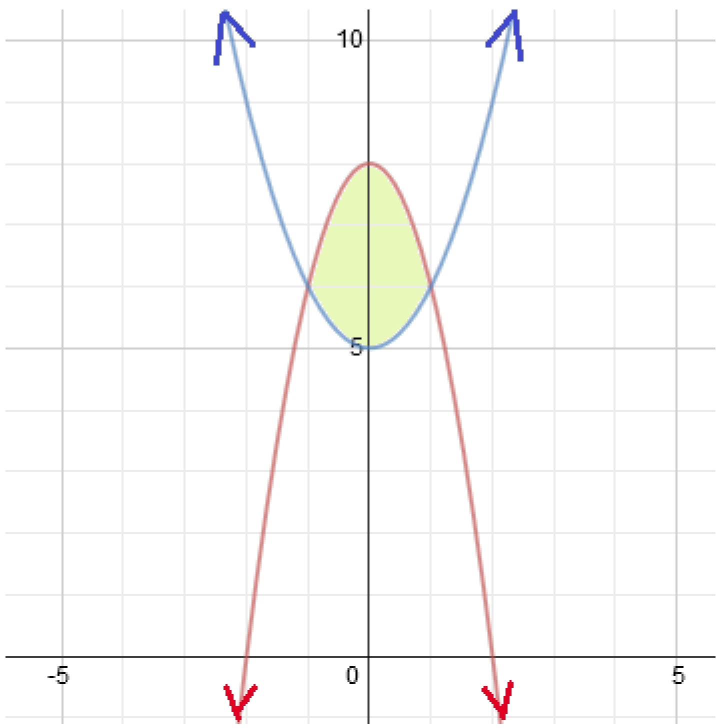
Calculus: Definite Integrals & Area between Curves



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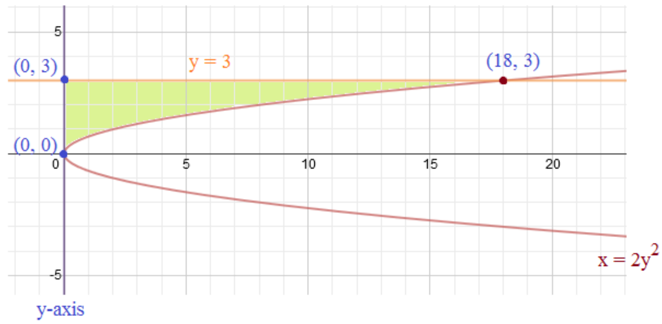
Finding the area between curves can be solved using integrals with either vertical or horizontal partitions....

Example: Find the area bounded by $y = 3$, the y-axis, and $x = 2y^2$

Step 1: Draw a quick sketch to see the region



Step 2: Determine the integral

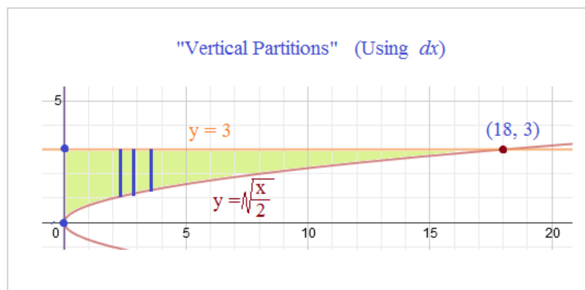
The left boundary is $x = 0$
The right boundary is the intersection of $y = 3$ and $x = 2y^2$ @ $x = 18$

$$\int_0^{18} \qquad dx$$

The upper boundary is $y = 3$

$$\int_0^{18} 3 \qquad dx$$

The lower boundary is $x = 2y^2 \longrightarrow y = \sqrt{\frac{x}{2}}$



Step 3: Solve

$$\int_0^{18} 3 - \sqrt{\frac{x}{2}}\, dx = 3x - \frac{2}{3}\cdot\frac{x^{3/2}}{\sqrt{2}}\Bigg|_0^{18} = 54 - \frac{2}{3}\cdot\frac{18^{3/2}}{2^{1/2}} = \boxed{18}$$

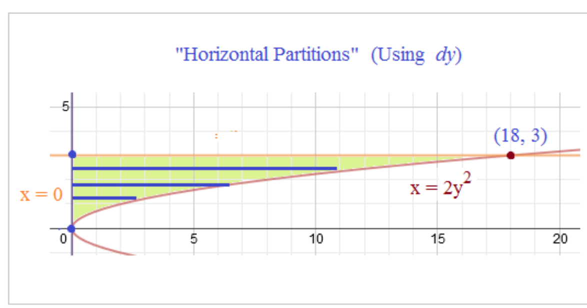
Step 2A: Determine the integral

The upper boundary is the intersection of $y = 3$ and $x = 2y^2$ @ $y = 3$
The lower boundary is the intersection of $x = 2y^2$ and the y-axis @ $y = 0$

$$\int_0^{3} \qquad dy$$

The "outer" boundary is the function on the right: $x = 2y^2$
The "inner" boundary is the function on the left: $x = 0$ (the y-axis)

$$\int_0^{3} 2y^2 - 0 \quad dy$$

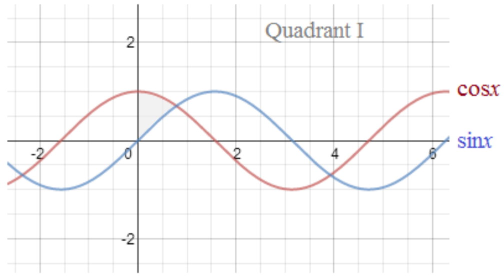


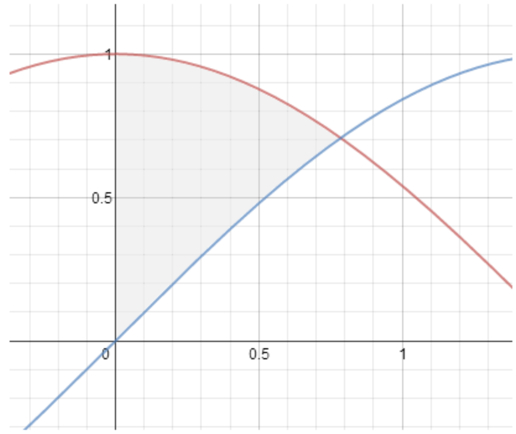
Step 3A: Solve

$$\int_0^{3} 2y^2 - 0 \quad dy = \int_0^{3} 2y^2 = \frac{2y^3}{3}\Bigg|_0^{3} = \frac{54}{3} - \frac{0}{3} = \boxed{18}$$

Example: Find the area in Quadrant I between cos*x* and sin*x*

Step 1: Sketch the graph





Step 2: Determine the boundaries

The intersection of cos*x* and sin*x* is $\frac{\pi}{4}$

$$\sin x = \cos x$$

$$\frac{\sin x}{\cos x} = 1$$

$$\tan x = 1$$

And, the intersections with Quadrant I imply a left boundary of $x = 0$

boundary of integral will be 0 and $\frac{\pi}{4}$

$$\int_0^{\frac{\pi}{4}} \quad dx$$

Step 3: Write the functions in the integral

Since cos*x* is above sin*x* in the interval $[0, \frac{\pi}{4}]$,

$$\int_0^{\frac{\pi}{4}} \cos x - \sin x \; dx$$

Step 4: Solve

$$\sin x + \cos x \Big|_0^{\frac{\pi}{4}} = \sin(\frac{\pi}{4}) + \cos(\frac{\pi}{4}) - \left(\sin(0) + \cos(0)\right)$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0 + 1)$$

$$= \sqrt{2} - 1 \cong \boxed{.414}$$

Example: The area of the region is 2.

If the equations that form the boundary are

$y = k\cos x$

$y = kx^2$

what is k?

Determine the boundary of the integral

$k\cos x = kx^2$

$x = .824$ and $-.824$

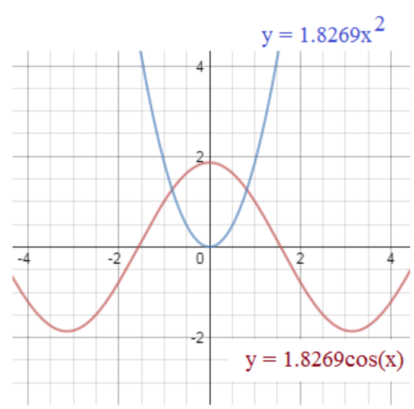
$$\int_{-.824}^{.824} k\cos x - kx^2 \; dx = 2$$

$$k \int_{-.824}^{.824} \cos x - x^2 \; dx = 2$$

$$\int_{-.824}^{.824} \cos x - x^2 \; dx = \frac{2}{k}$$

$$1.0948 = \frac{2}{k}$$

$$\boxed{k = 1.8269}$$



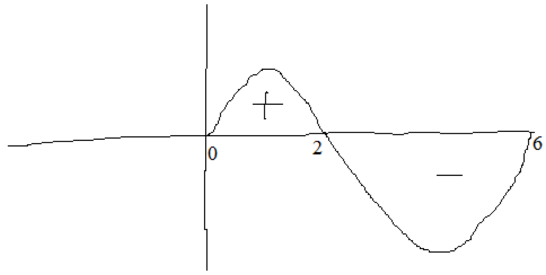
A sketch of the graph shows the answer is reasonable...

"Definite Integrals vs. Area under the curve"

When you find the "area under the curve", you're evaluating the integral and subtracting 0 (the x-axis)....

so, if the function is under the x-axis, the value would be *negative*.

Since area cannot be negative, the value of the definite integral must be turned positive...
Either "flip the sign" or use the absolute value...



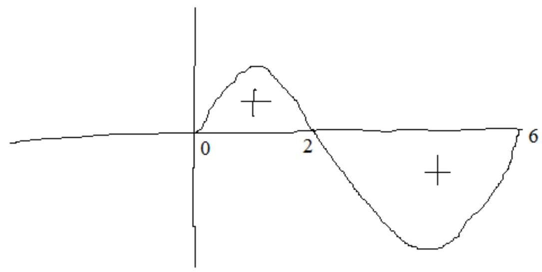
integral ⇒ $\int_0^6$

use absolute value or flip sign...

area betwen curve and x-axis ⇒ $\int_0^2 + \left| \int_2^6 \right|$ OR $\int_0^2 - \int_2^6$

To find the area between two functions, you take the integral of the "above" function and subtract the integral of the "below" function.

But, if the functions intersect, they swap places...



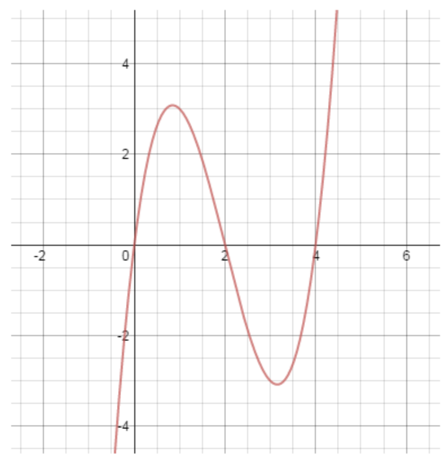
Example: Find the area between the x-axis and $f(x) = x^3 - 6x^2 + 8x$

We need to identify the boundaries of the integral(s)..
Find any intercepts..

$$x^3 - 6x^2 + 8x = 0$$

$$x(x-2)(x-4) = 0$$

The intercepts are at 0, 2, and 4 (See the graph)



$$\int_0^2 \qquad \int_2^4$$

And, since the 2nd half is below the x-axis, we'll flip the sign to create a positive value for area

$$\int_0^2 x^3 - 6x^2 + 8x \; dx - \int_2^4 x^3 - 6x^2 + 8x \; dx$$

$$\left. \frac{x^4}{4} - 2x^3 + 4x^2 \right|_0^2 - \left. \frac{x^4}{4} - 2x^3 + 4x^2 \right|_2^4 = \Big((4 - 16 + 16) - (0 - 0 + 0)\Big) - \Big((64 - 128 + 64) - (4 - 16 + 16)\Big) = \boxed{4}$$

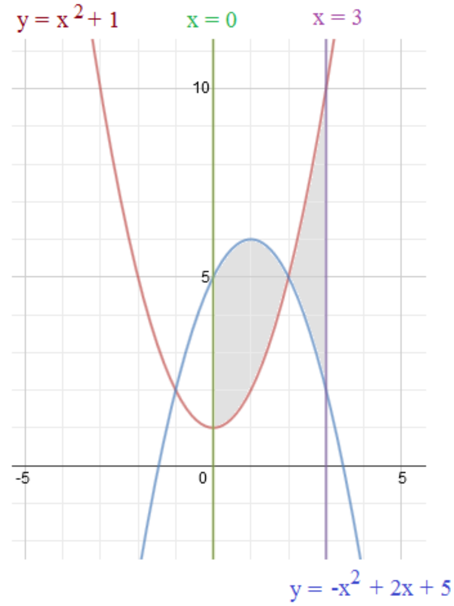
Note: the integral of the function, evaluated from 0 to 4, is

$$\int_0^4 x^3 - 6x^2 + 8x \; dx = \left. \frac{x^4}{4} - 2x^3 + 4x^2 \right|_0^4 = 64 - 128 + 64 - (0 - 0 + 0) = 0$$

The integral is 0..

Example: Find the area of the regions bounded by $y = x^2 + 1$, $y = -x^2 + 2x + 5$, $x = 0$, $x = 3$

Area between curves and boundaries



Since the functions 'cross over', it'll be helpful to divide the region into 2 parts...

Where do the functions cross? Find the intersection...

$$x^2 + 1 = -x^2 + 2x + 5$$

$$2x^2 - 2x - 4 = 0$$

$$2(x - 2)(x + 1) = 0$$

Intersections occur at (-1, 2) and (2, 5)

We'll evaluate the 'left' region from 0 to 2...
And, the 'right' region from 2 to 3....

$$\int_0^2 \left(-x^2 + 2x + 5\right) - \left(x^2 + 1\right)\,dx + \int_2^3 \left(x^2 + 1\right) - \left(-x^2 + 2x + 5\right)\,dx$$

left region: since downward parabola is above the upward facing parabola, it goes first...

right region: since upward parabola is above the downward facing parabola, it goes first in the integral..
(otherwise, value would be negative)

$$\int_0^2 -2x^2 + 2x + 4\,dx + \int_2^3 2x^2 - 2x - 4\,dx$$

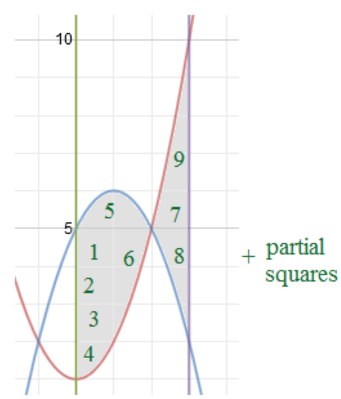
$$\left.\frac{-2x^3}{3} + x^2 + 4x\right|_0^2 + \left.\frac{2x^3}{3} - x^2 - 4x\right|_2^3$$

$$-16/3 + 4 + 8 - (0 + 0 + 0) + 18 - 9 - 12 - (16/3 - 4 - 8)$$

$$6\ 2/3 - 3 + 6\ 2/3 = \boxed{10\ 1/3}$$

Check for reasonableness:

If you count/estimate the number of shaded boxes in the graph, 10 1/3 square units seems reasonable!

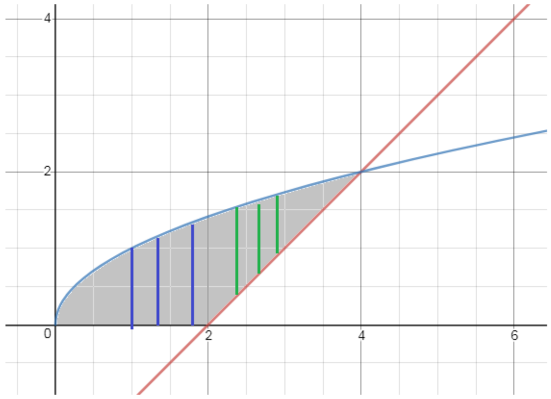


+ partial squares

Example: Find the area bounded by $y = \sqrt{x}$ and $y = x - 2$ that lies in quadrant 1

a) integrate with respect to x

b) integrate with respect to y



a) integrating *with respect to x*

In this case, we'll use *vertical partitions* and *dx*

First, identify the boundaries...

From the graph, x ranges from 0 to 4....

Note: You could solve algebraically,

$x - 2 = \sqrt{x}$

square both sides

$x^2 - 4x + 4 = x$

$x^2 - 5x + 4 = 0$

$(x - 1)(x - 4) = 0$

~~x = 1~~ and $x = 4$ ⟹ (4, 2)

extraneous

$y = \sqrt{x}$ and the x-axis

intersection at (0, 0)

$y = x - 2$ and the x-axis

intersection at (2, 0)

$$\int_0^2 \quad dx \qquad \int_2^4 \quad dx$$

Second, identify each partition, and write the equation...

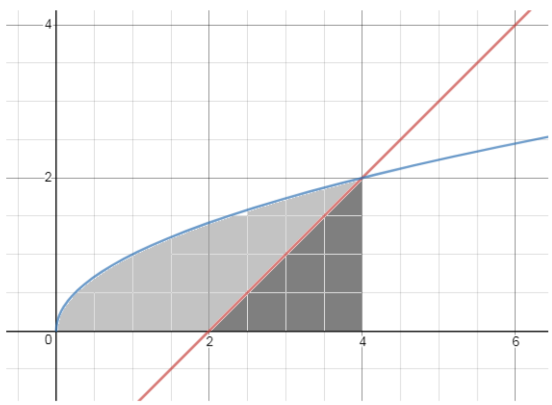
$$\int_0^2 \sqrt{x} - 0 \, dx + \int_2^4 \sqrt{x} - (x - 2) \, dx$$

Third, integrate...

$$\left. \frac{2x^{\frac{3}{2}}}{3} \right|_0^2 + \left. \frac{2x^{\frac{3}{2}}}{3} - \frac{x^2}{2} + 2x \right|_2^4$$

$$\frac{4\sqrt{2}}{3} + \frac{16}{3} - 8 + 8 - \left(\frac{4\sqrt{2}}{3} - 2 + 4 \right)$$

$$\frac{16}{3} - 2 = \boxed{\frac{10}{3}}$$

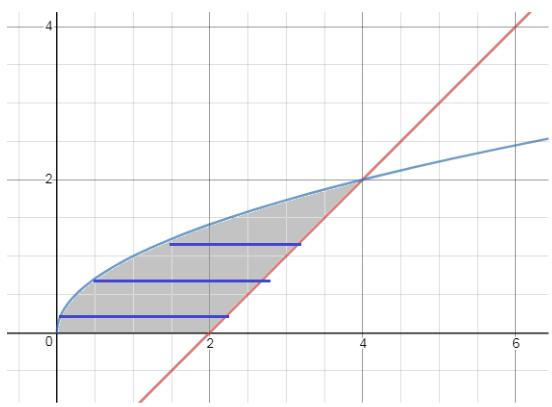


NOTE: You could integrate $\sqrt{x}$ from 0 to 4..

Then, subtract the right triangle...

$$\int_0^4 \sqrt{x} \, dx = \left. \frac{2x^{\frac{3}{2}}}{3} \right|_0^4 = \frac{16}{3}$$

then, subtract the area of the triangle (-2) $\boxed{\frac{10}{3}}$



b) integrating *with respect to y*

In this case, we'll use *horizontal partitions* and *dy*

Step 1: Write equations in terms of y

$y = \sqrt{x}$ ⟹ $x = y^2$

$y = x - 2$ ⟹ $x = y + 2$

Step 2: Establish the boundary for the integral

In quadrant I, the range y goes from 0 to 2

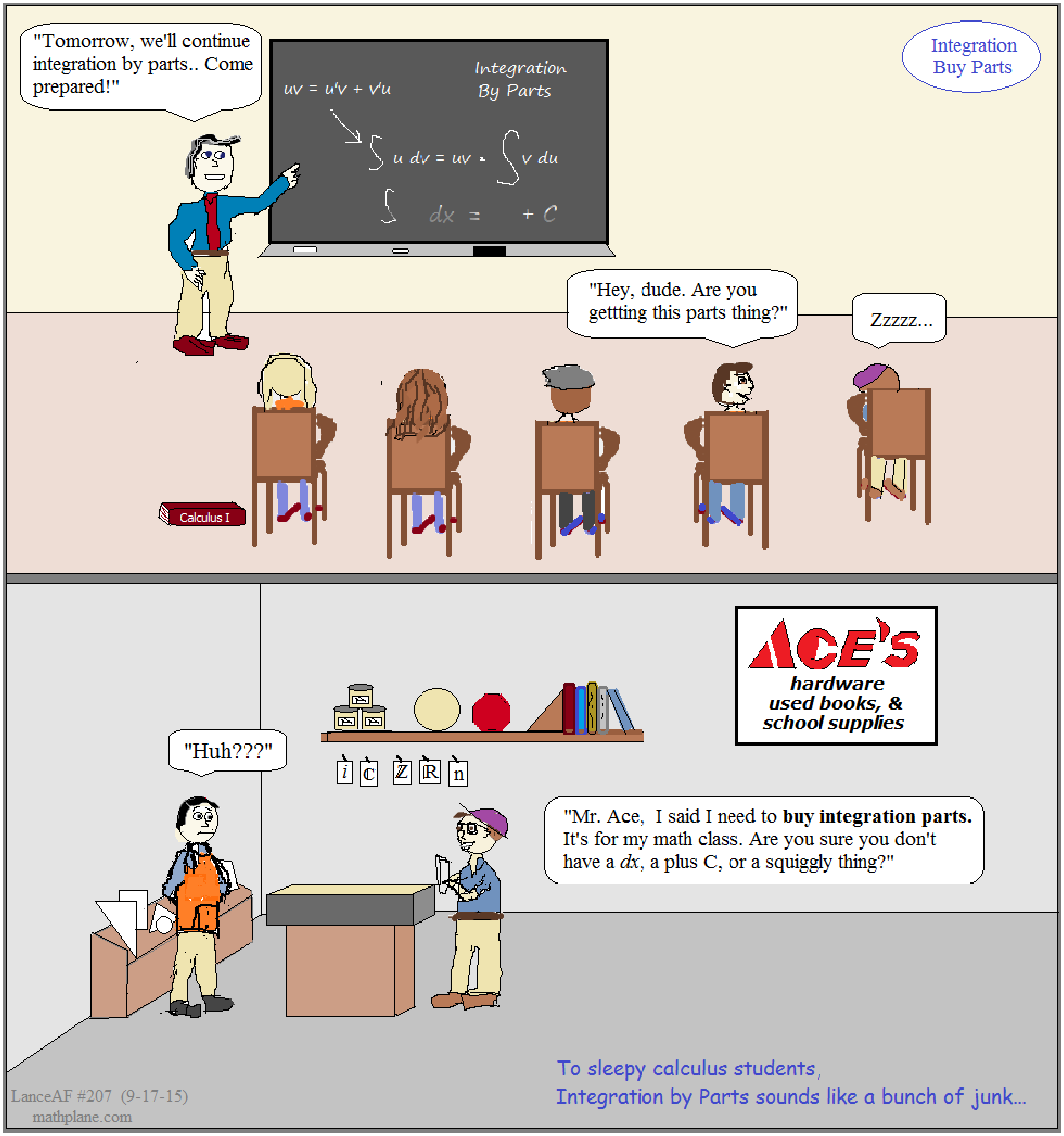
$$\int_0^2 \quad dy$$

Step 3: Identify each partition, and write the equation

$$\int_0^2 y + 2 - y^2 \, dy$$

Step 4: Solve...

$$\left. \frac{y^2}{2} + 2y - \frac{y^3}{3} \right|_0^2 = 2 + 4 - \frac{8}{3} = \boxed{\frac{10}{3}}$$



Integration Buy Parts
"Tomorrow, we'll continue integration by parts.. Come prepared!"
Integration By Parts
uv = u'v + v'u
∫ u dv = uv - ∫ v du
∫ dx = + C
"Hey, dude. Are you gettting this parts thing?"
Zzzzz...
Calculus I
ACE'S hardware used books, & school supplies
"Huh???"
i C Z R n
"Mr. Ace, I said I need to **buy integration parts.** It's for my math class. Are you sure you don't have a *dx*, a plus C, or a squiggly thing?"
To sleepy calculus students,
Integration by Parts sounds like a bunch of junk...
LanceAF #207 (9-17-15)
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Practice Questions-→

Definite Integrals: Area between Curves

1) Find the area of the region bounded by $y = x$

$y = 2 - x$

$y = 0$

2) Find the area of region created by the intersection of the following parabolas

$y = 8 - 2x^2$

$y = x^2 + 5$

3) Find the area between the functions $f(y) = y^2$

$g(y) = y + 2$

4) Find the area bounded by the functions

$f(x) = \frac{8}{x^2}$

$g(x) = 8x$

$h(x) = x$

5) Find the area between $x = y^2 - 4$ and $x = 6 - 3y$

6) Find the area enclosed by $f(x) = x^2$ and $g(x) = -x^2 + 2x + 24$

7) Find the area between the parabolas $x + y^2 = 0$

$x + 3y^2 = 2$

8) Find the area of the region surrounded by $y = \dfrac{10}{x}$

$x = 0$

$y = 2$

$y = 10$

9) Find the area of the region below $y = x$ and $y = 2$ and above the curve $y = \frac{1}{8}x^2$

10) Find the area of the region bordered by $y = 2\sin(x)$

$y = \sin(2x)$

11) Find the value using a) geometric areas

b) "split integrals"

$$\int_2^9 |x - 5| \; dx$$

12) Find the area of the region: Where $a > 0$, $y = x\sqrt{a^2 - x^2}$

$y = 0$

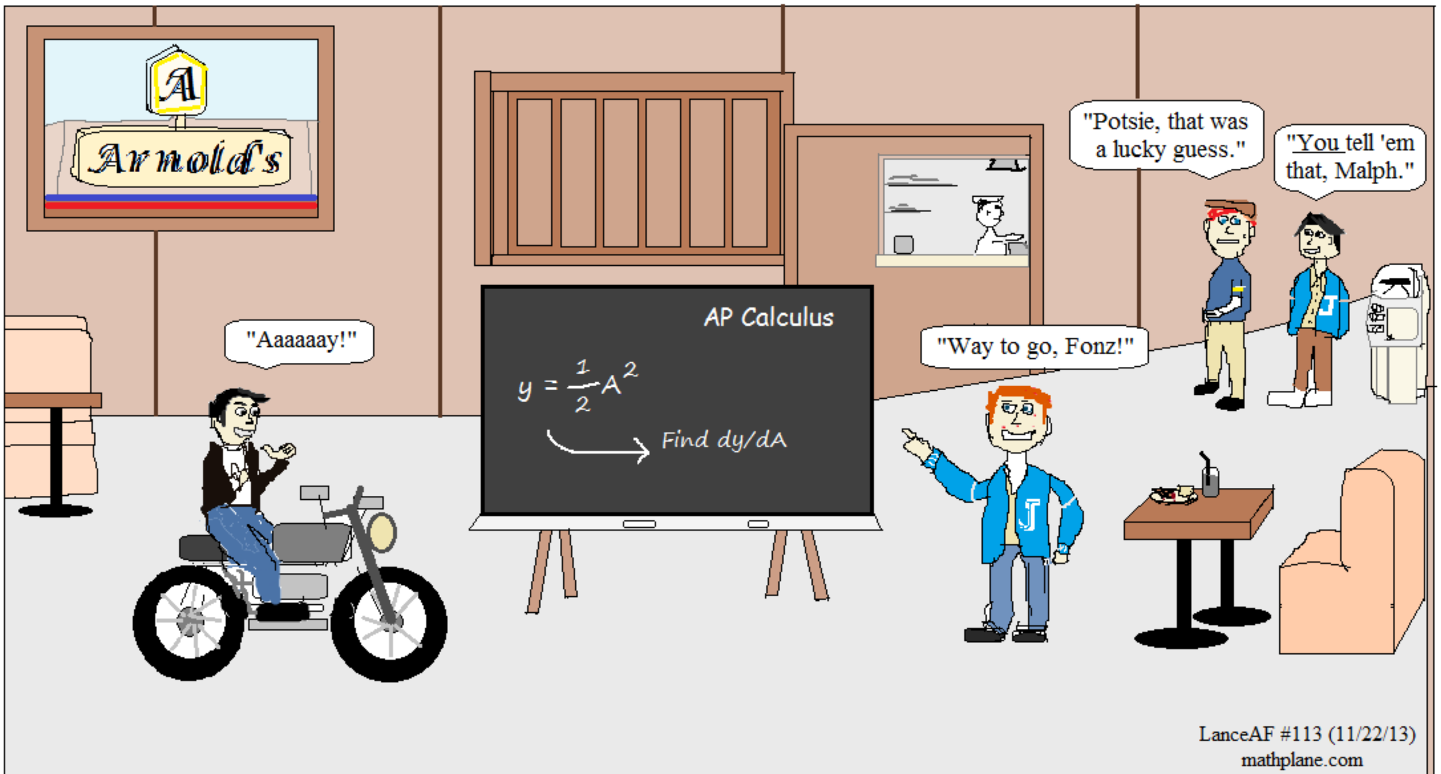
in Quadrant I

13) Find the area of the region bordered by $y = 3 - x^2$

$y = -1$

a) Solve using an integral with respect to x

b) Solve using an integral with respect to y



Happy Days

Despite Richie's help, Fonzie dropped out of Calculus.
(... although he did have some success with velocity and acceleration!)

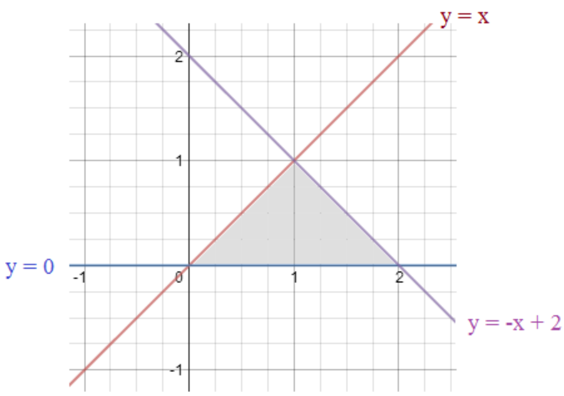
SOLUTIONS-→

1) Find the area of the region bounded by $y = x$

$y = 2 - x$

$y = 0$

First sketch the region:



Method 1: Find area of the triangle (geometry)

$$\text{Area} = \frac{1}{2}\,(\text{base})(\text{height})$$

$$= \frac{1}{2}\,(2)(1) \quad = \boxed{1}$$

Method 2: Use integrals (calculus)

shaded area under left half — shaded area under right half

$$\int_0^1 x - 0 \; dx \;+\; \int_1^2 (-x + 2) - 0 \; dx$$

$$\left.\frac{x^2}{2}\right|_0^1 \;+\; \left.\frac{-x^2}{2} + 2x\right|_1^2$$

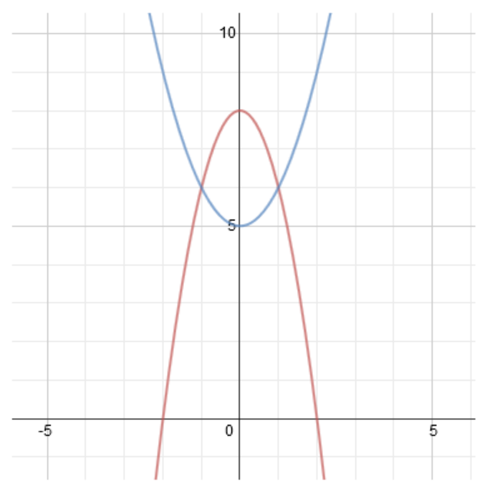
$$\frac{1}{2} - 0 \;+\; -2 + 4 \;-\; (-1/2 + 2) \;=\; \boxed{1}$$

2) Find the area of region created by the intersection of the following parabolas

$y = 8 - 2x^2$

$y = x^2 + 5$

Step 1: Sketch curves and identify area



Step 2: Find intersection to determine boundaries of integral

$$x^2 + 5 = 8 - 2x^2$$

$$3x^2 = 3 \qquad (1, 6)$$

$$x = 1 \text{ and } -1 \qquad (-1, 6)$$

Step 3: Solve definite integral

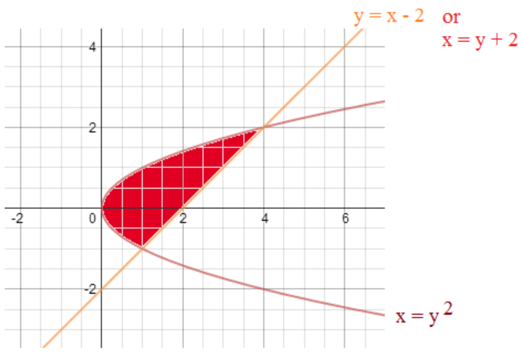
area under upper curve — area under lower curve

$$\int_{-1}^{1} 8 - 2x^2 \;-\; (x^2 + 5)\; dx$$

$$\int_{-1}^{1} -3x^2 + 3 \; dx \;=\; \left. -x^3 + 3x \right|_{-1}^{1} \qquad -1 + 3 - (1 + -3) = \boxed{4}$$

3) Find the area between the functions $f(y) = y^2$

$g(y) = y + 2$



Determine the region boundaries (points of intersection)

$y^2 = y + 2$

$y^2 - y - 2 = 0$

$(y - 2)(y + 1) = 0$

$y = -1, 2$

$x = y + 2$

$x = (-1) + 2 \quad x = 1 \quad (1, -1)$

$x = y + 2$

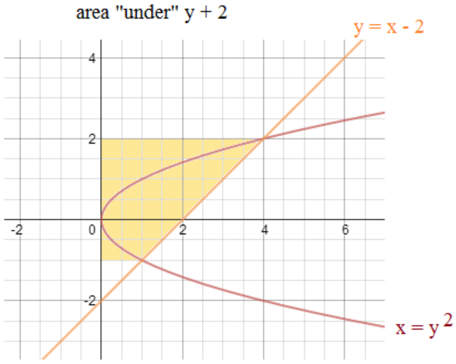
$x = (2) + 2 \quad x = 4 \quad (4, 2)$

$$\int_{-1}^{2} y + 2\, dy - \int_{-1}^{2} y^2\, dy$$

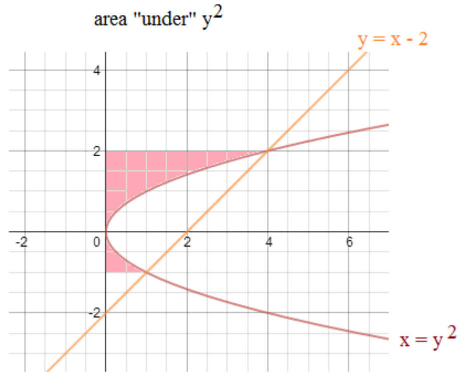
$$\left.\frac{y^2}{2} + 2y - \frac{y^3}{3}\right|_{-1}^{2} = 2 + 4 - \frac{8}{3} - (1/2 - 2 + 1/3)$$

$$= 8 - 9/3 - 1/2 = \boxed{4\tfrac{1}{2}}$$

area "under" y + 2



area "under" y^2



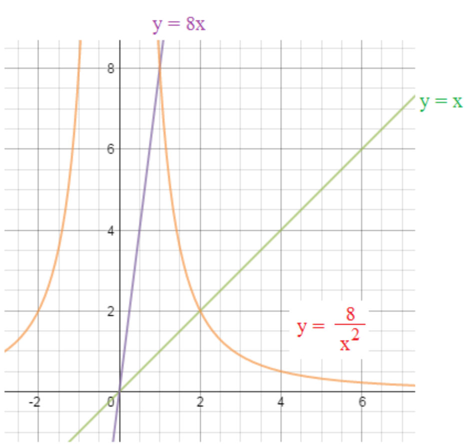
The difference between the yellow and pink areas is the region we are evaluating...

4) Find the area bounded by the functions

$f(x) = \dfrac{8}{x^2}$

$g(x) = 8x$

$h(x) = x$



Find intersection points: f and g (1, 8)

$\dfrac{8}{x^2} = 8x$

$8x^3 = 8$

$x = 1$

g and h (0, 0)

$8x = x$

$7x = 0$

$x = 0$

f and h (2, 4)

$\dfrac{8}{x^2} = x$

$x^3 = 8$

$x = 2$

Then, separate the areas..

$$\int_{0}^{1} 8x - x\, dx + \int_{1}^{2} \frac{8}{x^2} - x\, dx$$

(g above h) (f above h)

$$\left.\frac{7x^2}{2}\right|_0^1 + \left.\frac{-8}{x} - \frac{x^2}{2}\right|_1^2$$

$$\frac{7}{2} - 0 + (-4 - 2) - (-8 - \frac{1}{2})$$

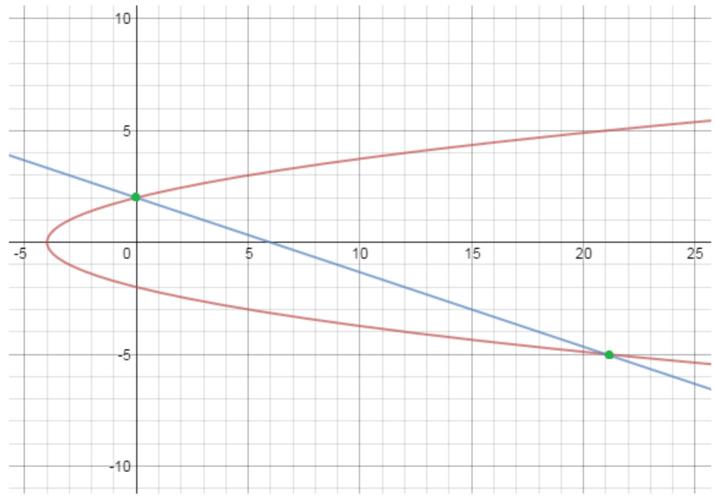
$$\frac{7}{2} + 2 + \frac{1}{2} = \boxed{6}$$

5) Find the area between $x = y^2 - 4$ and $x = 6 - 3y$

Find intersection(s) of equations to determine boundaries

$$y^2 - 4 = 6 - 3y$$
$$y^2 + 3y - 10 = 0$$
$$(y + 5)(y - 2) = 0$$

Intersections are (0, 2) and (21, -5)



Identify equation (and determine which function is "above" the other) or, in this case, "on the right"

$$\int_{-5}^{2} (6 - 3y) - (y^2 - 4) \; dy$$

(horizontal partitions, so we use dy)

$$\int_{-5}^{2} -y^2 - 3y + 10 \; dy$$

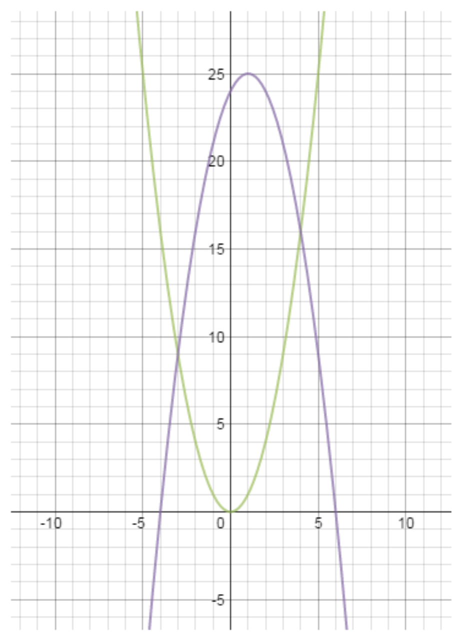
$$\left. \frac{-y^3}{3} - \frac{3y^2}{2} + 10y \right|_{-5}^{2} = (-8/3 - 6 + 20) - (125/3 - 75/2 - 50) = -133/3 + 75/2 + 64 = \boxed{57\tfrac{1}{6}}$$

6) Find the area enclosed by $f(x) = x^2$ and $g(x) = -x^2 + 2x + 24$

Find intersection(s) of equations to determine boundaries

$$x^2 = -x^2 + 2x + 24$$
$$2x^2 - 2x - 24 = 0$$
$$2(x - 4)(x + 3) = 0$$
$$x = -3, 4$$

Intersections occur at (-3, 9) and (4, 16)



Identify equation (and determine which function is "above" the other)

$$\int_{-3}^{4} \underbrace{-x^2 + 2x + 24}_{\text{above}} - \underbrace{x^2}_{\text{below}} \; dx$$

(vertical partitions, so we use dx)

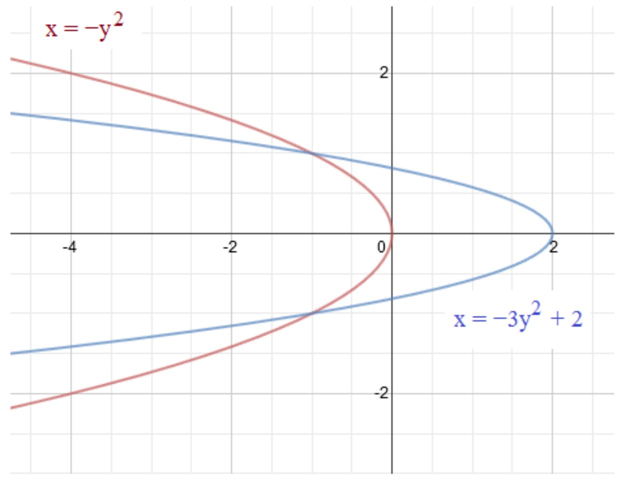
$$\int_{-3}^{4} -2x^2 + 2x + 24 \; dx$$

$$\left. \frac{-2x^3}{3} + x^2 + 24x \right|_{-3}^{4} = -128/3 + 16 + 96 - (18 + 9 - 72)$$

$$= -128/3 + 157 = \boxed{114\tfrac{1}{3}}$$

7) Find the area between the parabolas $x + y^2 = 0$

$x + 3y^2 = 2$



(two horizontal parabolas that open to the left)

$-y^2 = 2 - 3y^2$

$2y^2 = 2$ (-1, 1)

$y = 1, -1$ (-1, -1)

$$\int_{-1}^{1} 2 - 3y^2 - (-y^2)\, dy \qquad \int_{-1}^{1} 2 - 2y^2\, dy$$

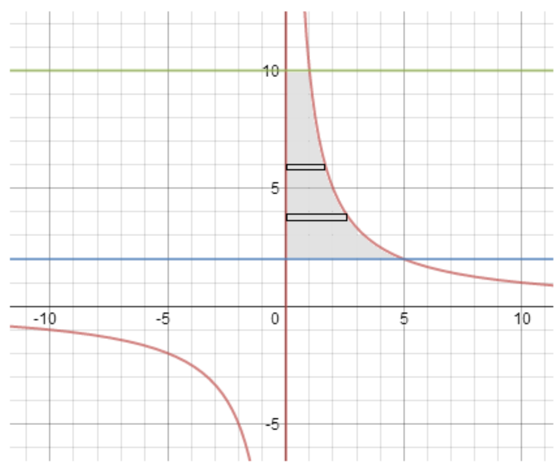
$$2y - \frac{2y^3}{3}\Bigg|_{-1}^{1} = 2 - \frac{2}{3} - \left(-2 + \frac{2}{3}\right) = \boxed{\frac{8}{3}}$$

8) Find the area of the region surrounded by $y = \frac{10}{x}$

$x = 0$

$y = 2$

$y = 10$



Use the graph or algebra to find the intersections

$y = \frac{10}{x}$ and $y = 2$ (5, 2)

$y = \frac{10}{x}$ and $y = 10$ (1, 10)

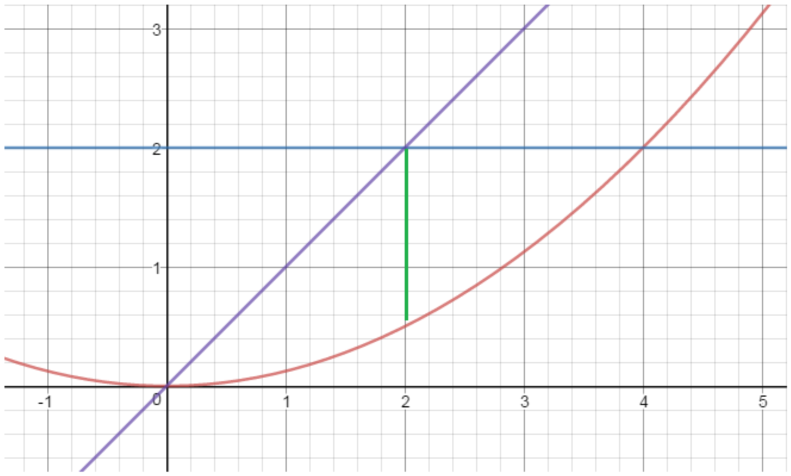
$x = \frac{10}{y}$

Note: *horizontal partitions* are used ---> *dy*

$$\int_{2}^{10} \frac{10}{y}\, dy = 10\ln y\Bigg|_{2}^{10} = 10\ln(10) - 10\ln(2) = \boxed{10\ln 5 \approx 16.1}$$

9) Find the area of the region below y = x and y = 2 and above the curve $y = \frac{1}{8}x^2$ SOLUTIONS

Definite Integrals: Area between Curves



By looking at the graph (or solving algebraically), we see the points of intersection are at x = 2 and x = 4..

And, since the border above changes at x = 2, we'll use 2 separate integrals...

$$\int_0^2 x - \frac{1}{8}x^2 \, dx + \int_2^4 2 - \frac{1}{8}x^2 \, dx$$

Left area Right area

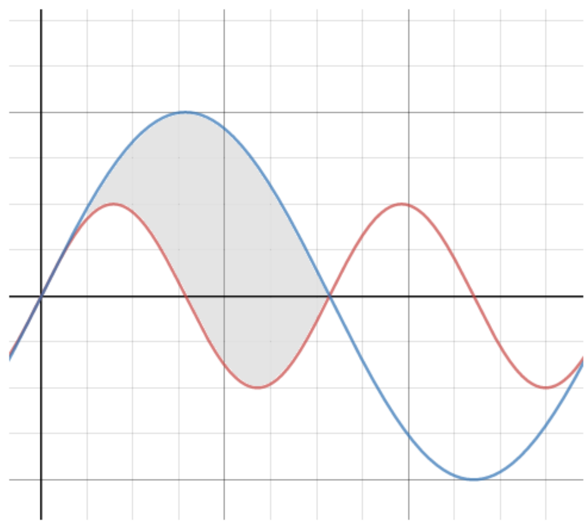
$$\frac{x^2}{2} - \frac{x^3}{24}\Big|_0^2 + 2x - \frac{x^3}{24}\Big|_2^4$$

$$2 - \frac{8}{24} + 8 - \frac{64}{24} - \left(4 - \frac{8}{24}\right)$$

$$6 - \frac{8}{3} = \boxed{10/3}$$

10) Find the area of the region bordered by y = 2sin(x)

y = sin(2x)



2sinx = sin2x

2sinx = 2sinxcosx

2sinxcosx − 2sinx = 0

2sinx(cosx − 1) = 0

2sinx = 0 x = 0 and π

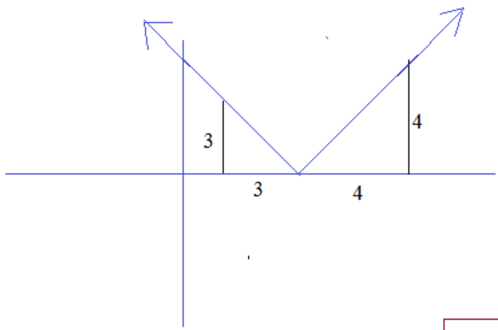
cosx − 1 = 0

cosx = 1 x = 0

$$\int_0^{\pi} 2\sin x + \sin(2x) \, dx \qquad -2\cos x - \frac{1}{2}\cos(2x)\Big|_0^{\pi} = -2(-1) - \frac{1}{2}(1) - \left(-2(1) - \frac{1}{2}(1)\right) = \boxed{4}$$

11) Find the value using a) geometric areas

b) "split integrals" $\int_2^9 |x-5|\,dx$



$9/2 + 16/2 = \boxed{12.5}$

$$\int_2^5 -(x-5)\,dx + \int_5^9 x-5\,dx$$

$$\left.-\frac{x^2}{2} + 5x\right|_2^5 + \left.\frac{x^2}{2} - 5x\right|_5^9$$

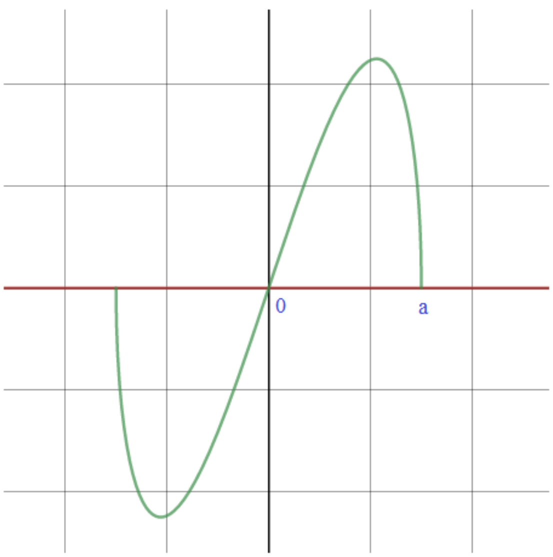
$$-25/2 + 25 - (-2+10) + 81/2 - 45 - (25/2 - 25)$$

$$-25/2 + 17 + 8 = \boxed{25/2}$$

12) Find the area of the region: Where $a > 0$, $y = x\sqrt{a^2 - x^2}$

$y = 0$

in Quadrant I



$$\int_0^a x(a^2 - x^2)^{\frac{1}{2}}\,dx$$

$$-\frac{1}{2}\int_0^a -2\,x(a^2 - x^2)^{\frac{1}{2}}\,dx$$

$$\left.-\frac{1}{2}\cdot\frac{(a^2 - x^2)^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^a = 0 - -\frac{1}{2}\cdot\frac{a^3}{\frac{3}{2}} = \boxed{\frac{a^3}{3}}$$

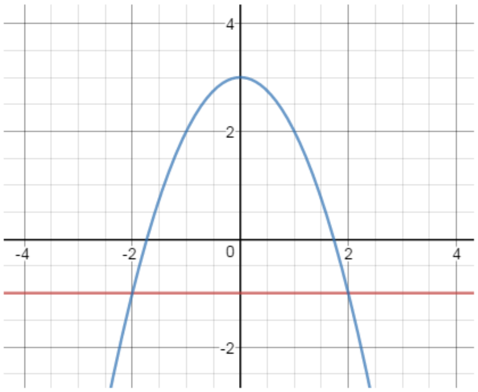
13) Find the area of the region bordered by $y = 3 - x^2$

$y = -1$

a) Solve using an integral with respect to x

$y = 3 - x^2$

$y = -1$



Find the intersections to determine the boundaries:

$3 - x^2 = -1$

$x^2 = 4$

$x = -2$ and 2

$$\int_{-2}^{2} 3 - x^2 - (-1)\ dx$$

upper curve *lower line*

$$\int_{-2}^{2} -x^2 + 4\ dx = \left.\frac{-x^3}{3} + 4x\right|_{-2}^{2}$$

$$\frac{-8}{3} + 8 - \left(\frac{8}{3} - 8\right)$$

$$\boxed{\frac{32}{3}}$$

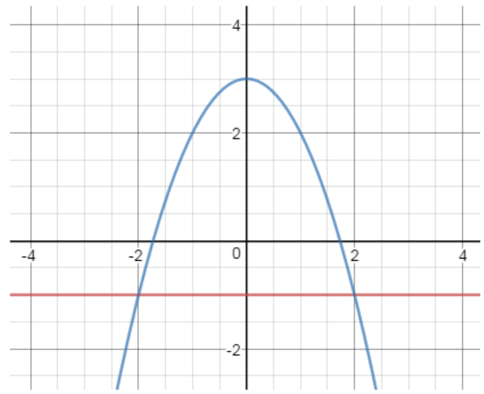
b) Solve using an integral with respect to y

$y = 3 - x^2$

$y - 3 = -x^2$

$3 - y = x^2$

$\pm\sqrt{(3 - y)} = x$



Looking at the graph, the boundaries are $y =$ -1 and 3

$$\int_{-1}^{3} \sqrt{(3 - y)} - 0\ dy$$

right curve *left line*

$$\int_{-1}^{3} \sqrt{(3 - y)}\ dy = -\int_{-1}^{3} -(3 - y)^{\frac{1}{2}}\ dy$$

$$\left.-\frac{2}{3}(3 - y)^{\frac{3}{2}}\right|_{-1}^{3}$$

$$0 - -\frac{2}{3}(3 - -1)^{\frac{3}{2}} = 16/3$$

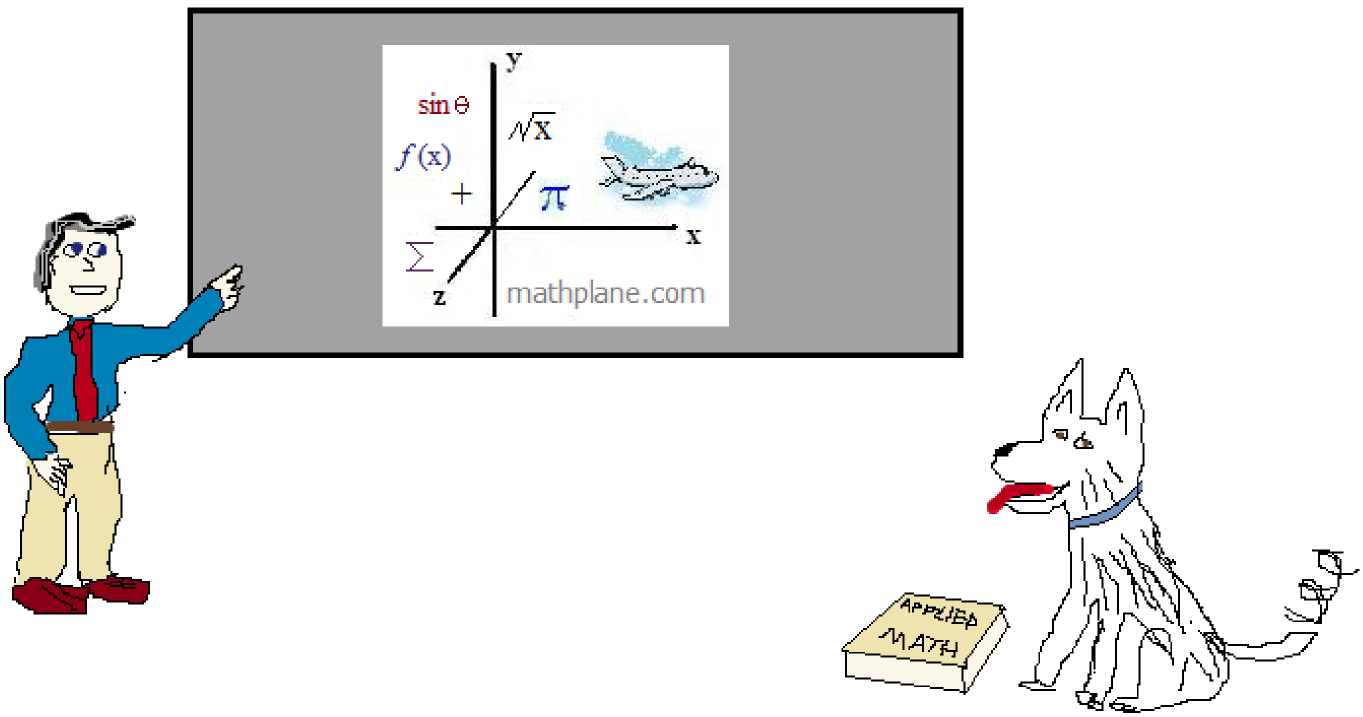
And, the left side is also 16/3...

total area: $\boxed{\frac{32}{3}}$

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know

Cheers



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