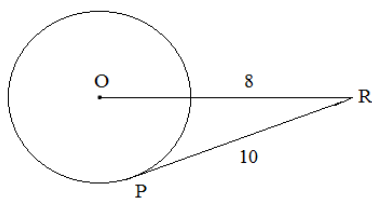# Circles Review V: Advanced Topics

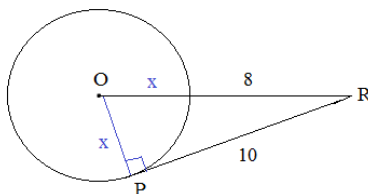## Notes, Examples, and Practice Test (with Solutions)

Topics include Angle-Arc Theorems, Power Theorems, Trigonometry, Special Right Triangles, and more.

*Example:*  What is the radius of circle O ?



Segment $\overline{RP}$ is tangent
to circle O at point P

**Method 1:  Tangent-radius Theorem**
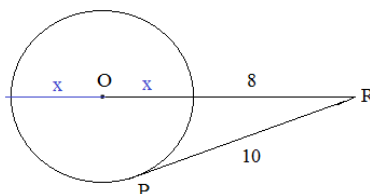**Pythagorean Theorem**



$$x^2 + 10^2 = (x + 8)^2$$

$$x^2 + 100 = x^2 + 16x + 64$$

$$36 = 16x$$

$$x = 9/4$$

**Method 2:  Tangent-Secant (power theorem)**
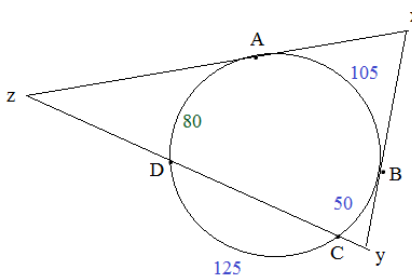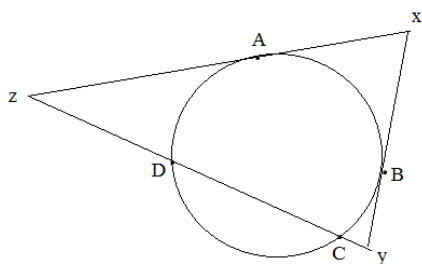


$$10^2 = 8 \cdot (8 + 2x)$$

$$100 = 64 + 16x$$

$$36 = 16x$$

$$x = 9/4$$

---

*Example:*   $\overset{\frown}{BC} = 50^0$

$\overset{\frown}{AB} = 105^\circ$

$\overset{\frown}{CD} = 125^{\circ}$

Find  x, y, z





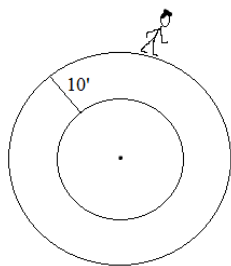Since AB = 105, then x = 75
(tangent-tangent, external angle)
supplementary

total arcs:  360
$105 + 50 + 125 + \overset{\frown}{AD} = 360$
$\overset{\frown}{AD} = 80$

$1/2(\overset{\frown}{ABC} - \overset{\frown}{AD}) = z$
(tangent-secant, external angle)
$1/2(155 - 80) = z$
$z = 37.5$

$x + y + z = 180$
$75 + y + 37.5 = 180$
$y = 67.5$

---

*Example:*



A hiker walked around a circle..  Then, he walked 10 feet
further from the center and walked around another circle.

If the total distance he walked was 2000 feet, what is
the radius of the inner circle?

inner circle        outer circle
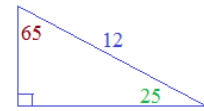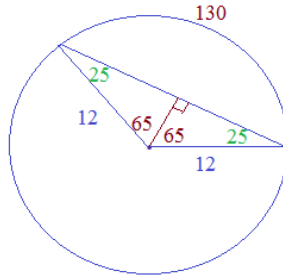circumference       circumference

$$2\pi r + 2\pi (r + 10) + 10 = 2000$$

don't forget the 10' the
hiker walked to get from
the inner to the outer circle!

$$4\pi r + 20\pi = 1990$$

$$4\pi r = 1927.1681$$

153.36 feet..

*Example:* In a circle with radius of 12 inches, what is the length of a chord that cuts off an arc measuring 130° ?

130

25

12

65 / 65   25

12

65   12

25

$$\cos(25°) = \frac{x}{12} \qquad x = 10.875$$

Then, $2x = 21.75$ is the length of the chord!

---
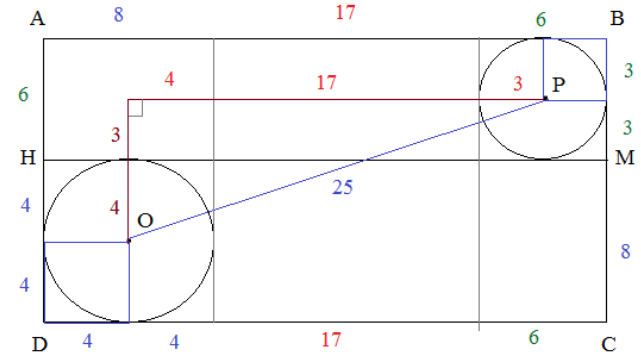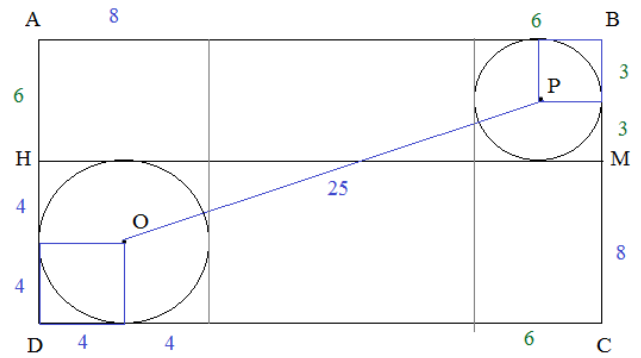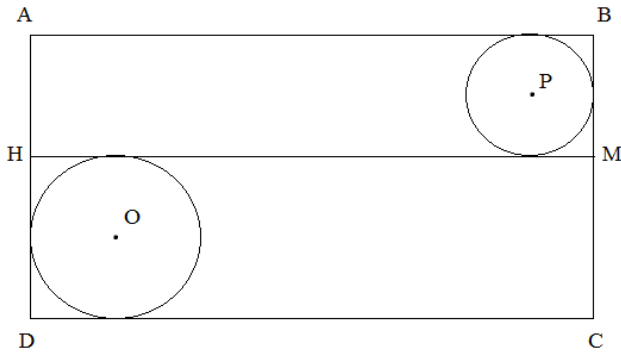
*Example:* Given: Rectangle ABCD

$\overline{HM} \parallel \overline{CD}$

Circle O has radius 4

Circle P has radius 3
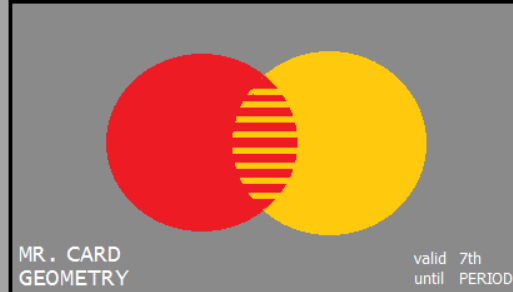
The distance between the centers of circles is 25
(i.e. the distance between O and P)

Find: the perimeter of rectangle ABCD

A                                    B

·P

H                                    M

O
·

D                                    C

---

A      8                    6      B

6                              P      3

H                              3      M

4        O

4                                    8

D    4        4              6      C

---

A      8          17        6      B

6        4        17      3  P      3

            3

H                              3      M

4      4    O          25

4                                    8

D    4      4      17        6      C

Perimeter of the rectangle = 31 + 14 + 31 + 14 = 90 units

"Bonus question:
what is the area of the
striped intersection of
the circles?"

Extra
Credit
Card

MR. CARD
GEOMETRY

valid  7th
until  PERIOD

Incomplete proofs:  minus 5 points...

Missed power theorems:  minus 10 points...

Incorrect circles answers:  minus 35 points...

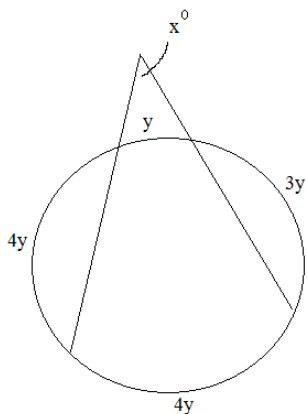Getting out of this geometry test with
a passing grade:  PRICELESS!

"There are some math grades you can't buy.
But, for everything else, there is extra credit from Mister Card."
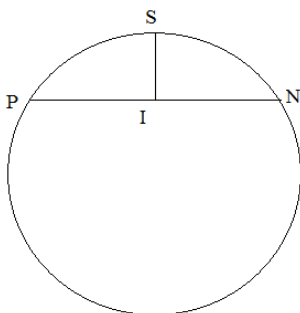
Practice Quiz-→

1) What is x?

x°

y

3y

4y

4y

2) $\overline{PN} = 12$

$\overline{SI} = 4$

$\overline{PN} \perp \overline{SI}$
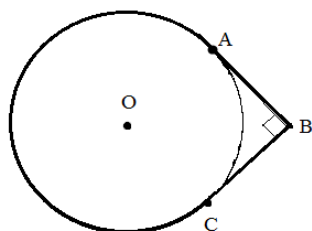
I is the midpoint of $\overline{PN}$

What is the circumference of the circle?

S

P    I    N

3) $\overline{AB}$ and $\overline{BC}$ are tangent segments
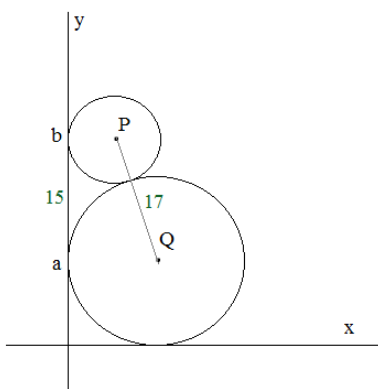
$\angle ABC$ is a right angle

If the length of $\overline{AB}$ is 6 units,
what is perimeter of the figure
(the angle and major arc)?

A

O

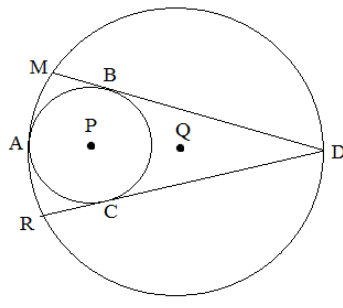B

C

4) a and b are points of tangency
of circles P and Q..

If the distance between a and b is 15,
and the distance between centers
P and Q is 17, then
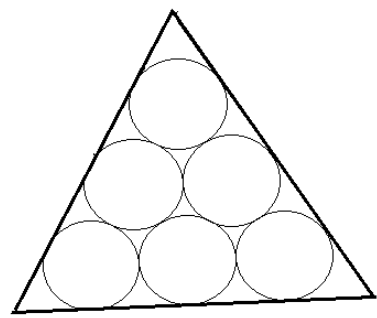
what are the coordinates of points P and Q?

y

b    P

15    17

a    Q

x

5) Circle P and Q are tangent at point A

$\overline{DM}$ and $\overline{DR}$ are external tangent lines
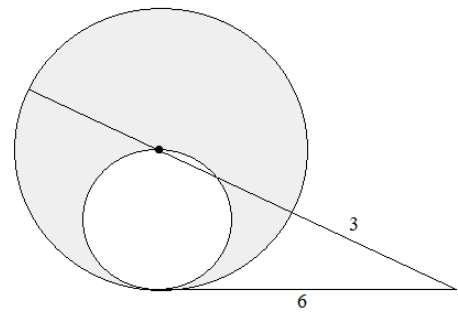
Arc $\overparen{BAC}$ is 224°

Can you determine the measure of arc $\overparen{MR}$?

6) Each of these 6 inscribed/tangent circles
are congruent with radius 8.

What is the perimeter of the triangle?

7) Can you find the shaded area?

8) At a pizzeria, a waiter and customer disagreed about a pizza order. .
Unfortunately, the customer had eaten most of the pizza.
However, the math chef came out and measured the remaining
portion and determined the size (diameter) of the original pizza.
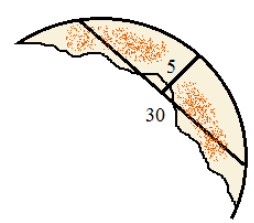
What was the diameter of the original, full pizza?

9) Two tangent segments have length 10 and form a 30 degree angle
   where they meet at external point H.
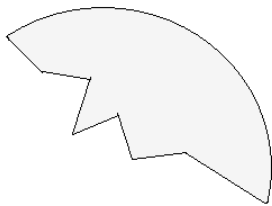
   How far is H from the circle?

10
30°
H
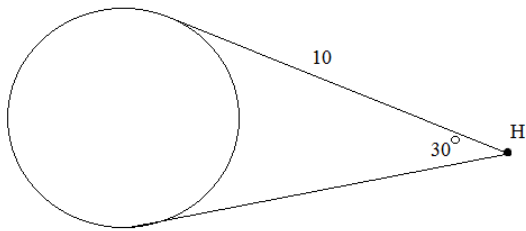
10) A pulley system includes two circles with radii 3 and 15 inches.
    If the centers of each circle are 12 inches apart,
    how much belt length is required to wrap around the circles?
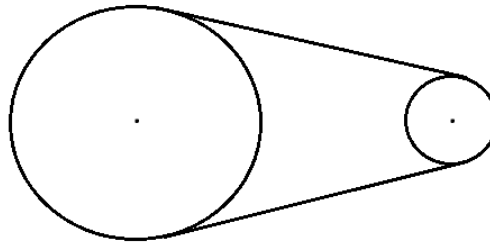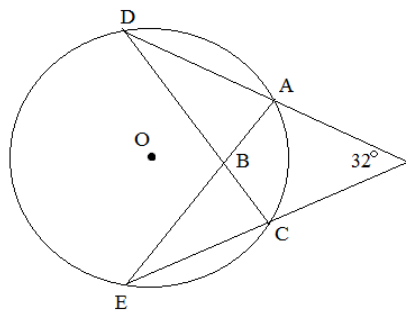
11) The figure is a piece of a broken plate.

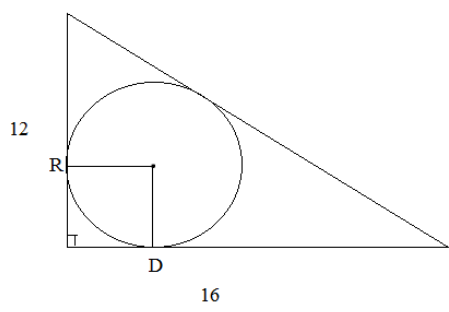    How can you find the circumference of the original plate?

12) $\angle ABC = 100°$

    $\overline{AD} = \overline{EC}$

    What are the individual arc
    measures in circle O?

D
A
O
B
32°
C
E

13) Find the arc length of $\overset{\frown}{RD}$

12
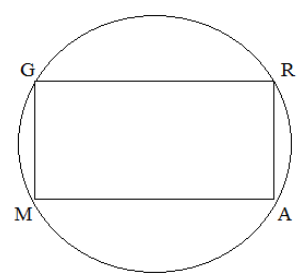
R

D

16

14) Parallelogram  GRAM

$\overset{\frown}{GR}$ = 120  degrees

$\overline{GM}$ = 12

Find the length of  $\overset{\frown}{GRM}$
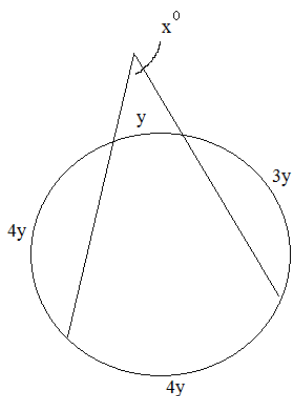
G          R

M          A

15)  The diameter of a circle has endpoints  (3, 1)  and  (11, 7).

If one of the chords has endpoints  (2, 4)  and  (7, −1), then
what is the distance from the chord to the center of the circle?
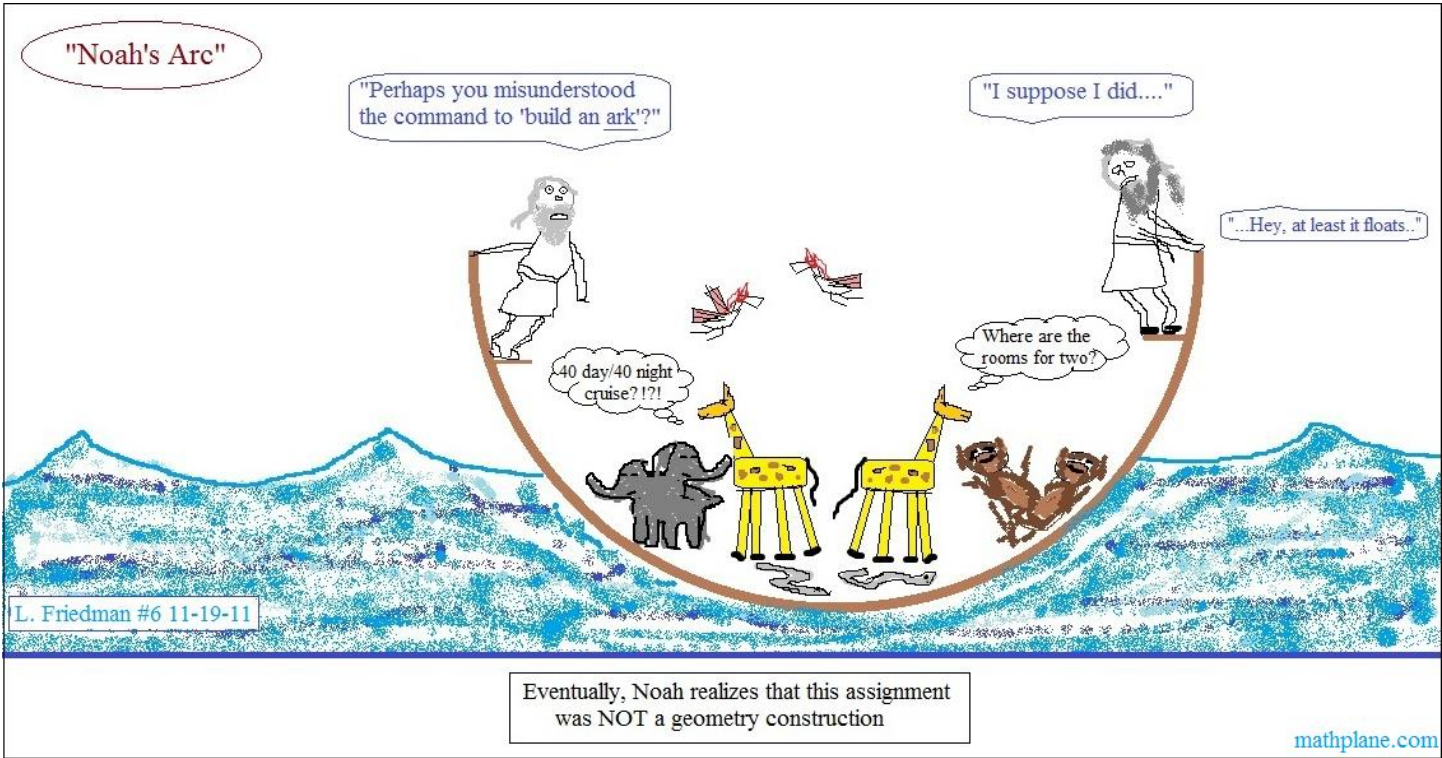
SOLUTIONS-→

1) What is x?

x°

y

3y

4y

4y

The sum of the arc lengths is 360 degrees...

$y + 3y + 4y + 4y = 360$

$12y = 360$

$y = 30$

**SOLUTIONS**

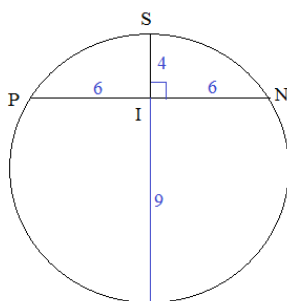$x = \frac{1}{2}(4y - y)$   (secant-secant power theorem)

$x = \frac{1}{2}(120 - 30)$

$\boxed{x = 45}$

---

2) $\overline{PN} = 12$

$\overline{SI} = 4$

$\overline{PN} \perp \overline{SI}$

I is the midpoint of $\overline{PN}$

What is the circumference of the circle?

S

4

P   6   6   N

I

9

Since SI and PN are perpendicular, and I is the midpoint, SI is a perpendicular bisector and goes through the chord and the center of the circle....

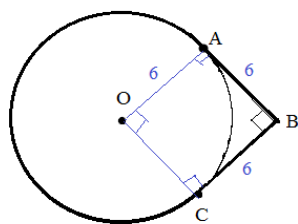Chord-Chord Power Theorem...   $6 \times 6 = 4 \times 9$

diameter of the circle is 13, so circumference is $\boxed{13\pi}$

---

3) $\overline{AB}$ and $\overline{BC}$ are tangent segments

$\angle ABC$ is a right angle

If the length of $\overline{AB}$ is 6 units, what is perimeter of the figure (the angle and major arc)?

A

6        6

O

B

6

C

$\overline{AB} = \overline{BC}$  (tangent-tangent theorem)

$\overline{AO} = \overline{CO}$  (all radii are congruent)

$\overline{AB}$ is perpendicular to $\overline{AO}$ ; $\overline{BC}$ is perpendicular to $\overline{CO}$ (tangent lines are perpedicular to radii at point of tangency)

Since they are all right angles and AB = BC, it forms a square..

Therefore, the radius is also 6 units..

$6 + 6 + \frac{270}{360} \cdot 12\pi = \boxed{12 + 9\pi}$

segment  segment  arc length

---

4) a and b are points of tangency of circles P and Q..

If the distance between a and b is 15, and the distance between centers P and Q is 17, then

what are the coordinates of points P and Q?

$\left.\begin{array}{l} x + y = 17 \\ x + 8 = y \end{array}\right\}$  system of equations
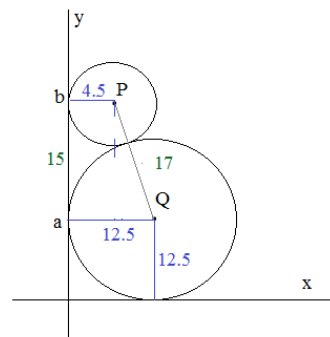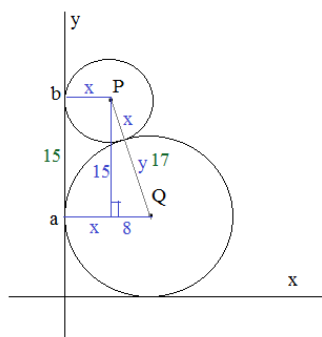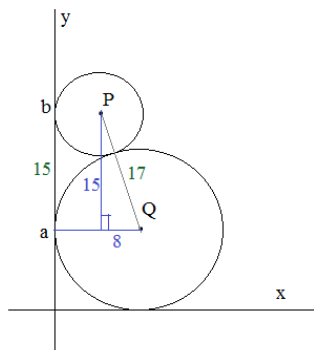
$x + (x + 8) = 17$   substitution

$2x = 9$

$x = 4.5$

radius of Q is 12.5
radius of P is 4.5

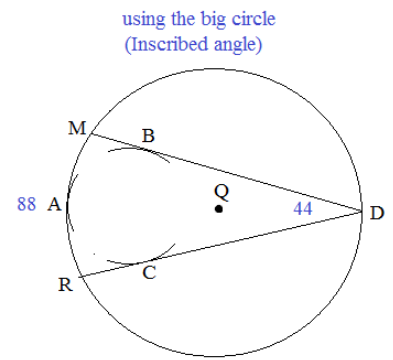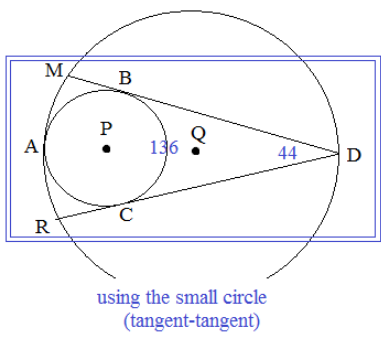coordinate P:
   (4.5, 27.5)

coordinate Q:
   (12.5, 12.5)

y

b   P

15   17
   15

a   Q
   8

x

y

b   x   P

15   x
   15   y   17

a   Q
   x   8

x

y

b   4.5   P

15        17

a   Q
   12.5
       12.5

x

mathplane.com

5) Circle P and Q are tangent at point A

$\overline{DM}$ and $\overline{DR}$ are external tangent lines

Arc $\overparen{BAC}$ is 224°

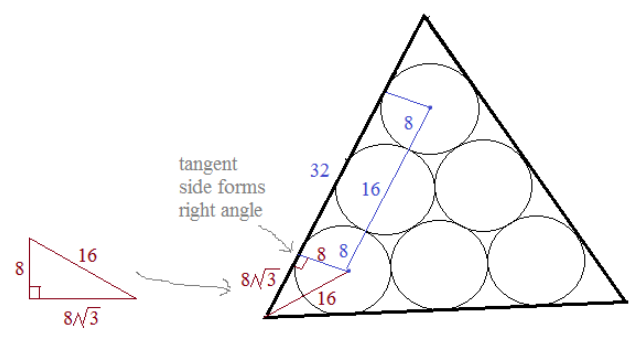Can you determine the measure of arc $\overparen{MR}$?

$\overparen{BC}$ is 136...

Then, angle D is 44...

So, arc $\overparen{MR}$ is 88...

using the small circle
(tangent-tangent)

using the big circle
(Inscribed angle)

6) Each of these 6 inscribed/tangent circles are congruent with radius 8.

What is the perimeter of the triangle?

$3(32) + 6(8\sqrt{3})$

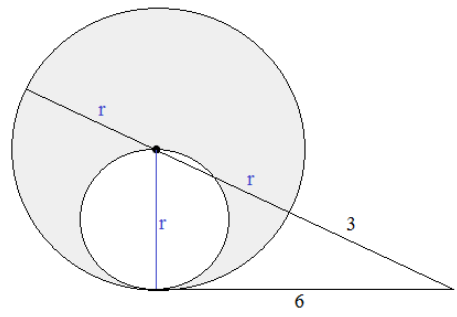$96 + 48\sqrt{3}$

tangent side forms right angle
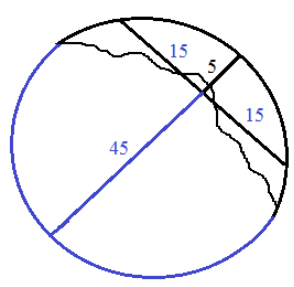
SOLUTIONS

7) Can you find the shaded area?

Using power theorem:

$6 \times 6 = 3 \times (3 + r + r)$

$12 = 2r + 3$

$r = 4.5$

So, r = 4.5 is the radius of the big circle..

And, 2.25 is the radius of the little circle..

$20.25\,\pi - 5.0625\,\pi = \boxed{15.1875\,\pi}$

8) At a pizzeria, a waiter and customer disagreed about a pizza order. .
Unfortunately, the customer had eaten most of the pizza.
However, the math chef came out and measured the remaining
portion and determined the size (diameter) of the original pizza.
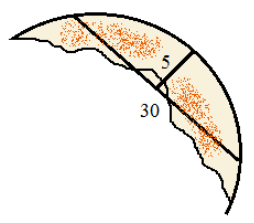
Distance of midpoint of arc
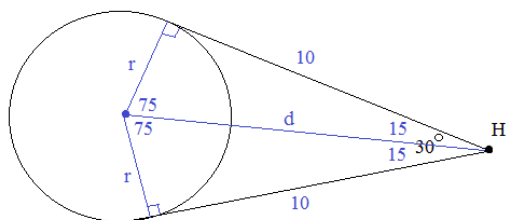to the midpoint of the chord is 5 cm.

Chord measures 30 cm.

$(15)(15) = (5)(X)$

chord-chord power theorem

$X = 45$

Diameter of pizza is 50 cm...

What was the diameter of the original, full pizza?

9) Two tangent segments have length 10 and form a 30 degree angle where they meet at external point H.

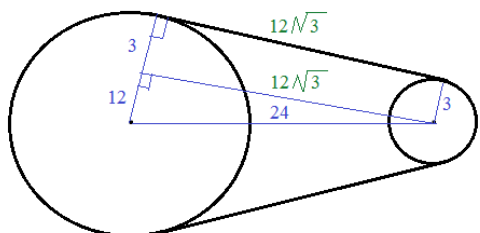How far is H from the circle?



SOLUTIONS

Using trigonometry we can find d and r

$$\cos(15) = \frac{10}{d} \qquad d = 10.35$$

$$\tan(15) = \frac{r}{10} \qquad r = 2.68$$

Distance from H to the circle is $d - r = \boxed{7.67}$

10) A pulley system includes two circles with radii 3 and 15 inches. If the centers of each circle are 12 inches apart, how much belt length is required to wrap around the circles?
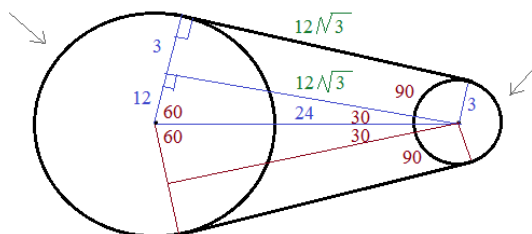


$\boxed{22\pi + 24\sqrt{3}}$

Note: circles and shapes are not drawn to scale

30-60-90 right triangles
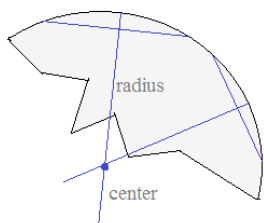finding lengths of
common external tangents

tangents are perpendicular to radius at
point of tangency



The relevant arc of the big circle is 240 degrees
and, the relevant arc of the small circle is 120 degrees

11) The figure is a piece of a broken plate.

How can you find the circumference of the original plate?



Answer:
(Using construction techniques)
Draw a chord
Then, construct the perpendicular bisector

Draw another chord
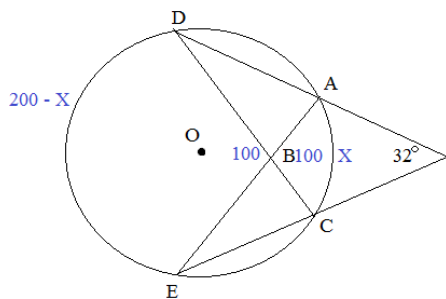Then, construct another perpendicular bisector

Since both perpendicular bisectors must go through
the center, the intersection would be the center
of the circle!

(The distance from the center to the circle is the radius)

12) $\angle ABC = 100^\circ$

$\overline{AD} = \overline{EC}$

What are the individual arc measures in circle O?



Using Chord-Chord Angle theorem:
since ABC is 100 degrees, $\overset{\frown}{AC} + \overset{\frown}{DE} = 200$

Then, using Secant-Secant Angle theorem:

$$\frac{1}{2}[(200 - x) - x] = 32$$

$$200 - 2x = 64$$

$$x = 68$$

then, AD = EC
and AD + EC = 160
80 and 80

$\boxed{\begin{array}{l} \overset{\frown}{DE} = 132 \\ \overset{\frown}{EC} = AD = 80 \\ \overset{\frown}{AC} = 68 \end{array}}$

13) Find the arc length of $\overarc{RD}$

x + 4

x + 4

20 (Pythagorean Theorem)

12

16 - x

R

x

x    D    16 - x

16
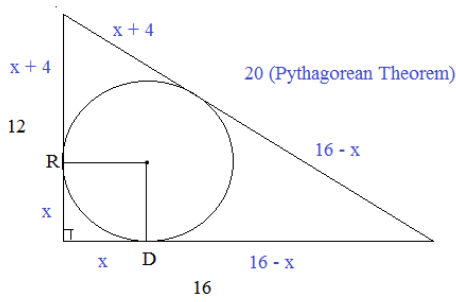
Since circle is inscribed, the
sides of the triangle are tangent..

("walk-about problem")

$x + (x + 4) = 12$

$x = 4$

We know the radii are congruent, AND
the radii are perpendicular to the tangents...
Therefore, the figure is a square with sides 4...

So, the radii of the inscribed circle are 4...
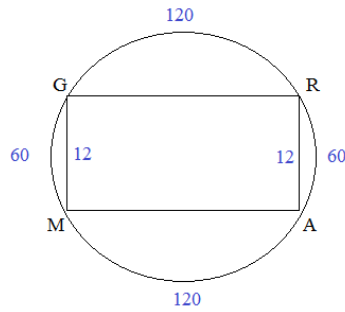And, the arc is 90 degrees..

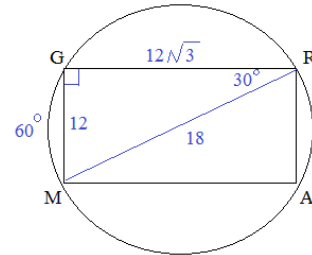Therefore, the arc length is $\dfrac{90}{360}(8\pi) = \boxed{2\pi}$

14) Parallelogram GRAM

$\overarc{GR} = 120$ degrees

$\overline{GM} = 12$

Find the length of $\overarc{GRM}$

120

G                      R

60   12          12   60

M                      A

120

Inscribed parallelogram must be a rectangle...

G    $12\sqrt{3}$    R
         30°
60°   12        18
M                A

GRM is inscribed angle ---> 1/2(60) = 30

30-60-90 right triangle ----> diameter is 24...

circumference of circle is 24 $\pi$

so, major arc $\overarc{GRM} = \dfrac{300}{360} \ 24\pi = \boxed{20\pi}$

15) The diameter of a circle has endpoints (3, 1) and (11, 7).

If one of the chords has endpoints (2, 4) and (7, −1), then
what is the distance from the chord to the center of the circle?

5

5

(7, 4)
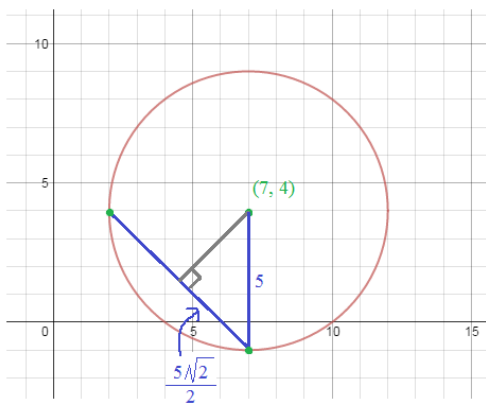
5

$\dfrac{5\sqrt{2}}{2}$
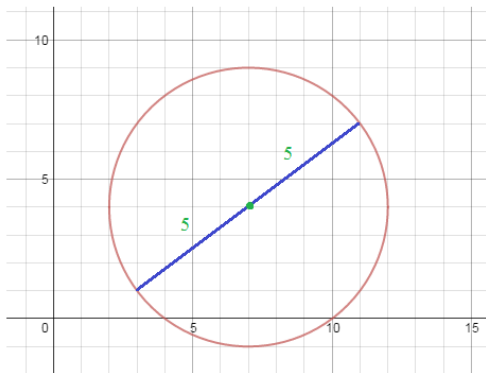
Since diameter's endpoints are (3, 1) and (11, 7),
the center is the midpoint (7, 4)

And, the length of the diameter (i.e. distance) is

$$\sqrt{(11-3)^2 + (7-1)^2} = 10$$

so, the radius is 5

$$(x-7)^2 + (y-4)^2 = 25$$

Since the chord's endpoints are (2, 4) and (7, −1),
the length of the chord is

$$\sqrt{(-1-4)^2 + (7-2)^2} = 5\sqrt{2}$$

and, 1/2 the chord is $\dfrac{5\sqrt{2}}{2}$

Then, Pythagorean Theorem

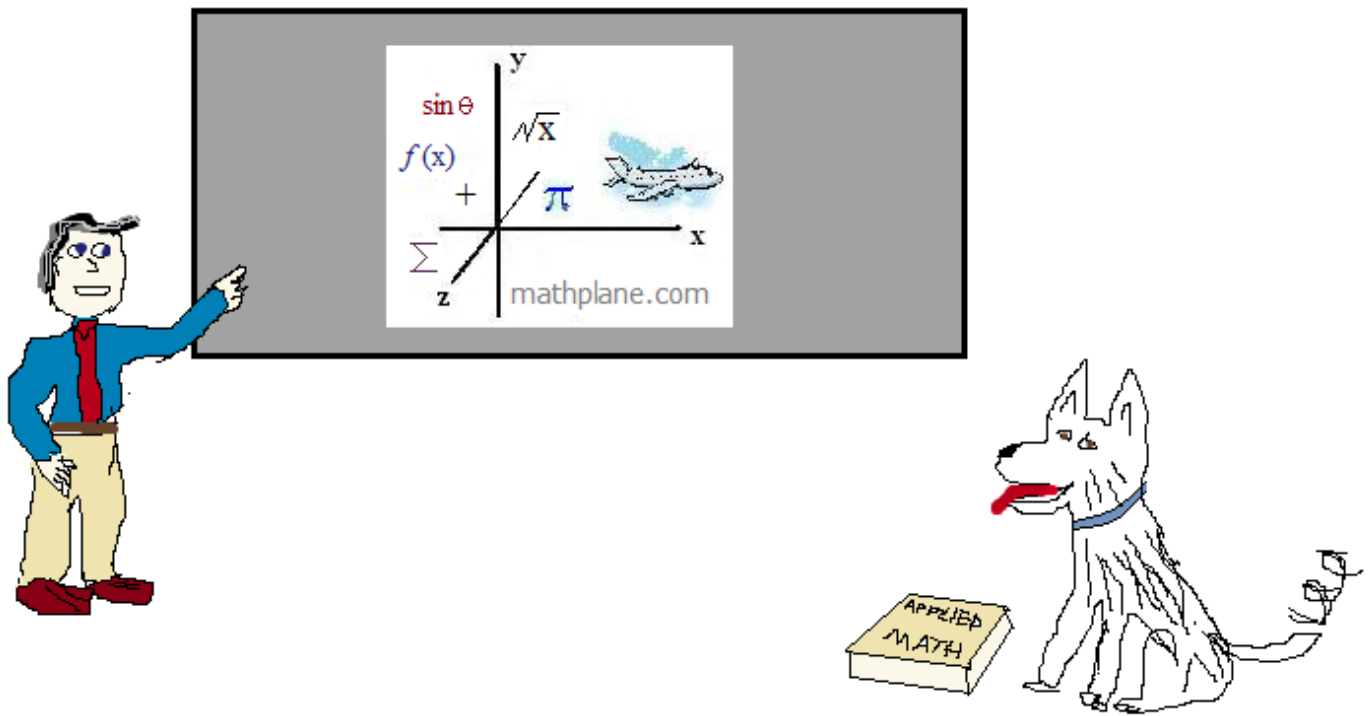$$\left(\dfrac{5\sqrt{2}}{2}\right)^2 + d^2 = 5^2$$

$$d^2 = 12.5$$

$$\boxed{d \approx 3.536}$$

mathplane.com

Thanks for visiting. (Hope it helped!)

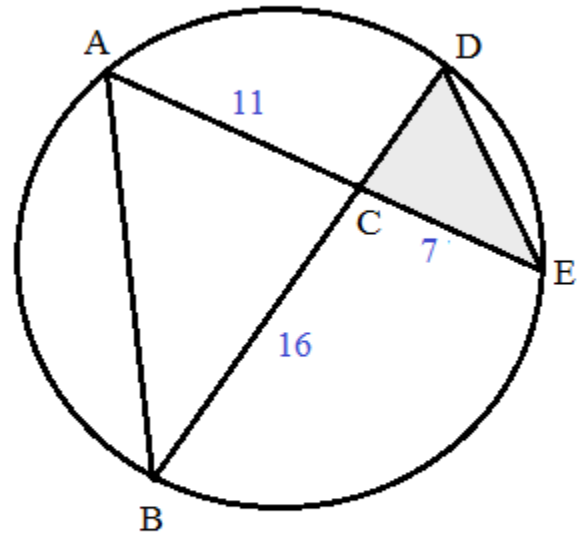If you have questions, suggestions, or requests, let us know.

Cheers

ONE MORE QUESTION-→

If the area of triangle ABC is 86,
what is the area of triangle CDE?

$\overline{AC} = 11$

$\overline{BC} = 16$

$\overline{CE} = 7$



ANSWER-→

If the area of triangle ABC is 86,
what is the area of triangle CDE?

AE and BD are intersecting chords...
therefore, $(AC)(CE) = (BC)(CD)$

$$(11)(7) = (16)(CD)$$

$$CD = 4.8125$$

$\angle$ BAE $\cong$ $\angle$ EDB

because the inscribed angles share
the same arc $\overset{\frown}{BE}$

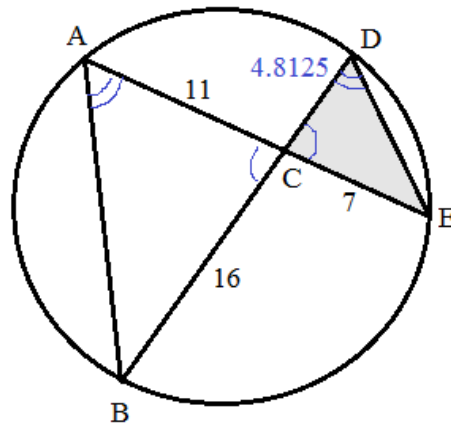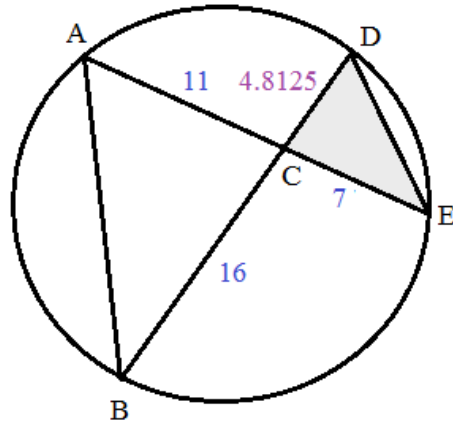$\angle$ ACB $\cong$ $\angle$ DCE

because vertical angles are congruent

Therefore, the triangles are similar (angle-angle)

Since the triangles are similar, the ratio of the
areas is "ratio of the squared sides"

$$\frac{(16)^2}{(7)^2} = \frac{(11)^2}{(4.8125)^2} = \frac{86}{\text{Area of CDE}}$$

$$= 16.46$$

Also, using trigonometry....

If the area of triangle ABC is 86,
what is the area of triangle CDE?

SOLUTION

AE and BD are intersecting chords...
therefore,   $(AC)(CE) = (BC)(CD)$

$(11)(7) = (16)(CD)$

$CD = 4.8125$

Area of triangle $= \dfrac{1}{2} ab(\sin C)$

$86 = \dfrac{1}{2} (11)(16)\sin C$

$.977 = \sin C$

$C = 77.76^{\circ}$

$\angle ACB = \angle DCE$   (vertical angles)

77.76 degrees

Area $\triangle CDE = \dfrac{1}{2} (4.8125)(7)\sin(77.76)$

$= 16.46$