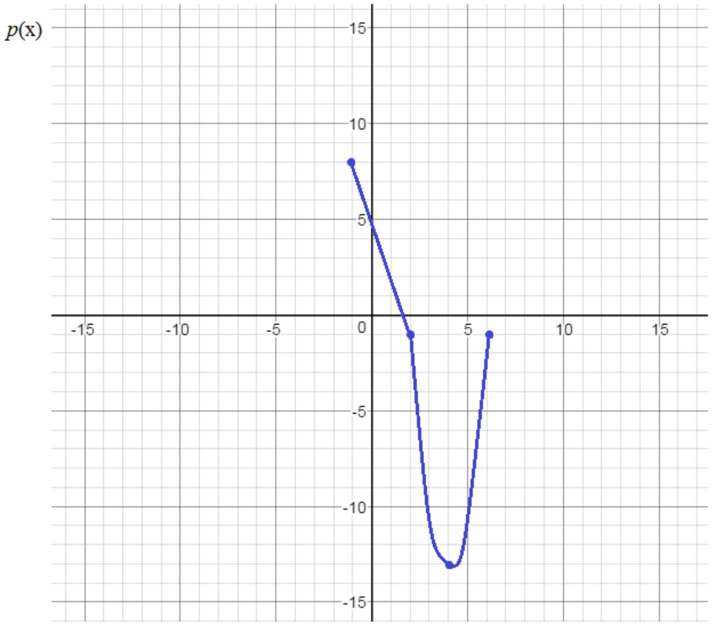
then, 0 − 1 = -1

inner affects the input (x)

w where does -2x + 4 = -8 ?

at x = 6

therefore, (-8, 0) transforms to (6, -1)



$-3h((-2)(x - 2)) - 1$

(0, 0) : reflect over y-axis
expand by factor of 1/2
shift right 2 units

reflect over x-axis
stretch by factor of 3
shift down 1

(2, -1)

(6, -3): reflect over y-axis
expand by factor of 1/2
shift right 2 units

(-1)

reflect over x-axis
stretch by factor of 3
shift down 1

(-1, 8)

Write the equation of the functions in each graph.

SOLUTIONS

parent function of the left part: $y = |x|$

shifted to the left 1 unit: $y = |x + 1|$

reflected down (over the x-axis): $y = -|x + 1|$

vertical stretch by factor of 2: $y = -2|x + 1|$
(slope of lines are 2 and -2)

vertical shift up 9 units: $y = -2|x + 1| + 9$

The interval is [-4, 3)

parent function of the right part is a semicircle (upper)

equation of the circle: $(x - 9)^2 + (y - 7)^2 = 16$

center: (9, 7) radius: 4

upper half (x-symmetry), so we'll solve for y:

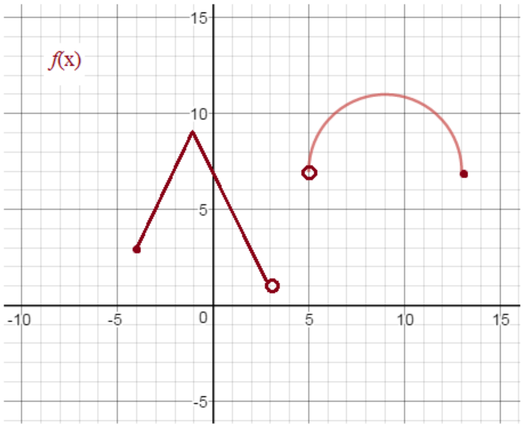
$$(y - 7)^2 = 16 - (x - 9)^2$$

$$y - 7 = \pm\sqrt{16 - (x - 9)^2}$$

Since only the upper half of the
circle is used, we ignore the negative...

$$y = +\sqrt{16 - (x - 9)^2} + 7$$

The interval is (5, 13]



$$f(x) = \begin{cases} -2|x + 1| + 9 & \text{if } -4 \le x < 3 \\ \sqrt{16 - (x - 9)^2} + 7 & \text{if } 5 < x \le 13 \end{cases}$$

parent function: $y = \sqrt[3]{x}$

vertical shift up 5 units: $\sqrt[3]{x} + 5$

horizontal shift 6 units to the left: $\sqrt[3]{x + 6} + 5$

reflected vertically (over the x-axis): $-\sqrt[3]{x + 6} + 5$

ordinarily, (0, 0) is between (-1, -1) and (1, 1)

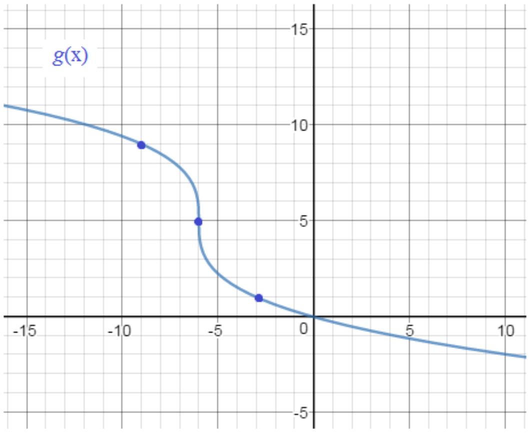
In this graph, (-6, 5) is between (-9, 9) and (-3, 1)

the y-value is up 4 and down 4 ---> vertical stretch by factor of 4

$$-4\sqrt[3]{x + 6} + 5$$

the x-value is left 3 and right 3 ---> horizontal expansion by factor of 1/3

$$-4\sqrt[3]{\frac{1}{3}(x + 6)} + 5$$

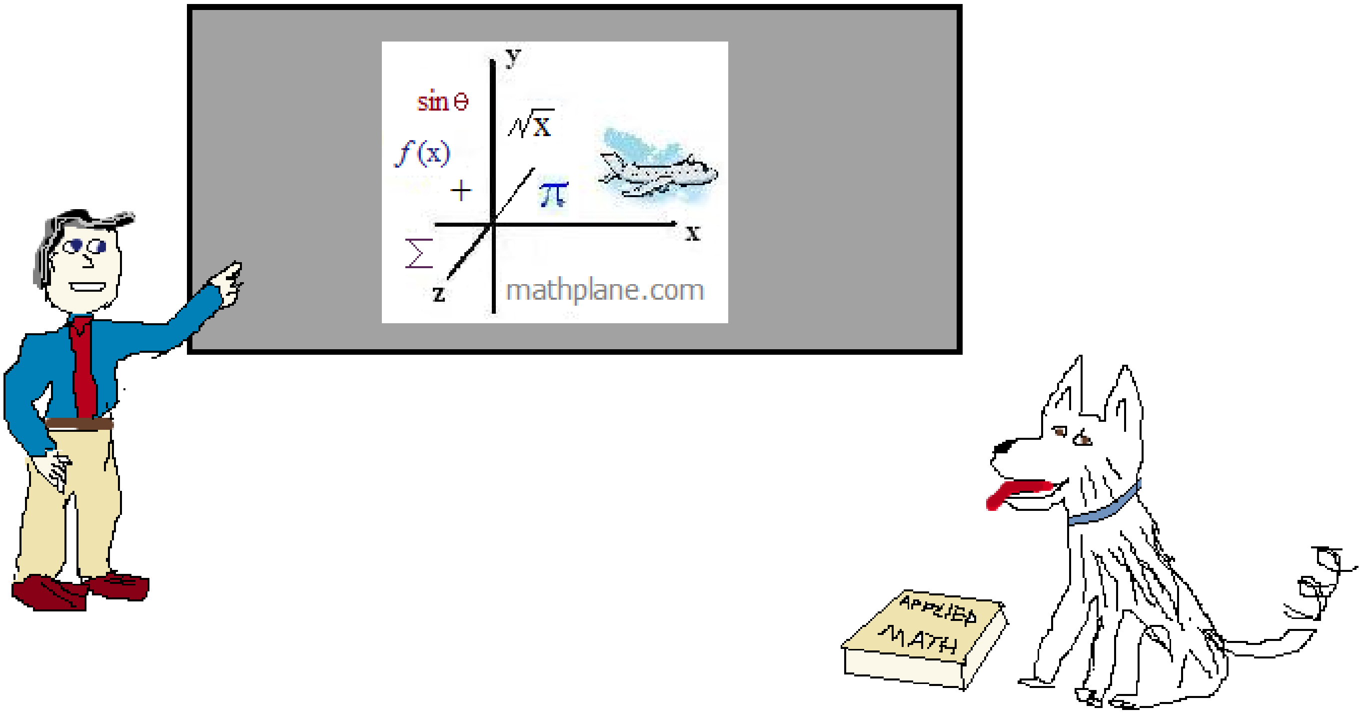


$$y = -4\sqrt[3]{(\frac{1}{3}x + 2)} + 5$$

Thanks for visiting. (Hope it helped!)

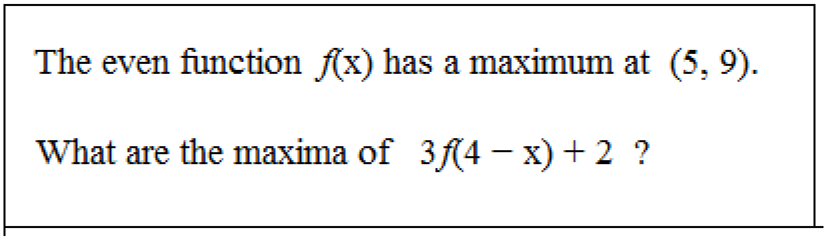
If you have questions, suggestions, or requests, let us know.

Cheers



Also, at Facebook, Google+, TES, TeachersPayTeachers, and Pinterest

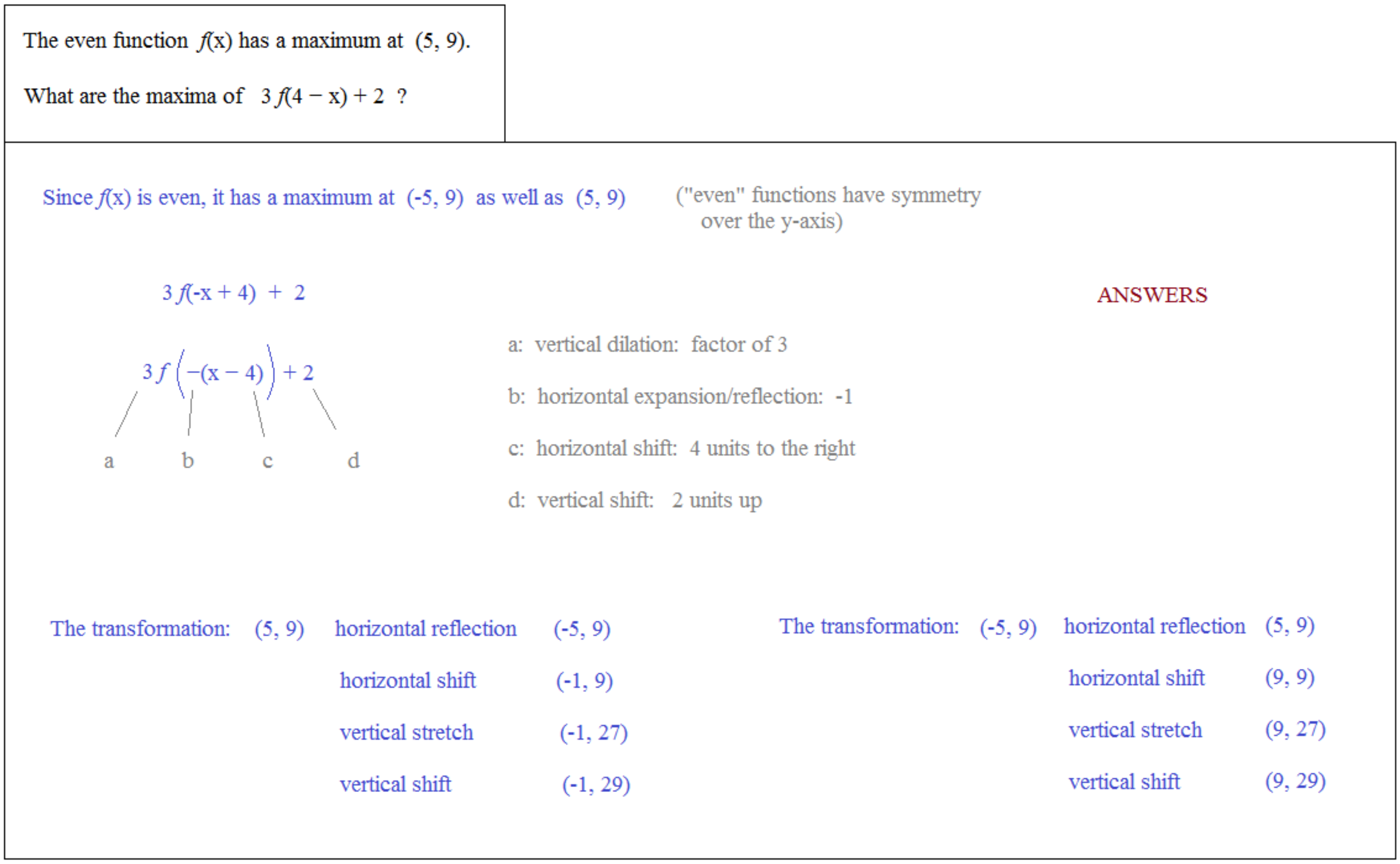
One More Question:



The even function $f(x)$ has a maximum at (5, 9).

What are the maxima of $3f(4 - x) + 2$?

ANSWER:



The even function $f(x)$ has a maximum at (5, 9).

What are the maxima of $3f(4 - x) + 2$?

Since $f(x)$ is even, it has a maximum at (-5, 9) as well as (5, 9) ("even" functions have symmetry over the y-axis)

$3f(-x + 4) + 2$

$3f\left(-(x - 4)\right) + 2$

a b c d

ANSWERS

a: vertical dilation: factor of 3

b: horizontal expansion/reflection: -1

c: horizontal shift: 4 units to the right

d: vertical shift: 2 units up

The transformation:	(5, 9)	horizontal reflection	(-5, 9)
		horizontal shift	(-1, 9)
		vertical stretch	(-1, 27)
		vertical shift	(-1, 29)

The transformation:	(-5, 9)	horizontal reflection	(5, 9)
		horizontal shift	(9, 9)
		vertical stretch	(9, 27)
		vertical shift	(9, 29)