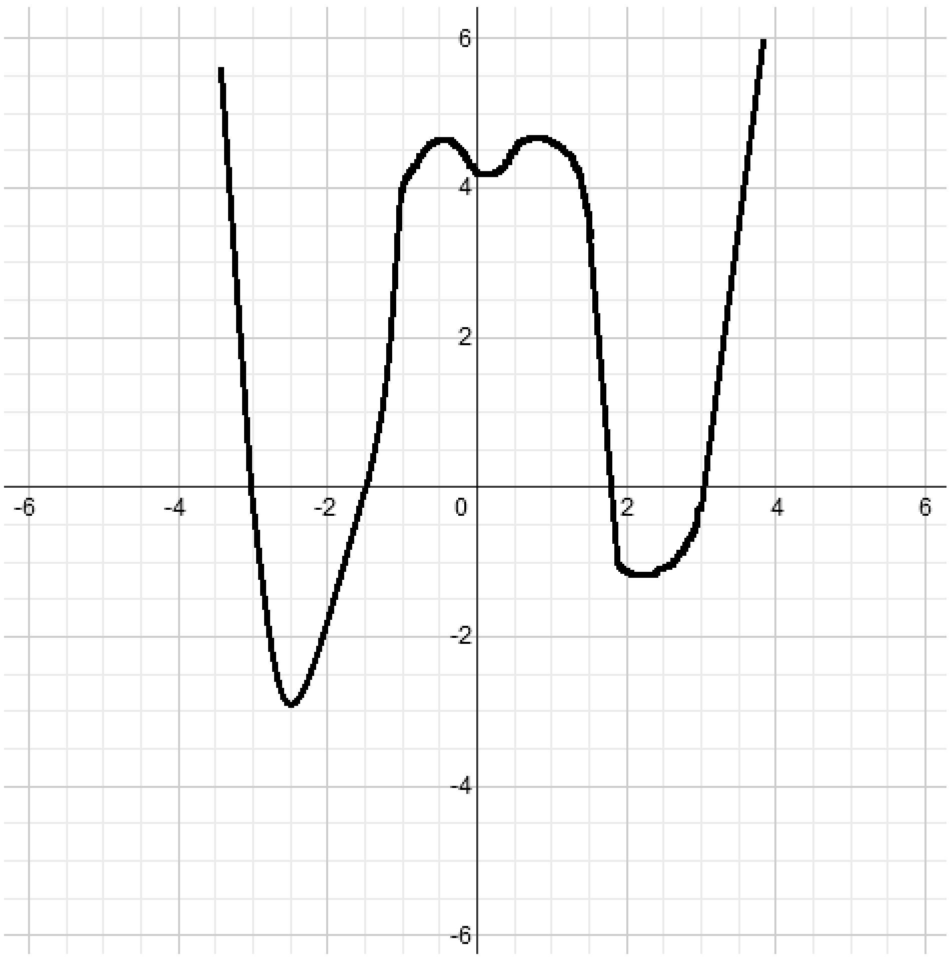
Polynomials 3: Factors, Roots, and Theorems (Honors)

Notes, Examples, and Practice Test (with Solutions)



Topics include interpreting graphs, synthetic division, intermediate value theorem, factoring, and more.

Mathplane.com

Example: Solve $x^4 = x$

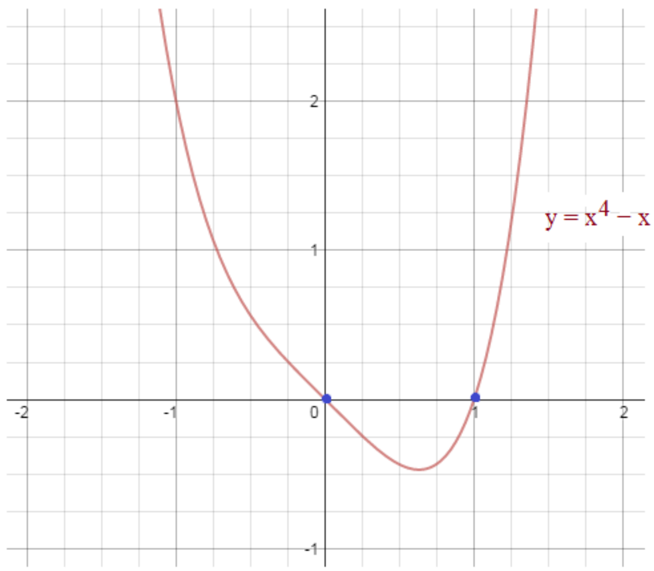
WRONG

Divide both sides by x

$x^3 = 1$

$x = 1$

when you divide by variables, you lose solutions...
(and, when you multiply by variables, you add (extraneous) solutions)



RIGHT

$x^4 - x = 0$ — Move terms to one side

$x(x^3 - 1) = 0$ — Factor

$x(x - 1)(x^2 + x + 1) = 0$ — Difference of Cubes

$x = 0 \quad x = 1$

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm i\sqrt{3}}{2}$$

two real and two imaginary solutions...

Example: Find the polynomial of degree 3, with zeros 1, -2, and 3, and the coefficient of x^2 is -4

If the polynomial has zeros 1, -2, and 3, then it has factors

$(x - 1)(x + 2)$ and $(x - 3)$

multiplying all 3 together:

$(x - 1)(x + 2) = x^2 + x - 2$

$(x^2 + x - 2)(x - 3) = x^3 + x^2 - 2x - 3x^2 - 3x + 6$

$= x^3 - 2x^2 - 5x + 6$

This is a cubic (polynomial with degree 3), and it has zeros 1, -2, and 3..

Now, to change the coefficient of x^2 to -4, we must multiply the function by 2!!

$2x^3 - 4x^2 - 10x + 12$

Example: $f(x) = x^5 - 3x^4 + 8x^3 - 8x^2 + 7x - 5$

If we know the roots of $f(x)$ include 1 and i, what are the other roots?

Since 1 is a root, we can use synthetic division to reduce $f(x)$...

$$\begin{array}{r|rrrrrr} 1 & 1 & -3 & 8 & -8 & 7 & -5 \\ & & 1 & -2 & 6 & -2 & 5 \\ \hline & 1 & -2 & 6 & -2 & 5 & \boxed{0} \end{array}$$

note: the remainder is 0,
confirming $(x - 1)$ is a factor

$$x^4 - 2x^3 + 6x^2 - 2x + 5$$

Since i is a root, we know $+i$ is a root (due to the 'conjugate pairs theorem')

So, we can reduce the remaining part by either A) synthetic division or B) Long Division

Method A: Synthetic Division

$$\begin{array}{r|rrrrr} i & 1 & -2 & 6 & -2 & 5 \\ & & i & -1-2i & 5i+2 & -5 \\ \hline & 1 & i-2 & -2i+5 & 5i & \boxed{0} \end{array}$$

check: since 0 is remainder and i is a root, the synthetic division is correct!

$$\begin{array}{r|rrrr} -i & 1 & i-2 & -2i+5 & 5i \\ & & -i & 2i & -5i \\ \hline & 1 & -2 & 5 & \boxed{0} \end{array}$$

$$x^2 - 2x + 5$$

Note: the discriminant $b^2 - 4ac = -16$

therefore, we'll use the quadratic formula to find the 2 complex/imaginary roots

Method B) Long Division

Since $(x + i)(x - i) = x^2 + 1$

$$\begin{array}{r|l} & x^2 - 2x + 5 \\ \hline x^2 + 1 & x^4 - 2x^3 + 6x^2 - 2x + 5 \\ & -\ (x^4 + x^2) \\ \hline & 0 - 2x^3 + 5x^2 - 2x + 5 \\ & -\ (-2x^3 - 2x) \\ \hline & 0 + 5x^2 \quad 0 \quad + 5 \\ & -\ (5x^2 + 5) \\ \hline & 0 \qquad 0 \end{array}$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$$

The 5 roots are 1, i, $-i$, $1 + 2i$, $1 - 2i$

Example: The graphs of $y = \frac{(x-k)}{(x+5)}$ and $y = \frac{(x-c)}{(x+2)}$ intersect at point (a, b)...

Find $(c - k)$ in terms of b

Since intersection at (a, b), we know..

$b = \frac{(a - k)}{(a+5)}$ and $b = \frac{(a - c)}{(a+2)}$

$b(a+5) = a - k$ $b(a+2) = a - c$

$\boxed{k = a - b(a+5)}$ $\boxed{c = a - b(a+2)}$

$(c - k) = a - b(a+2) - (a - b(a+5))$

$(c - k) = -b(a+2) + b(a+5)$

$(c - k) = -ba - 2b + ba + 5b$

$\boxed{c - k = 3b}$

Example: The function $y = \frac{x^3 - 6}{x^2 + 5}$ has one asymptote. Find the value of x at which the graph crosses that asymptote..

**Important: The graph NEVER crosses a vertical asymptote.. It may only cross a horizontal or oblique/slant asymptote!

So, this function potentially has 2 types of asymptotes...

1) Vertical asymptotes where denominator equals zero... (In this function, there are no vertical asymptotes!)

2) Horizontal asymptote...
If degree of numerator < degree of denominator ----> $y = 0$
If degree of numerator = degree of denominator ----> $y = \frac{\text{lead coefficient of numerator}}{\text{lead coefficient of denominator}}$

In this case, degree of numerator > degree of denominator ---> no horizontal asymptote..
HOWEVER, there is a slant asymptote because degree of num. is 3 and degree of denom. is 2 (they are 1 apart..)

So, the (slant) asymptote is $y = x$

Where does the graph cross?

simply solve:

$x = \frac{x^3 - 6}{x^2 + 5}$

$(x^2 + 5)(x) = x^3 - 6$

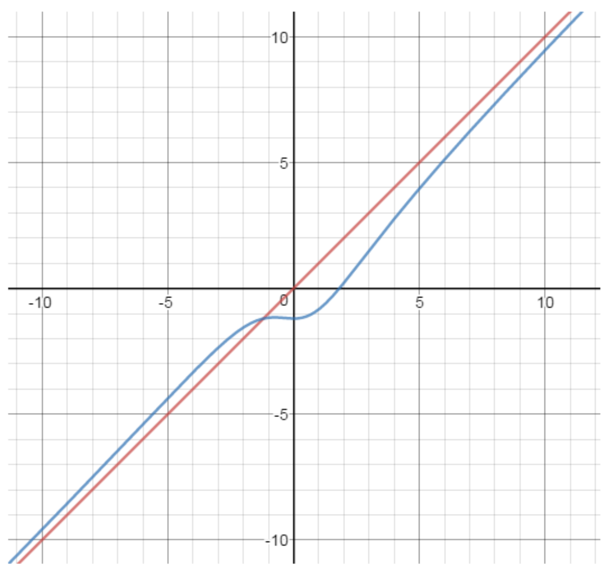
$x^3 + 5x = x^3 - 6$

$\boxed{x = -6/5}$

slant asymptote =

$$\begin{array}{r|l} & x \\ x^2 + 5 & x^3 + 0x^2 + 0x - 6 \\ & -(x^3 \quad\quad + 5x) \\ & \quad\quad -5x \end{array}$$

the remainder is irrelevant when looking at the asymptote



PRACTICE TEST-→

Warm-up

Polynomials: Factors, Roots, and Theorems III (Honors)

A) Write a polynomial $f(x)$ with real coefficients, given the degree and zeros.

1) Degree: 3
Zeros: 0, 1, 4

2) Degree: 4
Zeros: $3 + 2i$, 4 (with a multiplicity of 2)

B) Factor the following polynomial, where one of the zeros is $3i$:

$$3x^4 + 5x^3 + 25x^2 + 45x - 18$$

C) Solve $2x^4 + 2x^3 - 11x^2 + x - 6 = 0$

D) Simplify (using synthetic division) $\dfrac{2x^4 + 3x^2 - 9x}{x - 1}$

1) Find all complex solutions: $x^4 = 16$

2) $f(x) = x^3 - 2x^2 - 1$

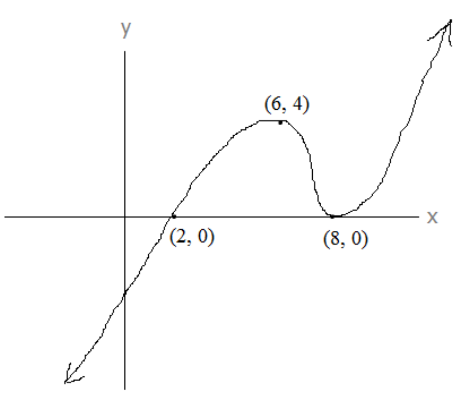
Does $f(x)$ have any rational roots?

Verify that there is a root between 2 and 3...

3) $x = 3$ is a triple root
$x = 0$ is a double root
$P(x)$ is a polynomial of degree 6
The remainder of $P(x) \div (x - 2)$ is 8

What is the polynomial $P(x)$ in factored form?

4) Find the y-intercept of the function:
(Note: the sketch may not be drawn to scale)



Polynomials: Factors, Roots, and Theorems III (Honors)

5) $P(x) = 3x^4 - 10x^3 + 2x^2 + bx + c$

$\frac{P(x)}{(x + 2)}$ has a remainder 167

$\frac{P(x)}{(x - 1)}$ has a remainder 11

Find b and c...

6) Polynomial $g(x) = x^3 + 4x^2 + bx + c$

When divided by $(x - 3)$, the remainder is 110.

When divided by $(x + 2)$, the remainder is 150.

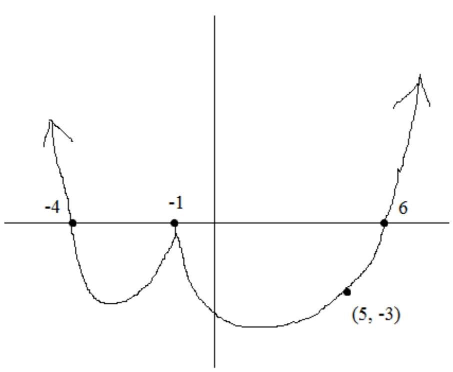
Find the polynomial factors of $g(x)$.

7) A polynomial has zeros at -1, 2, and 3...

a) If the linear term is 5, find the equation

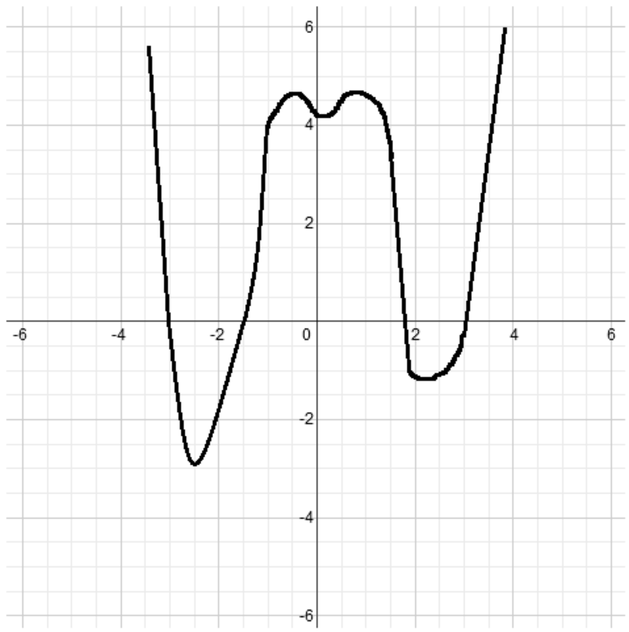
b) If the y-intercept is 16, find the equation

8) Find the polynomial shown in the graph:



9) Suppose the function $f(x) = x^6 + \ldots + 15$

is showed in the following graph:

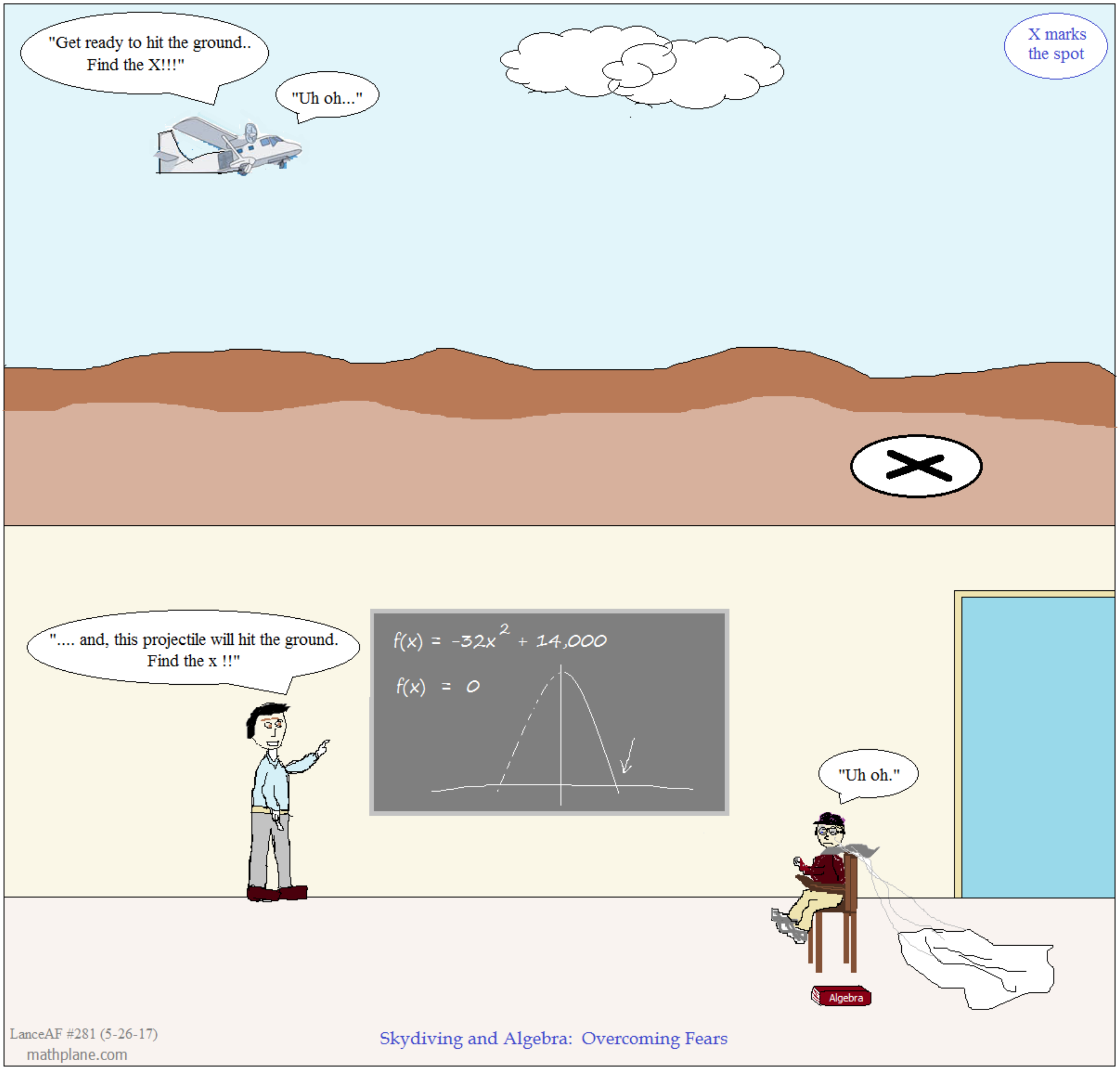


How many (real) rational roots are there?

How many (real) irrational roots are there?

How many non-real roots are there?

10) Find a degree 4 polynomial that has zeros 3 and $5i$ and passes through (0, 4)


"Get ready to hit the ground.. Find the X!!!"
"Uh oh..."
X marks the spot
"…. and, this projectile will hit the ground. Find the x !!"
f(x) = -32x² + 14,000
f(x) = 0
"Uh oh."
Algebra
LanceAF #281 (5-26-17)
mathplane.com
Skydiving and Algebra: Overcoming Fears

ANSWERS-→

Warm-up

SOLUTIONS

Polynomials: Factors, Roots, and Theorems III (Honors)

A) Write a polynomial $f(x)$ with real coefficients, given the degree and zeros.

1) Degree: 3
Zeros: 0, 1, 4

$(x - 0)(x - 1)(x - 4)$

$(x)(x^2 - 5x + 4)$

$f(x) = x^3 - 5x^2 + 4x$

Note: there are infinite possiblities for example,

$2x^3 - 10x^2 + 8x$ has degree 3, real coefficients, and zeros 0, 1, and 4

2) Degree: 4
Zeros: $3 + 2i$, 4 (with a multiplicity of 2)

$(x - 4)(x - 4)(x - (3 + 2i))(x - (3 - 2i))$

conjugate pair

$(x^2 - 8x + 16)(x^2 - 6x + 13)$

$f(x) = x^4 - 14x^3 + 77x^2 - 200x + 208$

$$\begin{array}{rrrr} & x - 3 - 2i & & \\ & x - 3 + 2i & & \\ \hline x^2 - 3x + 2ix & & & \\ -3x & + 9 - 6i & & \\ -2ix & + 6i & - 4i^2 & \\ \hline x^2 - 6x & + 9 & + 4 & \end{array}$$

$$\begin{array}{rrrrr} & & x^2 - 8x + 16 & & \\ & & x^2 - 6x + 13 & & \\ \hline x^4 - 6x^3 + 13x^2 & & & & \\ -8x^3 + 48x^2 - 104x & & & & \\ 16x^2 - 96x + 208 & & & & \\ \hline x^4 - 14x^3 + 77x^2 - 200x + 208 & & & & \end{array}$$

B) Factor the following polynomial, where one of the zeros is $3i$:

$3x^4 + 5x^3 + 25x^2 + 45x - 18$

Since $3i$ is a zero, we know $-3i$ is another zero (conjugate theorem)

Therefore, $(x - 3i)(x + 3i)$ is a factor...

$(x^2 + 9)$

Using long division:

$$\begin{array}{r|l} & 3x^2 + 5x - 2 \\ (x^2 + 9) & 3x^4 + 5x^3 + 25x^2 + 45x - 18 \\ & -\ (3x^4 + 27x^2) \\ & 5x^3 - 2x^2 \\ & -\ (5x^3 + 45x) \\ & -2x^2 \\ & -\ (-2x^2 - 18) \\ & 0 \end{array}$$

$3x^2 + 5x - 2$

$(3x - 1)(x + 2)$

$(3x - 1)(x + 2)(x^2 + 9)$

C) Solve $2x^4 + 2x^3 - 11x^2 + x - 6 = 0$

First, using "p's" and "q's"...

p's: factors of 6: 1, 2, 3, 6

q's: factors of 2: 1, 2

possible rational roots: 1, 2, 3, 6, 1/2, 3/2

-1, -2, -3, -6, -1/2, -3/2

factor theorem

try $x = 1$: $2 + 2 - 11 + 1 - 6 \neq 0$

try $x = 2$: $32 + 16 - 44 + 2 - 6 = 0$

2 is root... Use synthetic division:

$$\begin{array}{r|rrrrr} 2 & 2 & 2 & -11 & 1 & -6 \\ & & 4 & 12 & 2 & 6 \\ \hline & 2 & 6 & 1 & 3 & 0 \end{array}$$

$2x^3 + 6x^2 + x + 3$ Use grouping...

$2x^2(x + 3) + 1(x + 3)$

$(2x^2 + 1)(x + 3)$

and

$(x - 2)$

so, $x = 2, -3$ real

$x = i\frac{\sqrt{2}}{2}$ $-i\frac{\sqrt{2}}{2}$ imaginary

D) Simplify (using synthetic division) $\dfrac{2x^4 + 3x^2 - 9x}{x - 1}$

Note: using remainder theorem,

$2(1)^4 + 3(1)^2 - 9(1) = -4$

the remainder should be -4

$$\begin{array}{r|rrrrr} 1 & 2 & 0 & 3 & -9 & 0 \\ & & 2 & 2 & 5 & -4 \\ \hline & 2 & 2 & 5 & -4 & -4 \end{array}$$

$2x^3 + 2x^2 + 5x - 4 - \dfrac{4}{x - 1}$

Polynomials: Factors, Roots, and Theorems III (Honors)

SOLUTIONS

1) Find all complex solutions: $x^4 = 16$

$x^4 - 16 = 0$

$(x^2 - 4)(x^2 + 4) = 0$

$(x - 2)(x + 2)(x^2 + 4) = 0$

$x = 2, -2, 2i, -2i$

2) $f(x) = x^3 - 2x^2 - 1$

Does $f(x)$ have any rational roots?

NO... There is an irrational root... (and 2 imaginary roots)

Utilizing the rational root theorem: possible rational roots are 1, -1

$f(1) = -2$ $f(-1) = -4$ Since neither is 0, neither is a root...

(remainder/factor theorems)

Verify that there is a root between 2 and 3...

Using the Intermediate Value Theorem

$f(2) = -1$

$f(3) = 8$

Since $f(x)$ is continuous, $f(2)$ is under the x-axis, and $f(3)$ is below the x-axis, the graph must cross the x-axis somewhere between 1 and 2!

Therefore, there must be an x-intercept in the interval (2, 3) ✓

3) x = 3 is a triple root
x = 0 is a double root
$P(x)$ is a polynomial of degree 6
The remainder of $P(x) \div (x - 2)$ is 8

What is the polynomial $P(x)$ in factored form?

x = 3 is a triple root: $(x - 3)^3$

x = 0 is a double root: $(x)^2 (x - 3)^3$

$P(x)$ is a 6th degree polynomial: $(x)^2 (x - 3)^3 (x - B)$

B is the 6th root...

$P(2) = 8$ Remainder Theorem! $\frac{P(x)}{(x-2)}$ will have a remainder of 8

$(2)^2 (2 - 3)^3 (2 - B) = 8$

$4 \cdot (-1) \cdot (2 - B) = 8$

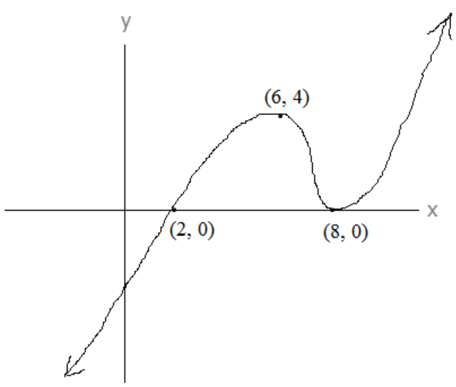
$2 - B = -2$

$B = 4$

$P(x) = (x)^2 (x - 3)^3 (x - 4)$

4) Find the y-intercept of the function:

(Note: the sketch may not be drawn to scale)



$y = a(x - 2)(x - 8)^2$

to find 'a', we'll insert (6, 4)

$4 = a(4)(4)$

$a = 1/4$

$y = \frac{1}{4}(x - 2)(x - 8)^2$ to find y-intercept, let x = 0

therefore, the y-intercept is (0, -32)

Polynomials: Factors, Roots, and Theorems III (Honors)

SOLUTIONS

5) $P(x) = 3x^4 - 10x^3 + 2x^2 + bx + c$

$\frac{P(x)}{(x+2)}$ has a remainder 167

$\frac{P(x)}{(x-1)}$ has a remainder 11

Find b and c...

Using the remainder theorem:

$P(1) = 11$

$P(1) = 3 - 10 + 2 + b + c$

$11 = -5 + b + c$

$b + c = 16$

$P(-2) = 167$

$P(-2) = 48 + 80 + 8 - 2b + c$

$167 = 136 - 2b + c$

$-2b + c = 31$

solve using elimination method:

$$\begin{aligned} b + c &= 16 \\ -\quad -2b + c &= 31 \\ \hline 3b &= -15 \end{aligned}$$

$b = -5$

$c = 21$

6) Polynomial $g(x) = x^3 + 4x^2 + bx + c$

When divided by $(x - 3)$, the remainder is 110.
When divided by $(x + 2)$, the remainder is 150.

Find the polynomial factors of $g(x)$.

$g(3) = 110 = (3)^3 + 4(3)^2 + b(3) + c$

$110 = 27 + 36 + 3b + c$

$3b + c = 47$

$g(-2) = 150 = (-2)^3 + 4(-2)^2 + b(-2) + c$

$150 = -8 + 16 - 2b + c$

$-2b + c = 142$

$$\begin{aligned} 3b + c &= 47 \\ -\quad -2b + c &= 142 \\ \hline 5b &= -95 \end{aligned}$$

$b = -19$

$c = 104$

Use Rational Root theorem to find factors...

POSSIBLE rational roots:

1, 2, 4, 8, 13, 26, 52, 104

(positive or negative)

$g(x) = x^3 + 4x^2 - 19x + 104$

-8	1	4	−19	104
		−8	32	−104
	1	−4	13	0

$(x + 8)(x^2 - 4x + 13)$

7) A polynomial has zeros at -1, 2, and 3...

a) If the linear term is 5, find the equation

$y = (x - (-1)(x - 2)(x - 3)$

$y = (x + 1)(x^2 - 5x + 6)$

$y = x^3 - 4x^2 + x + 6$

linear term

If the linear term is 5, then $y = 5x^3 - 20x^2 + 5x + 30$

b) If the y-intercept is 16, find the equation

$y = a(x - (-1)(x - 2)(x - 3)$

$y = a(x + 1)(x^2 - 5x + 6)$

(0, 16)

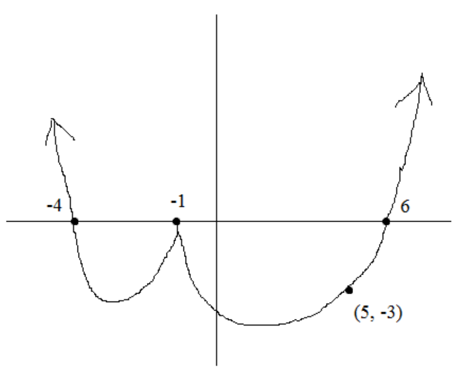
$16 = a(1)(6)$

$a = 8/3$

$y = \frac{8}{3}(x + 1)(x^2 - 5x + 6)$

(0, 16) is y-intercept

8) Find the polynomial shown in the graph:



The intercepts are -4, -1, and 6...

So, factors will include $(x + 4)$ $(x + 1)$ and $(x - 6)$
And, since there is a "bounce" at -1, the multiplicity of that factor is 2...

$y = a(x + 4)(x + 1)^2(x - 6)$

Then, to determine the shape of the curve, we'll use the other point...

$-3 = a(9)(36)(-1)$

$-3 = -324a$

$a = \frac{1}{108}$

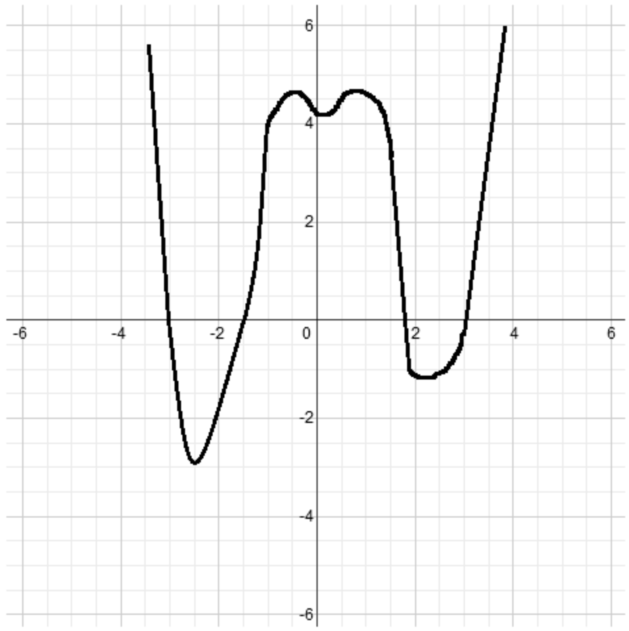
$y = \frac{1}{108}(x + 4)(x + 1)^2(x - 6)$

9) Suppose the function $f(x) = x^6 + \ldots + 15$

is showed in the following graph:

SOLUTIONS

Polynomials: Factors, Roots, and Theorems III (Honors)



How many (real) rational roots are there?

How many (real) irrational roots are there?

How many non-real roots are there?

according to rational root theorem, the possible rational zeros are

$\pm 1, \ \pm 3, \ \pm 5, \ \pm 15$

In the graph, 3 and -3 are zeros...

Then, any real root will be on the x-axis...

So, there are two other real roots, which are irrational...

then, since the degree of the polynomial is 6, there are a total of 6 roots...

So, the other 2 roots are non-real (and the graph shows two 'turns' above the x-axis representing roots)

10) Find a degree 4 polynomial that has zeros 3 and $5i$ and passes through (0, 4)

The zeros must be 3, $5i$, and $-5i$

and, since it is a 4th degree polynomial, 3 must have a multiplicity of 2

$$(x-3)^2(x-5i)(x+5i)$$

$$y = a(x-3)^2(x^2+25)$$

$$4 = a(9)(25)$$

$$4 = 225a$$

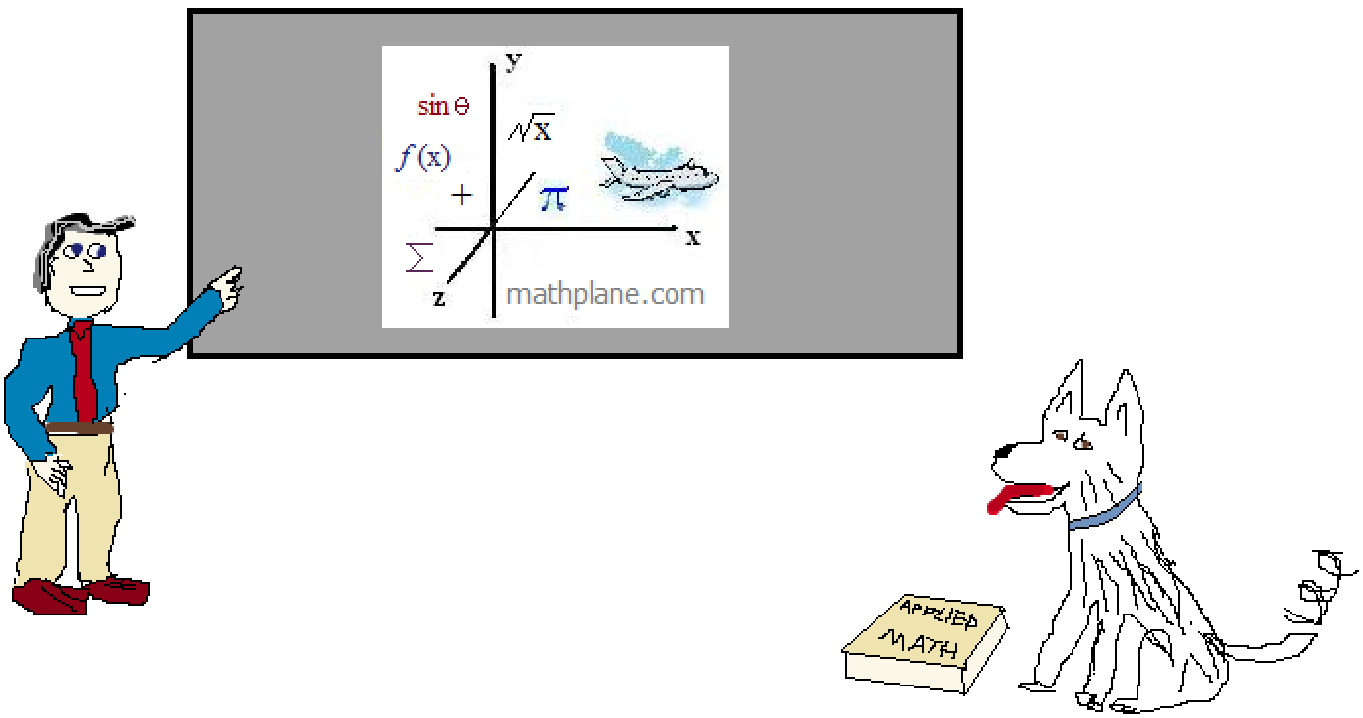
$$a = 4/225$$

$$y = \frac{4}{225}(x-3)^2(x^2+25)$$

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers



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