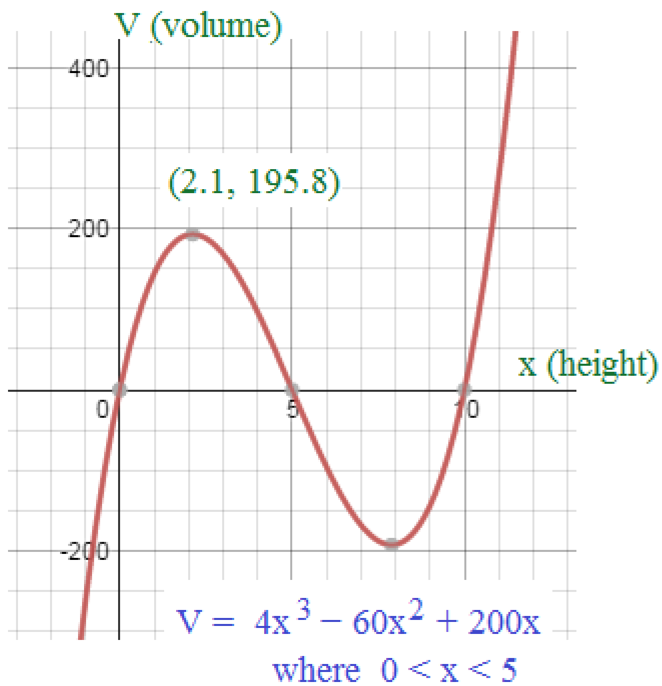
Modeling Functions using One Variable

Notes, graphs, examples, and practice quiz (with solutions)



Topics include parabola & vertex, area, volume, derivatives, revenue/cost, factoring quadratics, and more.

Modeling functions of one variable:
Area/Perimeter Problem

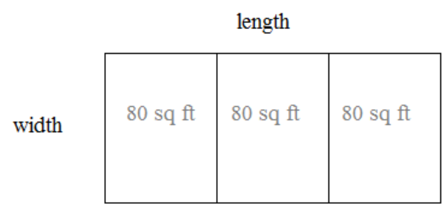
Example: A homeowner wishes to enclose/fence in 3 plots of land, 80 square feet each.

a) Write an equation that expresses *Perimeter as a function of x*

b) Write an equation that expresses *Area as a function of x*

c) If he has 88 feet of fence, what are the dimesions of each lot?

Step 1: Draw a picture and label parts



Step 2: Develop the equations

Let x = width
Let y = length

Area = xy
Perimeter = 2y + 4x

There is 88 feet of fence, so 88 feet = 4x + 2y
88 feet − 4x = 2y
or y = 44 − 2x

Each plot is 80 sq. feet, so 80 sq. feet = $x \cdot (1/3)y$

or $y = \dfrac{240}{x}$

Using the variables and equations on the left, we can create functions of one variable...

P = 2y + 4x.. this is perimeter as a function of x and y..
We need to get rid of the y!

$P = 2y + 4x$ and $y = \dfrac{240}{x}$

Using substitution,

a) $P(x) = \dfrac{480}{x} + 4x$

A = xy... this is area as a function of x and y
We want to get rid of the y variable...

$A = xy$ and $y = 44 - 2x$

Using substitution,

b) $A(x) = x(44 - 2x) = 44x - 2x^2$

Step 3: Solve

If he has 88 feet of fencing, we know that P(x) = 88

$$88 \text{ feet} = \frac{480}{x} + 4x$$

$$88x = 480 + 4x^2$$

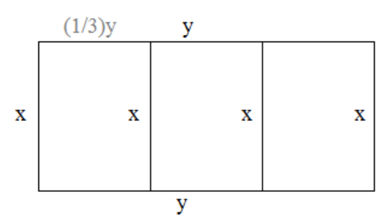
$$4x^2 - 88x + 480 = 0$$

$$x^2 - 22x + 120 = 0$$

$$(x - 10)(x - 12) = 0$$

$$x = 10 \text{ or } 12$$

c) The dimensions of each lot are
10 x 8 OR 12 x 20/3



Step 4: Answer the question and check

If x = 10, then y = 24
and (1/3)y = 8
10 x 8
area of each plot: 80
total fencing: 88

If x = 12, then y = 20
and (1/3) = 20/3
12 x 20/3
area of each plot: 12 x (20/3) = 80
total fencing: 4(12) + 2(20) = 88

Example: An open box with a square base has volume 15 feet^3 .

a) Write a function that models the surface area of the box (as a function of the base side (x)).

b) What is the domain of x?

c) What dimensions would minimize the amount of material needed to build the box?

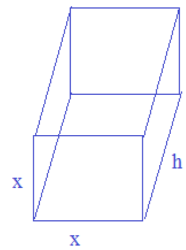
d) What is the surface area of the minimized box?

Modeling: creating functions of one variable

Step 1: Draw a diagram and label

The base is a square, so the four sides are equal (x)

And, the height is (h)



Step 2: Write the relevant functions

Volume = (length)(width)(height) = 15 feet^3

So, a square prism's volume is (side)(side)(height) = 15 feet^3

$15 = x^2 h$

Surface Area of square prism = (side)(side) + 4(side)(height)
(with open top)

$SA = x^2 + 4xh$

Step 3: Use substitution to get the desired function

Our desired function is "surface area as a function of the base side"...
Right now, we have surface area as a function of x and h...
We need to get rid of the h...

since $15 = x^2 h$, $h = \dfrac{15}{x^2}$ therefore, $SA = x^2 + 4x\left(\dfrac{15}{x^2}\right)$

a) $SA = x^2 + \dfrac{60}{x}$

If we look at the surface area and volume, we can determine the domain of x...

$SA = x^2 + \dfrac{60}{x}$ so, x cannot be 0...

since x is a length, x cannot be negative...

b) domain of SA(x) is $(0, \infty)$

Step 4: Solve equation/find the minimum value

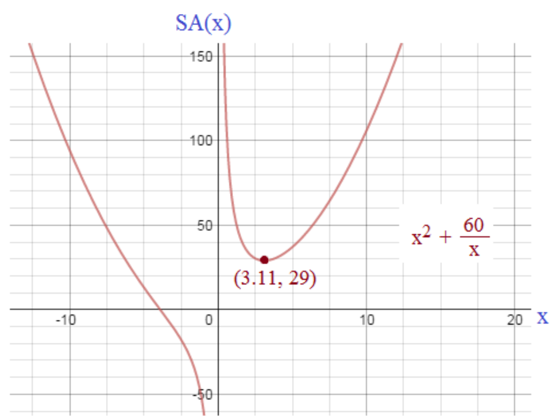
method 1: use a graphing calculator...

At the right, is the graph of the function...
We elimate the minimum values left of the y-axis
(because they are not within the domain)..

The minimum value is when x = 3.11

If x = 3.11, then h = 1.55 because $(3.11)^2(1.55) = 15$

c) So, the dimensions are 3.11 x 3.11 x 1.55



method 2: use calculus

Find 1st derivative: $SA'(x) = 2x - \dfrac{60}{x^2}$

Then, determine 'critical values'...

$$0 = 2x - \frac{60}{x^2}$$

$$0 = 2x^3 - 60$$

SA'(x) is undefined at x = 0
and,
$SA'(x) = 0$ when $x = \sqrt[3]{30} = 3.107$

$SA(3.107) = (3.107)^2 + \dfrac{60}{(3.107)} = 28.96$

$SA = (3.107)^2 + 4(3.107)(1.55) = 28.91$

d) the minimum surface area is 29 feet^2 approximately

Example: A math center charges $400 for a course, and they get 750 students.
For every $25 increase in price, they lose 30 students.
What price would maximize revenue?
What is the domain and range?

Quadratic vertex example (finding maximum)

Since we want to 'maximize revenue', we start with a revenue function..

Revenue = (Price)(Quantity)

Then, use the details in the question to fill in the variables and constraints...

$$\underbrace{(400 + 25x)}_{\text{price}}\underbrace{(750 - 30x)}_{\text{quantity}} = \underbrace{R(x)}_{\text{revenue}}$$

(where x is the number of $25 increases)

method 1: use midpoint of zeros

$400 + 25x = 0 \qquad x = -16$

$750 - 30x = 0 \qquad x = 25$

axis of symmetry of 4.5

method 2: -b/2a

change to standard form and find (-b/2a, f(-b/2a))

$$300000 + 6750x - 750x^2$$

$$\frac{-6750}{2(-750)} = 4.5$$

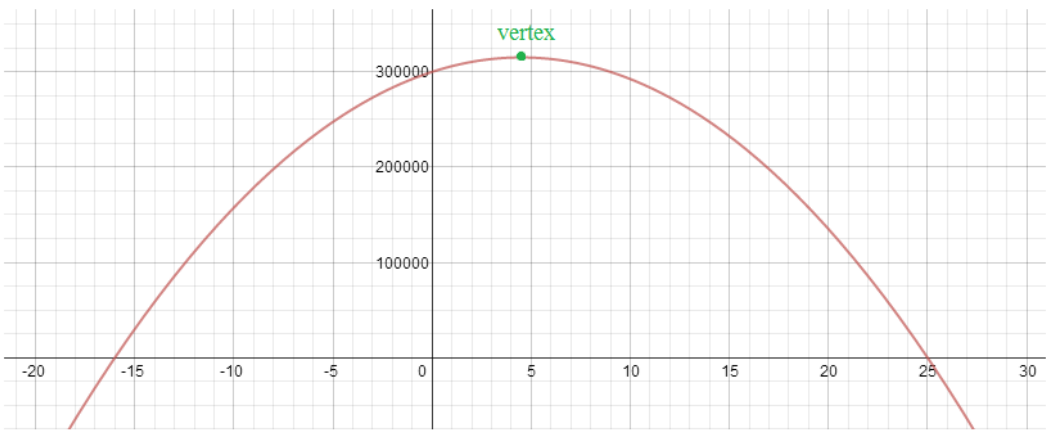
4.5 increases would lead to a price of

$(400 + 25(4.5)) = 512.50$

4.5 increases would lead to a quantity of

$(750 + 30(4.5)) = 615$

revenue = $315,187.5



domain: $-16 < x < 25$

After 16 price decreases, the price would be 0... (Free items don't have revenue!)
After 25 price increases, there will be no sales...

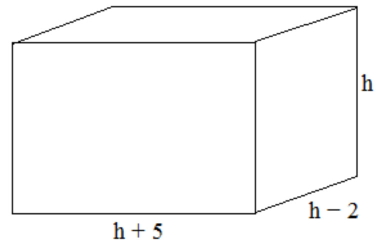
range: $0 < y < 315,187.5$

Example: The volume of a metal box is 30 cubic feet. If the length is 5 feet *greater than the height* and the width is 2 feet *less than the height*, what are the dimensions of the box?

Modeling function of one variable: *Volume*

Volume = length x width x height

$30\text{ ft}^3 = (h + 5)\text{ feet} \cdot (h - 2)\text{ feet} \cdot (h)\text{ feet}$



$h(h + 5)(h - 2) = 30$	combine 1st and 2nd terms
$(h^2 + 5h)(h - 2) = 30$	FOIL
$h^3 + 3h^2 - 10h = 30$	Set equal to zero
$\underbrace{h^3 + 3h^2}\ \underbrace{- 10h - 30} = 0$	solve for h (by grouping)

$h^2(h + 3) - 10(h + 3) = 0$ $\qquad h + 3 = 0 \qquad h = -3$

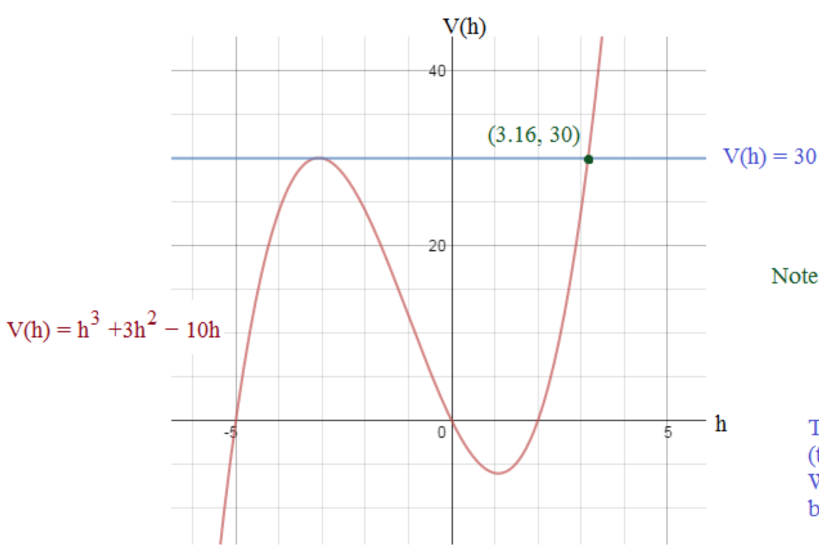
$(h + 3)(h^2 - 10) = 0$ $\qquad h^2 - 10 = 0 \qquad h = \sqrt{10}$

$h = -\sqrt{10}$

Since height cannot be negative, our solution is h =

the solution is $h = \sqrt{10}$ (approximately 3.16)

so, the dimensions of the box are approximately 8.16 x 1.16 x 3.16



Note: Height (h) cannot be negative, and Volume (V(h)) cannot be negative

Therefore, the domain is $h > 2$

The volume 30 occurs when h = 3.16 (the intersection of the line and the curve) We ignore -3.16, because height cannot be negative...

mat

m

Example: The profit model of a manufacturer is $P(x) = 75x + 3x^2 - .015x^3 - 1960$
where x is the number of toys that are produced

Analyzing a cubic model of a toy manufacturer

a) What is the break-even production level?

(16.3, 0) is the first x-intercept... So, the company loses money on the 16th toy... But, makes money producing 17 toys...

(Also, the company 'breaks even' at producing the 220 toys, although it's unlikely a rational company would produce that many toys.)

b) What is the maximum profit?

The maximum profit occurs at the local maximum..

26,261

c) What is the domain of this function?

The domain of this curve is all real numbers.. HOWEVER, there are no "negative" numbers of toys... And, presumably the manufacturer will stop producing toys when it becomes a loss.. [0, 220]

(also, a company doesn't have enough resources to produce infinite toys!)

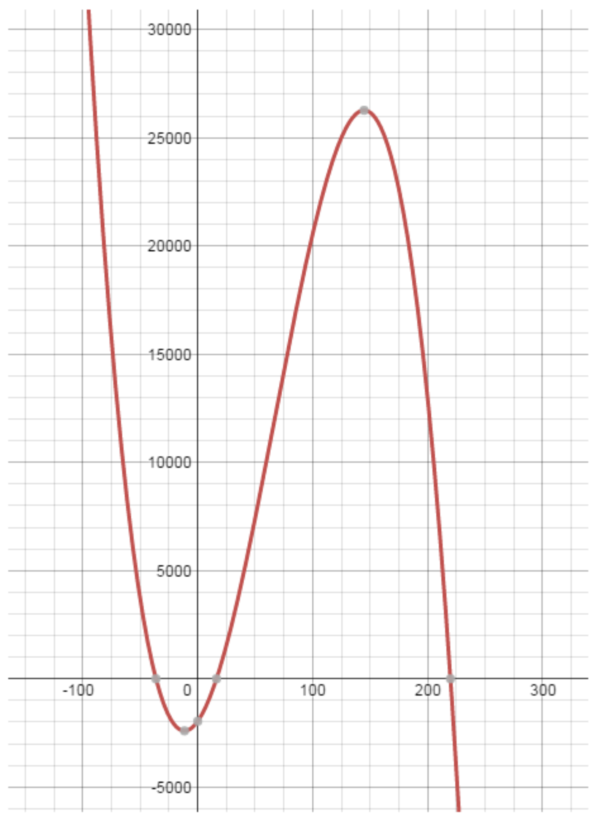
d) What levels of production does this manufacturer lose money?

$0 < x < 17$ or $x > 220$

(where x is an integer)

e) What are the manufacturer's fixed costs?

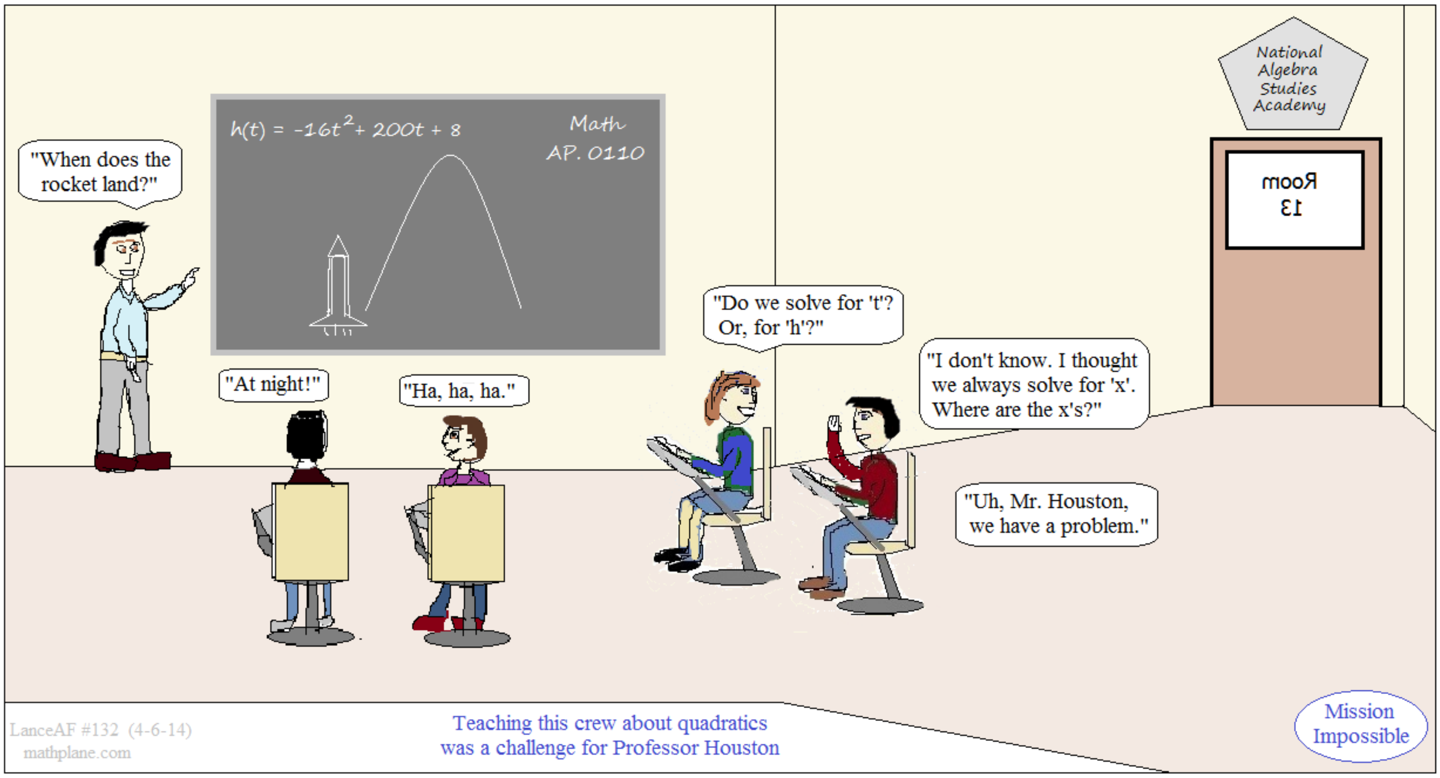
$1960 is the fixed cost, because that is the amount the manufacturer loses if it produces 0 toys.
(the y-intercept)



x-intercepts: (16.33, 0) and (220.03, 0)

y-intercept: (0, -1960)

local max: (145, 26,261)



Practice Quiz-→

Modeling Quiz

1) The quantity $Q = 2x^2 + 3y^2$ is subject to the constraint $x + y = 5$.

   a) Express the quantity Q as a *function of only x.*

   b) What is the minimum quantity of Q?

2) Assume you have 60 feet of fencing material.

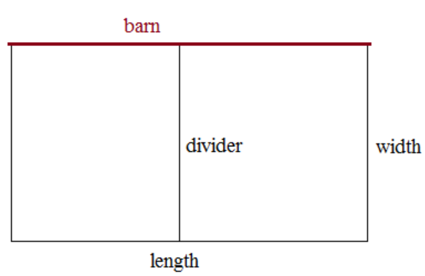
   a) What is the maximum area you could enclose?

   b) What are the dimensions of the closed area?

3) A math farmer is going to build a pen using 240 meters of wood. One side of the pen will border a barn, and there will be a wooden divider to separate the pen into 2 parts.

   a) Write a function A(x) expressing area as a function of the width x

   b) What is the maximum area of the pen?



4) You're contracted to build a *square-based* 600 cubic foot container made of steel.
Assuming the construction is an *open-top* container,

Modeling Quiz

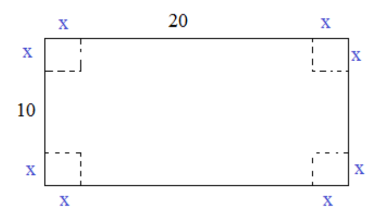
a) What are the dimensions of the container that will minimize the weight?

b) What is the surface area of the container?

5) Suppose you have a 10' x 20' piece of cardboard.
If you wanted to make an open rectangular box (by cutting out the corners & folding up the sides),

a) what dimensions would create a box with the largest volume?

b) what is the maximum volume?



Modeling Quiz

6) You operate a tour company that accommodates 60 - 90 people.
You charge the following rates:

$200 per person

If more than 60 people sign up, the rate drops *$2 per person for each additional person* after 60...

If less than 60 people sign up, the tour is cancelled.

The cost of each tour is $6000 plus $32 per person.

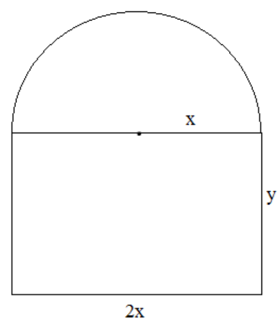
a) Write an equation that shows *profit as a function of people*

b) How many people would maximize your profit?

c) What is your maximum profit?

7) A building's window frame shaped as a rectangle with a semicircle is constructed with 28 feet of wood.

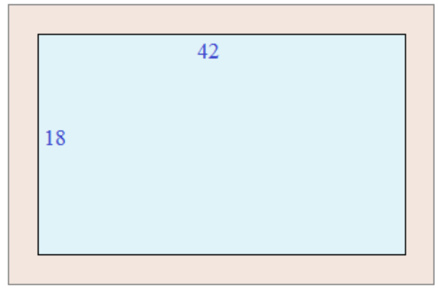
Which dimensions would maximize the light shining inside the building?



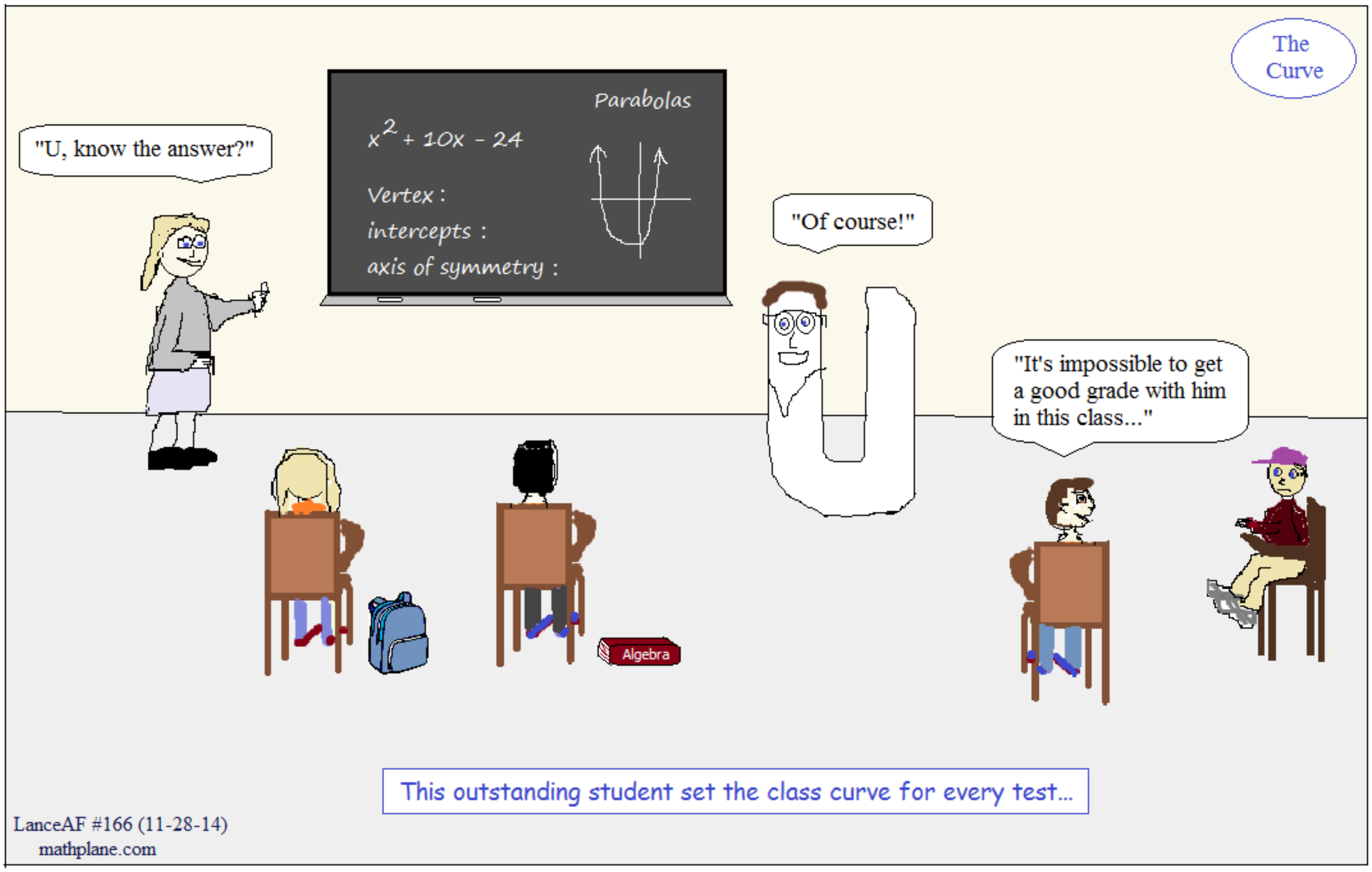
8) A rectangular swimming pool in the backyard is 18' x 42'.

Modeling Quiz

You want to embed a brick border around the entire pool.
You have 240 square feet of bricks, and the border will have a uniform width,
what will the width of the border be?



9) A golf ball travels in a parabolic arch for 270 yards from the tee to the point of contact on the green.
If the ball was 60 feet high when it had traveled 50 yards, what was the golf ball's maximum height?


The Curve
"U, know the answer?"
Parabolas
x^2 + 10x - 24
Vertex :
intercepts :
axis of symmetry :
"Of course!"
"It's impossible to get a good grade with him in this class..."
Algebra
This outstanding student set the class curve for every test...
LanceAF #166 (11-28-14)
mathplane.com

SOLUTIONS-→

Modeling Quiz

SOLUTIONS

1) The quantity $Q = 2x^2 + 3y^2$ is subject to the constraint $x + y = 5$.

a) Express the quantity Q as a *function of only x*.

b) What is the minimum quantity of Q?

a) Since Q is a function of x and y, let's change to 1 variable....

$x + y = 5 \longrightarrow y = 5 - x$ then, substitute into the main equation...

$Q = 2x^2 + 3(5 - x)^2$

$Q = 2x^2 + 75 - 30x + 3x^2$

$\boxed{Q = 5x^2 - 30x + 75}$

b) $Q = 5x^2 - 30x + 75$ is a parabola that faces up.

The minimum quantity will occur at the vertex!

The axis of symmetry: $\dfrac{-b}{2a} = \dfrac{-(-30)}{2(5)} = 3$

Since the minimum occurs at x = 3,

$\boxed{\text{the minimum quantity is } 5(3)^2 - 30(3) + 75 = 30}$

Quick Check:
minimum occurs at x = 3
and, therefore, y = 2
because x + y = 5

$Q = 2(3)^2 + 3(2)^2$

$\boxed{Q = 30}$

If x = 2 and y = 3,
then, Q = 35

If x = 4 and y = 1
then, Q = 35

2) Assume you have 60 feet of fencing material.

a) What is the maximum area you could enclose?

b) What are the dimensions of the closed area?

Step 1: Establish variables and formulas

Area = xy

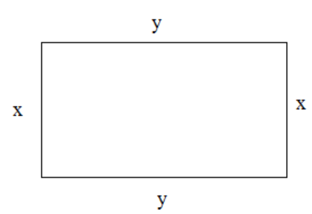
Perimeter = 60
Perimeter = 2x + 2y

60 = 2x + 2y

30 = x + y

y = 30 − x

Step 1a: Draw a diagram



Step 2: Write the equation you wish to maximize.

We want to maximize the area:
The following is the Area as a function of x,

$A(x) = xy$

$A(x) = x(30 - x)$

$A(x) = -x^2 + 30x$

Step 3: Solve and answer the questions

This is a parabola that faces down, so the maximum occurs at the vertex...

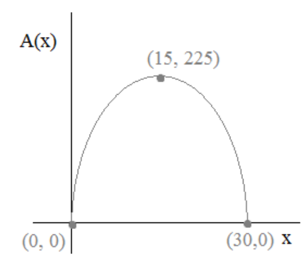
Since the zeros (x-intercepts) are at 0 and 30, the vertex occurs at the midpoint: 15

Since x = 15, y = 15

a) Area = 225 square feet

b) dimensions: 15' x 15'

Graph of one fence side
and the corresponding area



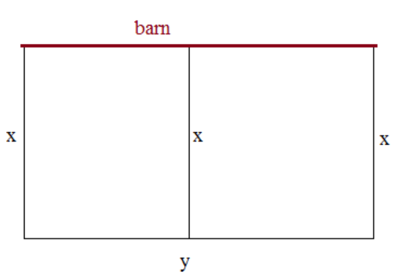
$A(x) = x(30 - x)$

where domain is (0, 30)

3) A math farmer is going to build a pen using 240 meters of wood. One side of the pen will border a barn, and there will be a wooden divider to separate the pen into 2 parts.

a) Write a function A(x) expressing area as a function of the width x

b) What is the maximum area of the pen?



Area = xy 'Main Function' that we want to maximize

240 = 3x + y 'Constraint Function'

Using substitution, we make Area *as a function of x*

Area = x(240 − 3x)

$A = 240x - 3x^2$

$\boxed{\text{a) } A(x) = -3x^2 + 240x \quad \text{where x is the width...}}$

To find the maximum algebraically, we look for the vertex:

$\dfrac{-b}{2a} = \dfrac{-240}{2(-3)} = 40$

Since x = 40, $A(40) = -3(40)^2 + 240(40) = 4800$

OR,

Since x = 40, 240 = 3(40) + y... So y = 120...
therefore, the Area = (40)(120) = 4800..

b) Maximum Area of pen: 4800 square meters

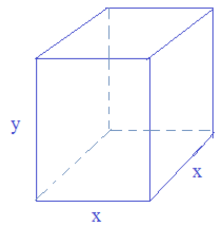
Modeling Quiz

4) You're contracted to build a *square-based* 600 cubic foot container made of steel.
Assuming the construction is an *open-top* container,

a) What are the dimensions of the container that will minimize the weight?

b) What is the surface area of the container?

Step 1: Draw a diagram and label



(since it is square-based, the length and width are equal)

Step 2: Establish formulas

Volume = (length)(width)(height)

$V = x \cdot x \cdot y = x^2 y$

$\boxed{600\text{ft}^3 = x^2 y}$

Surface Area = 4$\cdot$(area of each side) + (area of bottom)

$\boxed{SA = 4xy + x^2}$

**Since we are trying to minimize surface area, we will try to set up SA in terms of one variable (x)

$y = \dfrac{600}{x^2}$ therefore, $\boxed{SA = 4x\left(\dfrac{600}{x^2}\right) + x^2}$

Step 3: Find minimum of function!

$SA = \dfrac{2400}{x} + x^2$

Using Calculus $SA' = \dfrac{-2400}{x^2} + 2x$

(set derivative equal to zero) to find critical values

$\dfrac{-2400}{x^2} + 2x = 0$ (multiply by x^2)

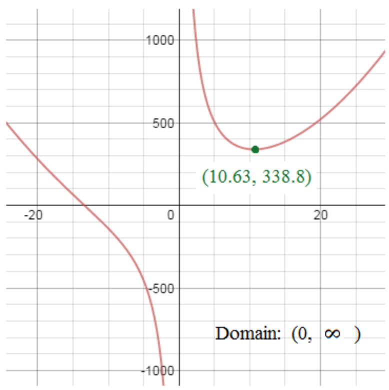
$2x^3 - 2400 = 0$

$x^3 = 1200$

$x = 2\sqrt[3]{150}$ feet

$x = 10.63$ feet (approximately)

Using a Graph



Step 4: Answer the questions:

a) What are the dimensions?

Since $x = 10.63$, $y = \dfrac{600}{(10.63)^2} = 5.31$ (approximately)

$\boxed{10.63' \text{ x } 10.63' \text{ x } 5.31'}$ → slightly more than 600 cubic feet

b) What is the surface area of the container?

$SA = 4xy + x^2$

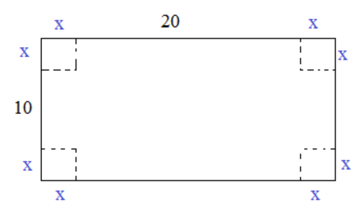
$= 4(10.63)(5.31) + (10.63)^2$

$= \boxed{338.78 \text{ square feet}}$

5) Suppose you have a 10' x 20' piece of cardboard.
If you wanted to make an open rectangular box (by cutting out the corners & folding up the sides),

a) what dimensions would create a box with the largest volume?

b) what is the maximum volume?



(note: the cut-out corners must be squares; otherwise, the top of the open box will be uneven)

Step 1: Label the diagram and write formulas

Area of cardboard box = (length)(width)

(original length)(original width) = 200 sq. ft.

Volume of open box = (length)(width)(height)

$(20 - 2x)(10 - 2x)(x)$

Step 2: Establish function

Since we want to maximize volume,

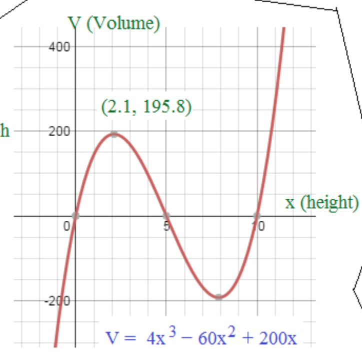
$V = (20 - 2x)(10 - 2x)(x)$

$V = (20 - 2x)(10x - 2x^2)$

$V = 200x - 40x^2 - 20x^2 + 4x^3$

$V = 4x^3 - 60x^2 + 200x$

Solve with Graph



$V = 4x^3 - 60x^2 + 200x$

where $0 < x < 5$
(all other values are impossible)

Solve with Calculus

(To find critical value -- max/min --- set first derivative equal to zero)

$V' = 12x^2 - 120x + 200 = 0$

$3x^2 - 30x + 50 = 0$

(quadratic formula)

$x = \dfrac{30 \pm \sqrt{900 - 600}}{6}$

$= 5 + \dfrac{5\sqrt{3}}{3} = \cancel{7.9}$

extraneous!
2 x 7.9 = 15.8
15.8 > the width!

$5 - \dfrac{5\sqrt{3}}{3} = 2.1$

Step 3: Answer the questions

a) What dimension creates the maximum volume?

$x = 2.1$ $\boxed{\text{(cut out 2.1' x 2.1' in each corner)}}$

length = 20 + 2x = 15.8 feet

width = 10 + 2x = 5.9 feet

b) Volume = length x width x height

15.8' x 5.9' x 2.1' = $\boxed{195.8 \text{ cubic feet}}$

6) You operate a tour company that accommodates 60 - 90 people.

SOLUTIONS

Modeling Quiz

You charge the following rates: $200 per person

If more than 60 people sign up, the rate drops *$2 per person for each additional person* after 60...

If less than 60 people sign up, the tour is cancelled.

The cost of each tour is $6000 plus $32 per person.

a) Write an equation that shows *profit as a function of people*

b) How many people would maximize your profit?

c) What is your maximum profit?

Step 1: Transform above description into math equations:

The domain is [60, 90]

Profit = Revenue − Cost

let p = # of people

Cost = $6000 + $32p

Revenue = p($200 − $2(p − 60))

Step 2: Write the equation you're seeking to maximize

$$\text{Profit} = p(\$200 - \$2(p-60)) - [\$6000 + \$32p]$$

$$= \$200p - \$2p^2 + \$120p - \$6000 - \$32p$$

$$= \$-2p^2 + \$288p - \$6000$$

$$\boxed{\text{a) } P(p) = \begin{cases} 0 & \text{if } 0 \le p < 60 \\ -2p^2 + 288p - 6000 & \text{if } 60 \le p \le 90 \\ 0 & \text{if } 90 < p \end{cases}}$$

Take derivative: $\dfrac{d\text{Profit}}{dp} = -4p + 288$

Set equal to zero to find max/min: $-4p + 288 = 0$

$p = 72$

OR, find the vertex in the parabola...

$$\frac{-b}{2a} = \frac{-(288)}{2(-2)} = 72$$

Step 3: answer questions

b) What is the optimal number of people: $\boxed{p = 72}$

c) What is your maximum profit?

Revenue: $200 x 72 = $14,400

Discount: 12 people over 60 ---- $24 discount/person

$24 x 72 = $1728

Total revenue: $12,672

Cost: $6000 + $32(72 people) = $8304

Profit: $12,672 − $8304 = $\boxed{\$4,368}$

Step 4: Check your answer

71 tourists: Revenue: $200 x 71 = $14,200
Discount: (11 x $2) x 71 = −$1562
Cost: $6000 + ($32 x 71) = −$8272
Profit: $4366

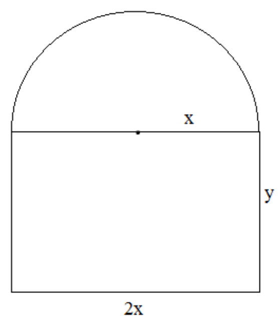
72 tourists: Profit: $4368

73 tourists: Revenue: $200 x 73 = $14,600
Discount: (13 x $2) x 73 = $1898
Cost: $6000 + ($32 x 73) = $8336
Profit: $4366

7) A building's window frame shaped as a rectangle with a semicircle is constructed with 28 feet of wood.

Which dimensions would maximize the light shining inside the building?

To maximize the light, we need to maximize the area of the window.



'Main function to optimize': $\text{Area} = \underbrace{2xy}_{\text{rectangle}} + \underbrace{\tfrac{1}{2}\pi x^2}_{\text{semicircle}}$

'Constraint function' $\text{Perimeter} = 2x + 2y + \frac{1}{2}\cdot 2\pi x$

$$28' = 2x + 2y + \pi x$$

$$y = \frac{28' - 2x - \pi x}{2}$$

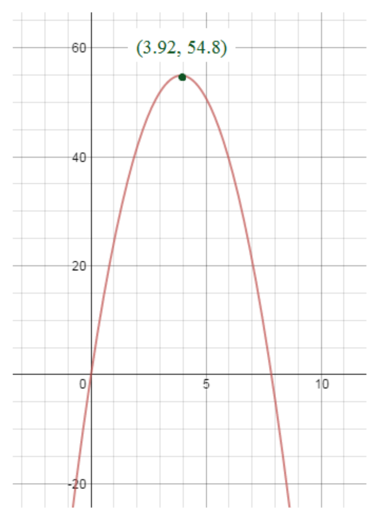
Use substitution and combine the equations

$$\text{Area} = 2x\left(\frac{28 - 2x - \pi x}{2}\right) + \frac{1}{2}\pi x^2$$

$$\text{Area} = 28x - 2x^2 - \pi x^2 + \frac{1}{2}\pi x^2$$

$$A(x) = 28x - 2x^2 - \frac{1}{2}\pi x^2$$

$$\frac{-b}{2a} = \frac{-28}{2(-3.57)} = 3.92$$



To maximize, set derivative equal to zero....

(calculus)

$$A' = 28 - 4x - \pi x$$

$$0 = 28 - 4x - \pi x$$

$$28 = 4x + \pi x$$

$$x = \frac{28}{(4 + \pi)} = 3.92 \text{ feet}$$

$$y = \frac{28' - 2(3.92') - \pi(3.92')}{2}$$

$$= \frac{28' - 7.84' - 12.32'}{2} = 3.92 \text{ feet}$$

Area of rectangle:

(2x)(y)
2(3.92)(3.92) = 30.7

Area of semicircle:

$\frac{1}{2}\pi(3.92)^2 = 24.1$

Total Area: 54.8

Dimensions:
Bottom: 7.84 feet
Left side: 3.92 feet
Right side: 3.92 feet
Arch: 3.92 π feet

total: 28 feet

Modeling Quiz

SOLUTIONS

8) A rectangular swimming pool in the backyard is 18' x 42'.

You want to embed a brick border around the entire pool.
You have 240 square feet of bricks, and the border will have a uniform width,
what will the width of the border be?

area of border — area of the pool WITH bricks — area of pool ALONE

Area = $(18 + 2x)(42 + 2x) - (18 \times 42)$

$240 = (18 + 2x)(42 + 2x) - 756$

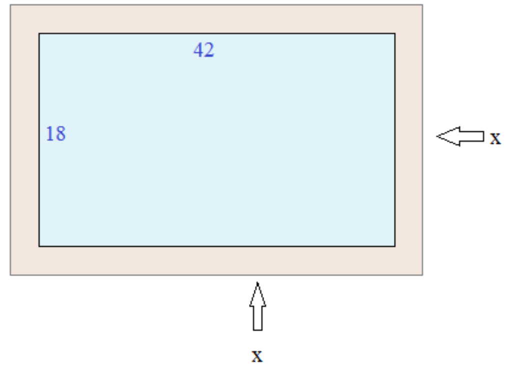
$240 = 756 + 84x + 36x + 4x^2 - 756$

$4x^2 + 120x - 240 = 0$

$x^2 + 30x - 60 = 0$

(quadratic formula) $x = \cancel{-31.88}$ or $\boxed{1.88}$

border must be > 0



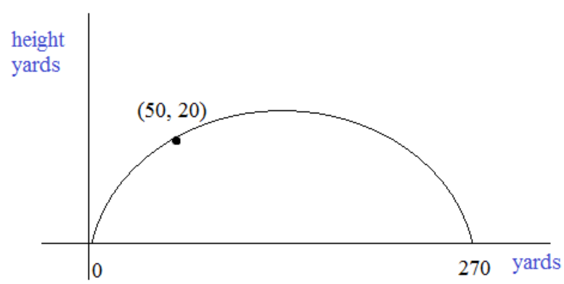
9) A golf ball travels in a parabolic arch for 270 yards from the tee to the point of contact on the green.
If the ball was 60 feet high when it had traveled 50 yards, what was the golf ball's maximum height?

Here's what we know... The golf ball travels in a *parabolic* arch...
It goes 270 yards (x)...
When it is 50 YARDS away, it was 60 FEET high...
(note the units!)

The sketch shows 3 points.. (0, 0), (50, 20), and (270, 0)

So, we have the two intercepts: (0, 0) and (270, 0)

and, a point: (50, 20)



So, we'll create a quadratic in *intercept form*....

$y = a(x - 0)(x - 270)$

to find a, we'll substitute (50, 20):

$20 = a(50 - 0)(50 - 270)$

$20 = -11{,}000a$

$a = \frac{-1}{550}$

equation of parabola:

$y = \frac{-1}{550}(x)(x - 270)$

Note: (0, 0), (270, 0) and (50, 20) all fit in the equation...

Once we have the equation, we can see the vertex (max height of the ball) occurs at the midpoint of 0 and 270 ----> 135

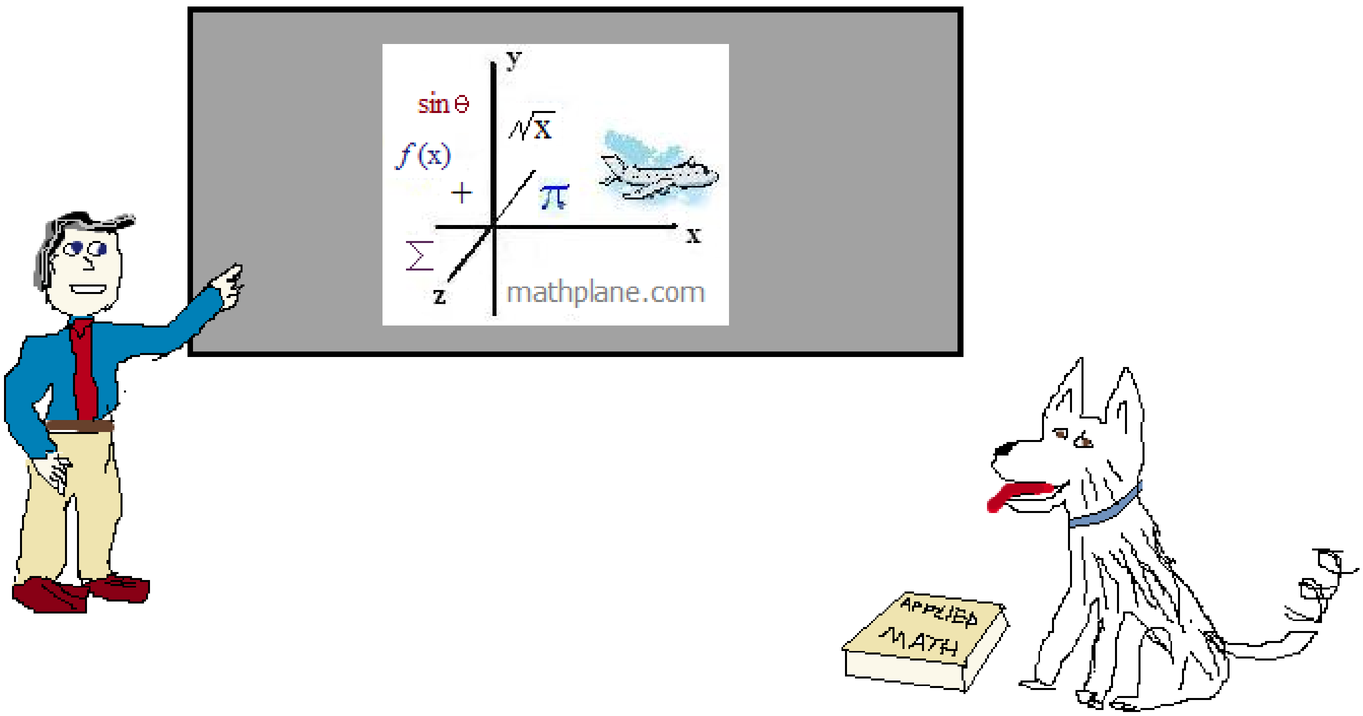
The last step:; If $x = 135$, what is y?

$y = \frac{-1}{550}(135)(-135) = \boxed{33.136 \text{ yards...}}$

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, and requests, let us know.

Cheers



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