Binomial Expansion

Notes, Examples, Formulas, and Practice

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{(n-k)} b^k$$

Topics include factorials, combinations, polynomial multiplication, Pascal's Triangle, and more

Binomial (Expansion) Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{(n-k)} b^k$$

where

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = {}_nC_k$$

Example: Expand the binomial $(2x+y)^4$

"Place the first terms" $\quad (2x)^4 + (2x)^3 + (2x)^2 + (2x)^1 + (2x)^0$

"Place the second terms" $\quad (2x)^4 (y)^0 + (2x)^3 (y)^1 + (2x)^2 (y)^2 + (2x)^1 (y)^3 + (2x)^0 (y)^4$

"Add the coefficients" $\quad \binom{4}{0}(2x)^4 (y)^0 + \binom{4}{1}(2x)^3 (y)^1 + \binom{4}{2}(2x)^2 (y)^2 + \binom{4}{3}(2x)^1 (y)^3 + \binom{4}{4}(2x)^0 (y)^4$

Simplify $\quad (1)(16x^4)(1) + (4)(8x^3)(y) + (6)(4x^2)(y^2) + (4)(2x)(y^3) + (1)(1)(y^4)$

$$16x^4 + 32x^3 y + 24x^2 y^2 + 8xy^3 + y^4$$

Finding the term:

where $n \geq (r-1)$

The r^{th} term of the expansion $(a+b)^n$ is

$$\binom{n}{r-1} a^{n-(r-1)} b^{(r-1)}$$

Example: Find the <u>15th term</u> of $(x^2+y)^{22}$

$$\binom{22}{15-1} (x^2)^{22-(15-1)} (y)^{(15-1)}$$

$$\binom{22}{14} (x^2)^8 (y)^{14}$$

$$319{,}770x^{16}y^{14}$$

Alternative method (if you forget the formula)...
"Begin the binomial expansion and determine the pattern!"

1st term $\quad$ 2nd term $\quad$ 3rd term

$$\binom{22}{0}(x^2)^{22}(y)^0 + \binom{22}{1}(x^2)^{21}(y)^1 + \binom{22}{2}(x^2)^{20}(y)^2$$

so, the 15th term will be... $\binom{22}{14}$

$(x^2)^8$

$(y)^{14}$

(observe: the difference between the 1st term and 15th term is 14 terms...)

Example: Find the constant term in the binomial expansion of $(x^2 + \frac{3}{x})^{15}$

a = x^2, b = $\frac{3}{x}$

since the 'a' and 'b' terms are multiplied, we need to figure out which term would cancel the variable x...
In other words, which term will have an exponent in the 3/x term that is double the exponent in the x^2 term

$${}_{15}C_5 \cdot (x^2)^5 (\frac{3}{x})^{10}$$

$$3003 \cdot x^{10} \cdot \frac{59049}{x^{10}} = 177324147$$

Pascal's Triangle

```
                1
              1   1
            1   2   1
          1   3   3   1
        1   4   6   4   1
      1   5  10  10   5   1
    1   6  15  20  15   6   1
  1   7  21  35  35  21   7   1
1   8  28  56  70  56  28   8   1
```

Note: each number is the sum of the adjoining number(s) above...

$\binom{6}{2}$ ⇒ 7th row, 3rd position... 15

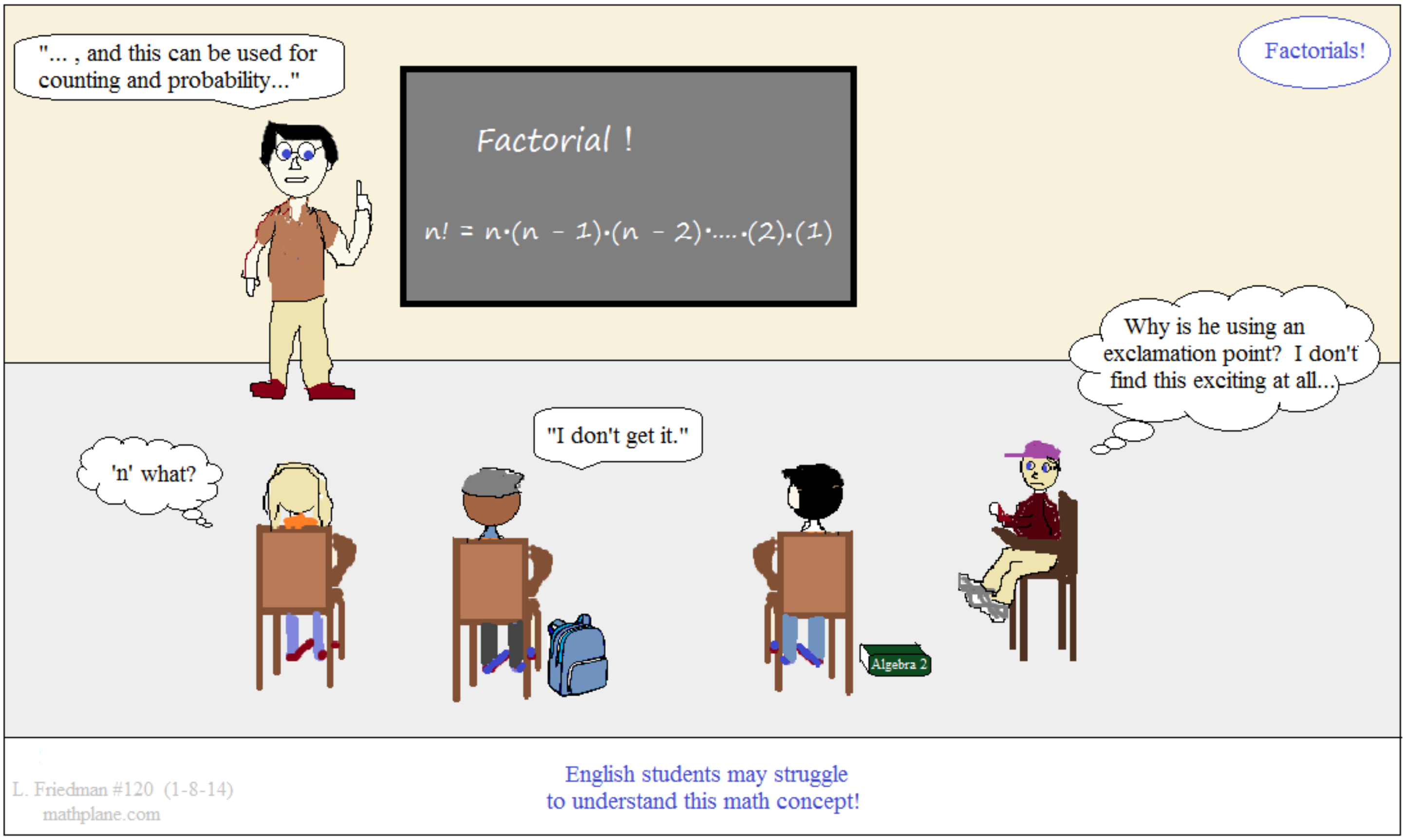
$$\frac{6!}{4!\,2!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{(4 \times 3 \times 2 \times 1) \times 2 \times 1} = \frac{30}{2} = 15$$

Example: What is $\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}$?

This is the 6th row in Pascal's Triangle...

```
                1
              1   1
            1   2   1
          1   3   3   1
        1   4   6   4   1
      1   5  10  10   5   1
    1   6  15  20  15   6   1
  1   7  21  35  35  21   7   1
1   8  28  56  70  56  28   8   1
```

1 + 5 + 10 + 10 + 5 + 1 = 32



Practice Exercises -→

Binomial Expansion Quiz

I. Expand the following:

1) $(x + y)^5 =$

2) $(m - p)^7 =$

3) $(b + 2)^4 =$

4) $(2x - 3)^5 =$

5) $(x^2 + y^3)^6 =$

II. Find the term:

1) $(x + 4y)^7$ 5th term

2) $(3x - 2)^{12}$ 6th term

3) $(2m + p^2)^{24}$ 9th term

III. More Questions and Concepts

Binomial Expansion Quiz

A) Condense/Simplify the following Binomial Expansions...

1) $\binom{5}{0}a^{10} + \binom{5}{1}3a^{8}b + \binom{5}{2}9a^{6}b^{2} + \binom{5}{3}27a^{4}b^{3} + \binom{5}{4}81a^{2}b^{4} + \binom{5}{5}243b^{5}$

2) $\binom{4}{0}5^{4} + \binom{4}{1}5^{3}(-2)^{1} + \binom{4}{2}5^{2}(-2)^{2} + \binom{4}{3}5^{1}(-2)^{3} + \binom{4}{4}(-2)^{4}$

B) Use binomial expansion to find:

1) $(1.001)^{4}$ Hint: $.001 = 10^{-3}$

2) $(.998)^{3}$ Hint: $.002 = 2(10)^{-3}$

C) Miscellaneous

1) What is the 5th term in the expansion of $(x + 3y^{2})^{5}$?

2) What is the coefficient of the st^{2} term in the expansion of $(s - 5t)^{3}$?

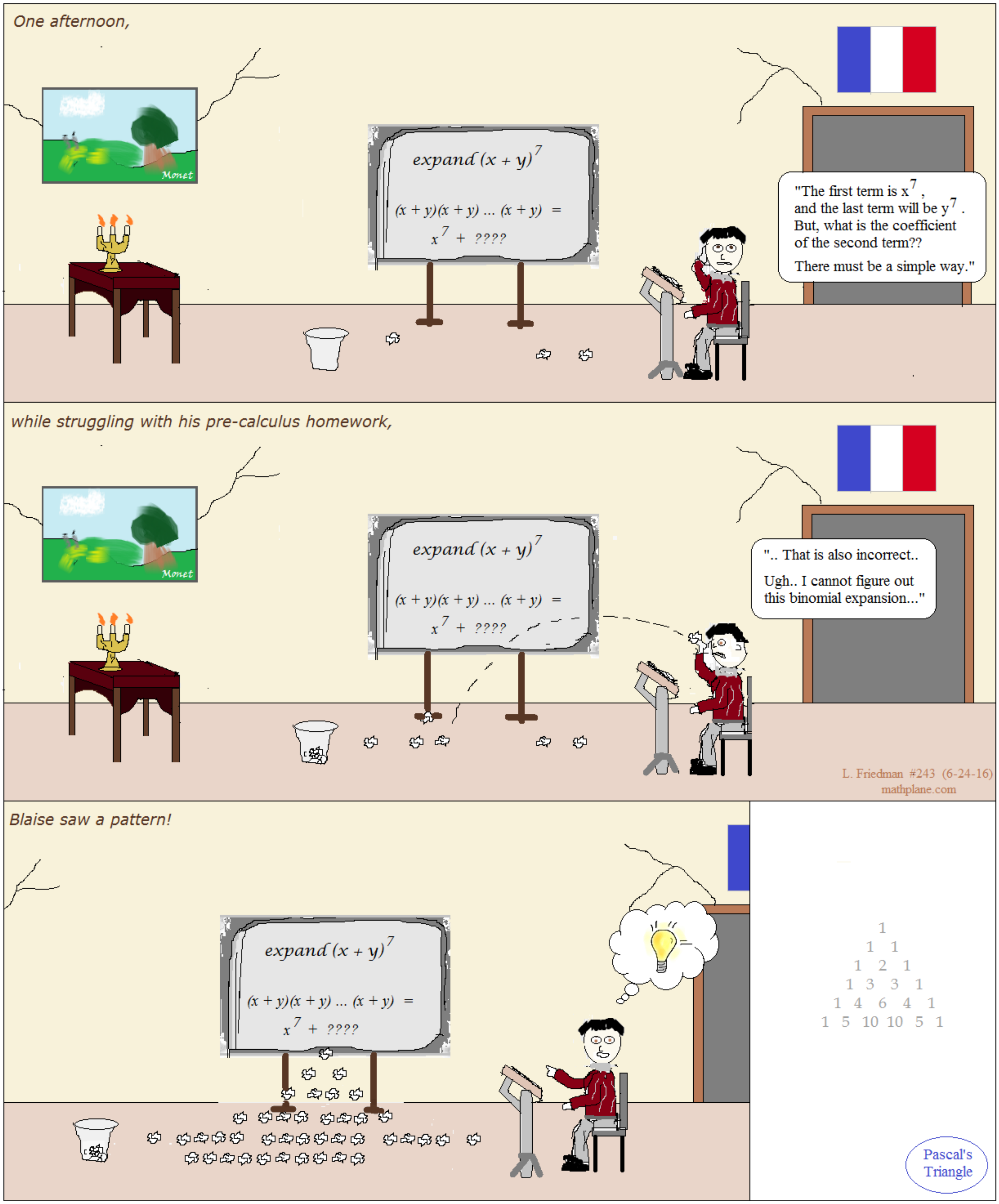
3) In the expansion of $(2k + 2)^{18}$, what is the term that includes k^{7} ?

4) Expand the following $(2 + 3i)^5$

5) Find the x^3 term from the expansion $(2x + \frac{8}{x})^7$

6) Solve the following: find n

$$\binom{n}{6} = 3\binom{n-1}{5}$$



One afternoon,
expand (x + y)^7
(x + y)(x + y) ... (x + y) =
x^7 + ????
Monet
"The first term is x^7 , and the last term will be y^7 . But, what is the coefficient of the second term?? There must be a simple way."
while struggling with his pre-calculus homework,
expand (x + y)^7
(x + y)(x + y) ... (x + y) =
x^7 + ????
Monet
".. That is also incorrect.. Ugh.. I cannot figure out this binomial expansion..."
L. Friedman #243 (6-24-16)
mathplane.com
Blaise saw a pattern!
expand (x + y)^7
(x + y)(x + y) ... (x + y) =
x^7 + ????
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
Pascal's Triangle

Binomial Expansion Quiz

I. Expand the following: SOLUTIONS

1) $(x+y)^5 =$

step 1: x terms $x^5 \quad + x^4 \quad + x^3 \quad + x^2 \quad + x^1 \quad + x^0$

step 2: y terms $y^0 + \quad y^1 + \quad y^2 + \quad y^3 + \quad y^4 + \quad y^5$

step 3: coefficients $\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$

$$x^5 + 5x^4y + 10x^3y^3 + 10x^2y^3 + 5xy^4 + y^5$$

2) $(m-p)^7 =$

$$m^7 - 7m^6p + 21m^5p^2 - 35m^4p^3 + 35m^3p^4 - 21m^2p^5 + 7m^6p - p^7$$

3) $(b+2)^4 =$

step 1: first terms --- $b^4 \quad b^3 \quad b^2 \quad b^1 \quad b^0$

step 2: second terms --- $2^0 \quad 2^1 \quad 2^2 \quad 2^3 \quad 2^4$

step 3: coefficients --- $\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$

combine and simplify --- $b^4 + 8b^3 + 24b^2 + 32b + 16$

4) $(2x-3)^5 =$

first term: $\binom{5}{0}(2x)^5(-3)^0 = \frac{5!}{5!\,0!}(32x^5)(1)$

second term: $\binom{5}{1}(2x)^4(-3)^1 = \frac{5!}{4!\,1!}(16x^5)(-3) = -15(16x^5)$

etc....

$$32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243$$

5) $(x^2+y^3)^6 =$

$$x^{12} + 6x^{10}y^3 + 15x^8y^6 + 20x^6y^9 + 15x^4y^{12} + 6x^2y^{15} + y^{18}$$

Binomial (Expansion) Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{(n-k)} b^k$$

II. Find the term: using the formula:

1) $(x+4y)^7$ 5th term

$\binom{7}{5-1} x^{7-(5-1)}(4y)^{5-1}$

$\binom{7}{4} x^3(4y)^4 = 35x^3(256y^4) = 8960x^3y^4$

2) $(3x-2)^{12}$ 6th term find the pattern:

1st term: $\binom{12}{0}(3x)^{12}(-2)^0$ the 6th term will be $\binom{12}{5} \quad (3x)^7 \quad (-2)^5$

2nd term: $\binom{12}{1}(3x)^{11}(-2)^1$ $\quad 792 \quad 2187x^7 \quad -32$ → $-55427328x^7$

3) $(2m+p^2)^{24}$ 9th term

$\binom{24}{8}(2m)^{16}(p^2)^8 = 735{,}471 \quad 65{,}536m^{16} \quad p^{16}$

$= 48{,}199{,}827{,}456m^{16}p^{16}$

The r^{th} term of the expansion $(a+b)^n$

$$\binom{n}{r-1} a^{n-(r-1)} b^{(r-1)}$$

III. More Questions and Concepts

Binomial Expansion Quiz

SOLUTIONS

A) Condense/Simplify the following Binomial Expansions...

1) $\binom{5}{0}a^{10} + \binom{5}{1}3a^{8}b + \binom{5}{2}9a^{6}b^{2} + \binom{5}{3}27a^{4}b^{3} + \binom{5}{4}81a^{2}b^{4} + \binom{5}{5}243b^{5}$ $\boxed{(a^2 + 3b)^5}$

2) $\binom{4}{0}5^4 + \binom{4}{1}5^3(-2)^1 + \binom{4}{2}5^2(-2)^2 + \binom{4}{3}5^1(-2)^3 + \binom{4}{4}(-2)^4$ $(5 + (-2))^4 = \boxed{81}$

B) Use binomial expansion to find:

1) $(1.001)^4$ Hint: $.001 = 10^{-3}$

From Pascal's triangle the coefficients will be 1, 4, 6, 4, 1

$(1 + 10^{-3})^4$

$1 \cdot 1^4 \cdot 10^0 + 4 \cdot 1^3 \cdot 10^{-3} + 6 \cdot 1^2 \cdot 10^{-6}\ \ 4 \cdot 1^3 \cdot 10^{-9}\ \ 1 \cdot 1^4 \cdot 10^{-12}$

$1 + .004 + .000006 + .000000004 + .000000000001$

$\boxed{1.004006004001}$

2) $(.998)^3$ Hint: $.002 = 2(10)^{-3}$

From the 4th row in Pascal's triangle, the coefficients will be 1, 3, 3, 1

$(1 - (2)10^{-3})^3$

$1 \cdot 1^3 \cdot ((-2)10^{-3})^0 + 3 \cdot 1^2 \cdot ((-2)10^{-3})^1 + 3 \cdot 1^1 \cdot ((-2)10^{-3})^2 + 1 \cdot 1^0 \cdot ((-2)10^{-3})^3$

$1 + -.006 + .000012 + -.000000008$

$\boxed{.994011992}$

C) Miscellaneous

1) What is the 5th term in the expansion of $(x + 3y^2)^5$?

Note, since x^5 is in the first term, x^1 is in the fifth term...
Remember, a power of 5 means there will be six terms...

Coefficients for 5th power: 1, 5, 10, 10, 5, 1

$\binom{5}{0}x^5(3y^2)^0 + \dots + \binom{5}{4}x^1(3y^2)^4 + \binom{5}{5}x^0(3y^2)^5$

(first term) (fifth term) (sixth term)

$5 \cdot x^1 \cdot 81y^8 = \boxed{405xy^8}$

2) What is the coefficient of the st^2 term in the expansion of $(s - 5t)^3$?

The st^2 term occurs when the coefficient is 3...

Coefficients for 3rd power: 1, 3, 3, 1

$1s^3(-5t)^0 + 3s^2(-5t)^1 + \boxed{3s^1(-5t)^2} + 1s^0(-5)^3$

$s^3 - 15s^2t + 75st^2 - 125$ $\boxed{\text{coefficient is } 75}$

3) In the expansion of $(2k + 2)^{18}$, what is the term that includes k^7 ?

$\dots + \binom{18}{11}(2k)^7(2)^{11} + \dots$

$31824 \cdot 2^{18}k^7$

$\boxed{8342470656k^7}$

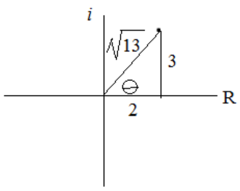
Binomial Expansion Quiz

SOLUTIONS

4) Expand the following $(2 + 3i)^5$

Method 1: DeMoivre's Theorem

Step 1: convert into polar *cis* form



$r = \sqrt{13}$

$\theta = 56.3099$

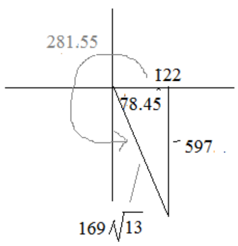
$\sqrt{13}\ cis\ 56.3099^\circ$

Step 2: apply DeMoivre's Theorem

$$[\,r cis \theta\,]^n = r^n\ cis(n\theta)$$

$$[\sqrt{13}\ cis\ 56.3099\,]^5 = 169\sqrt{13}\ cis\ 281.55$$

Step 3: convert to rectangular complex form



$121 - 597i$

Method 2: Binomial Expansion Theorem

$$(2 + 3i)^5$$

Step 1: Apply first part of binomial expansion

$$2^5(3i)^0 + 2^4(3i)^1 + 2^3(3i)^2 + 2^2(3i)^3 + 2^1(3i)^4 + 2^0(3i)^5$$

Step 2: Add the coefficients (using Pascal's triangle or combinations)

$$\binom{5}{0}2^5(3i)^0 + \binom{5}{1}2^4(3i)^1 + \binom{5}{2}2^3(3i)^2 + \binom{5}{3}2^2(3i)^3 + \binom{5}{4}2^1(3i)^4 + \binom{5}{5}2^0(3i)^5$$

$$32 + 80(3i) + 80(9i^2) + 40(27i^3) + 10(81i^4) + 243i^5$$

$$32 + 240i - 720 - 1080i + 810 + 243i$$

$121 - 597i$

5) Find the x^3 term from the expansion $(2x + \frac{8}{x})^7$

$(2x)^7(\frac{8}{x})^0 \Rightarrow x^7$ $(2x)^4(\frac{8}{x})^3 \Rightarrow x^1$

$(2x)^6(\frac{8}{x})^1 \Rightarrow x^5$ $(2x)^3(\frac{8}{x})^4 \Rightarrow x^{-1}$

$(2x)^5(\frac{8}{x})^2 \Rightarrow x^3$ etc...

occurs when $\binom{7}{5}(2x)^5(\frac{8}{x})^2 \Rightarrow 21(32x^5)(\frac{64}{x^2})$

$43008x^3$

6) Solve the following: find n

$$\binom{n}{6} = 3\binom{n-1}{5}$$

$$\frac{n!}{6!\,(n-6)!} = \frac{3(n-1)!}{(n-6)!\cdot 5!}$$

$$\frac{n!}{6!\,\cancel{(n-6)!}} = \frac{3(n-1)!}{\cancel{(n-6)!}\cdot 5!}$$

$$\frac{n!}{\cancel{6!}\ 6} = \frac{3(n-1)!}{\cancel{5!}}$$

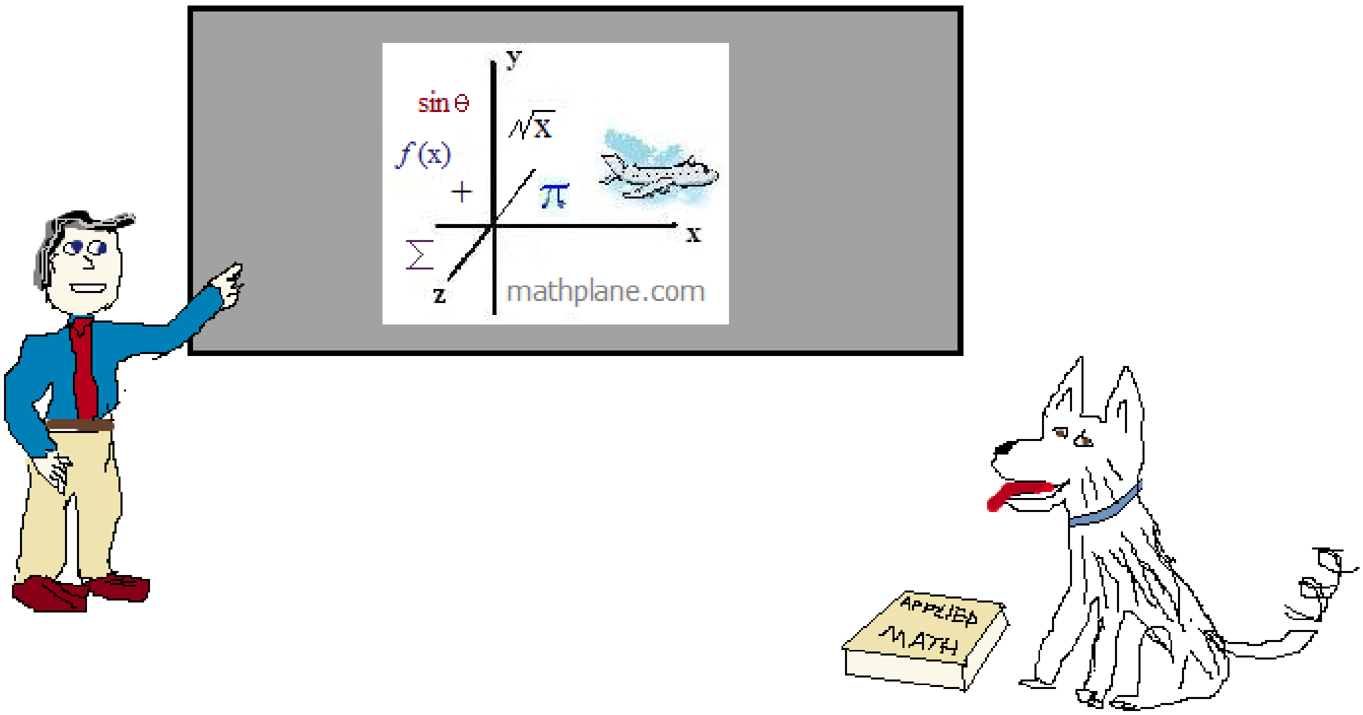
$$\frac{\cancel{n!}\ n}{6} = \frac{3\cancel{(n-1)!}}{1}$$

answer: $n = 18$

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers



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