Implicit Differentiation

Notes, examples, applications, and practice test (with solutions)

Topics include logarithms, inverse trig, tangent lines, graphing, related rates, and more.

Implicit Differentiation Notes and Examples

Explicit vs. Implicit Form:

Equations involving 2 variables are generally expressed in *explicit form*

$$y = f(x)$$

In other words, one of the two variables is explicitly given in terms of the other.

$y = 3x + 5$

$d = (.05t)^2 + 20t - 7$

$s = \sqrt{r+1}$

(explicit form: put in the input variable, and easily get the other)

Equations where relationships are not given explicitly are in *implicit form*.

$2x - y = 4$

$xy = 1$

$x^3 + 2xy + y^2 = 0$

(implicit form: the relationship between x and y isn't easily seen)

Sometimes it is possible to change the form from implicit to explicit...

$2x - y = 4 \longrightarrow y = 2x - 4$

$xy = 1 \longrightarrow y = \frac{1}{x}$

.... But, other times it is very difficult or impossible to express in explicit form.

$x^2 + 2xy + y^2 = 0 \longrightarrow y = ?$

So, to find the derivative, implicit differentiation is an easier approach.

Implicit Differentiation:

Method:

1) Take derivatives
2) When taking derivative of y, insert $\frac{dy}{dx}$ (or y')
3) Solve for $\frac{dy}{dx}$ (or y')

Implicit Differentiation Example:

$x^2 - 2y^3 + 4x = 2$

$2x - 6y^2\frac{dy}{dx} + 4 = 0$

$\frac{dy}{dx} = \frac{-2x - 4}{-6y^2}$

$\frac{dy}{dx} = \frac{x+2}{3y^2}$

Example: Find the derivative with respect to x of

$x^2 + 2xy + y^2 = 0$

$2x + 2(1y + 1y'x) + 2yy' = 0$

$2x + 2y + 2xy' + 2yy' = 0$

$2x + 2y = -2xy' - 2yy'$

$x + y = y'(-x - y)$

$y' = \frac{x+y}{-(x+y)} = -1$

Verifying Implicit Differentiation: An Example

Find the derivative of $x^2 + y^2 = 25$

Implicit Differentiation:

$x^2 + y^2 = 25$

Take derivatives, inserting y' next to derivatives of y..

$2x + 2yy' = 0$

$2yy' = -2x$

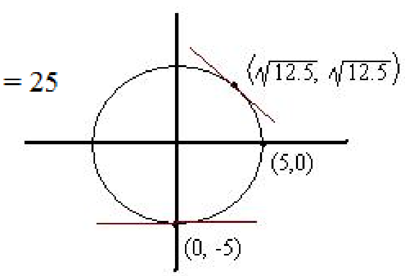
Solve for y'

$y' = \frac{-x}{y}$

Notice that implicit differentiation used fewer steps and easier equations!

Graph:

$x^2 + y^2 = 25$



slope (rate of change) at

$(0, -5)$: $\frac{-(0)}{(5)} = 0$

$(5, 0)$: $\frac{-(5)}{(0)}$ = undefined

$(\sqrt{12.5}, \sqrt{12.5})$: $\frac{-\sqrt{12.5}}{\sqrt{12.5}} = -1$

Explicit Differentiation:

$x^2 + y^2 = 25$

Change to implicit form

$y^2 = 25 - x^2$

$y = \pm\sqrt{25 - x^2}$

Find derivative (using power rule)

$y = (25 - x^2)^{\frac{1}{2}}$

$y' = \frac{1}{2}(25 - x^2)^{\frac{-1}{2}}(-2x)$

Simplify the result.

(note: we could substitute the denominator for y)

$y' = \frac{-x}{(25 - x^2)^{\frac{1}{2}}} = \frac{-x}{y}$

Then, for $y = -\sqrt{25 - x^2}$ $\quad y' = \frac{-x}{y}$

$y' = (-1)\frac{1}{2}(25 - x^2)^{\frac{-1}{2}}(-2x)$

$y' = \frac{x}{(25 - x^2)^{\frac{1}{2}}} = \frac{x}{-y}$ <u>Same solution</u>

Implicit differentiation notes and examples

Example: Given $x^3 + xy + y^2 = 8$ Find $\frac{dy}{dx}$

Derivative uses [product rule]
Inserting $\frac{dy}{dx}$ (or y')

$$3x^2 + \left[1 \cdot y + x \cdot 1\left(\frac{dy}{dx}\right)\right] + 2y\left(\frac{dy}{dx}\right) = 0$$

Simplify

$$3x^2 + y + x\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = 0$$

("Move everything without a dy/dx to the other side")

$$x\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = -3x^2 - y$$

("Factor out dy/dx")

$$\left(\frac{dy}{dx}\right)x + 2y = -3x^2 - y$$

Divide to finish

$$\left(\frac{dy}{dx}\right) = \frac{-3x^2 - y}{x + 2y}$$

Example: Given $x^3 - 2x^2y + 3xy^2 = 38$
Evaluate the derivative at (2,3)

Use implicit differentiation

$$3x^2 - 2\,[2xy + 1 \cdot y' x^2] + 3\,[1 \cdot y^2 + 2y\,y'\,x] = 0$$

Simplify

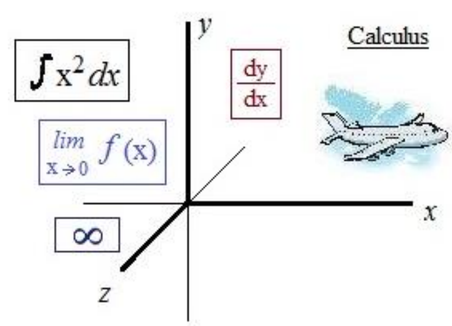
$$3x^2 - 4xy - 2y'x^2 + 3y^2 + 6y\,y'\,x = 0$$

$$6y\,y'\,x - 2y'x^2 = -3x^2 + 4xy - 3y^2$$

$$y' = \frac{-3x^2 + 4xy - 3y^2}{6xy - 2x^2}$$

Plug in (2, 3)

$$\frac{-3(4) + 4(6) - 3(9)}{6(6) - 2(4)} = \frac{-15}{28}$$



Implicit Differentiation and Tangent Lines

Find the equation of the line tangent to $x^2 + xy - y^2 = 1$ @ (2, 3)

To find the equation of a line, we need the slope and a point.
The point is given: (2, 3)
And, the slope is the *instantaneous rate of change (IROC)* at the given point.
Use implicit differentiation to find the *IROC*

$$2x + (1)y + x(1)\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} - 2y\frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx}(x - 2y) = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x - y}{(x - 2y)}$$

then, the IROC at (2, 3) is

$$\frac{-2(2) - (3)}{(2) - 2(3)} = \frac{-7}{-4}$$

So, the slope is $\frac{7}{4}$

$$(y - 3) = \frac{7}{4}(x - 2)$$

$$\boxed{y = \frac{7}{4}x - \frac{1}{2}}$$

Find the equation of the line tangent to $2xy + \pi \sin y = 2\pi$ @ $(1, \frac{\pi}{2})$

Use implicit differentiation to find the IROC, which is the slope of the tangent lines.

$$2[(1)y + x(1)\frac{dy}{dx}] + \pi(\cos y)\frac{dy}{dx} = 0$$

$$2y + 2x\frac{dy}{dx} + \pi(\cos y)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2x + \pi(\cos y)) = -2y$$

$$\frac{dy}{dx} = \frac{-2y}{2x + \pi(\cos y)}$$

then, the slope is

$$\frac{-2(\frac{\pi}{2})}{2(1) + \pi(\cos\frac{\pi}{2})} = \frac{-\pi}{2 + 0}$$

and, the equation of the tangent line is:

$$(y - \frac{\pi}{2}) = -\frac{\pi}{2}(x - 1)$$

$$\boxed{y = -\frac{\pi}{2}x + \pi}$$

Implicit differentiation to find slope

Example: Find slope of $x^2 + y^2 = 13$ at (-2, 3) and at (3, 2)

Find y':

$$2x + 2yy' = 0$$

$$2yy' = -2x$$

$$y' = \frac{-2x}{2y}$$

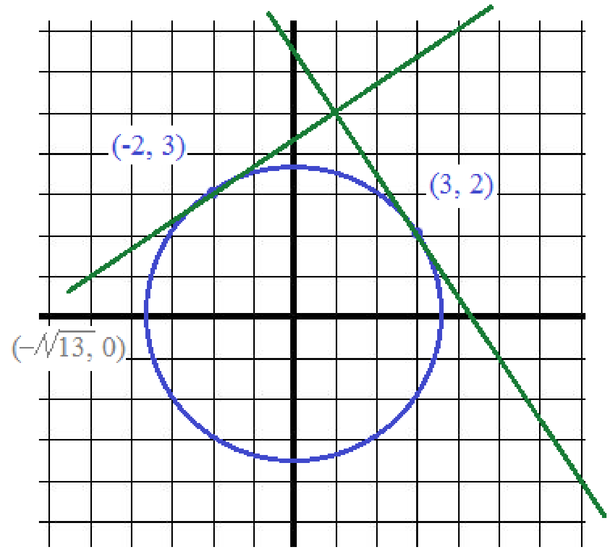
$$y' = \frac{-x}{y}$$

Then, to find slope at (-2, 3):

$$y' = \frac{-(-2)}{(3)} = \frac{2}{3}$$

at (3, 2):

$$y' = \frac{-(3)}{(2)}$$



Example: Find lines that are tangent and normal to $x^2 y^2 = 9$ at the point (-1, 3)

Utilize implicit differentiation to find y'

(product rule)

$$2xy^2 + 2yy'x^2 = 0$$

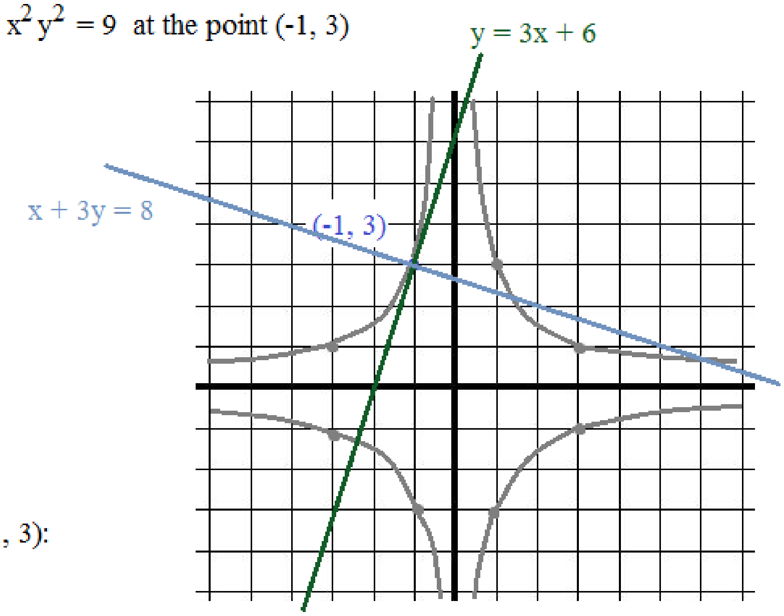
(solve for y')

$$2yy'x^2 = -2xy^2$$

$$y' = \frac{-2xy^2}{2yx^2} = \frac{-y}{x}$$

Instantaneous rate of change (dy/dx) at (-1, 3):

$$\frac{-(3)}{(-1)} = 3$$



Slope of tangent: 3 Equation of tangent line: $(y - 3) = 3(x - (-1))$

$$y = 3x + 6$$

Slope of normal: -1/3 Equation of normal line: $(y - 3) = -1/3(x - (-1))$

$$y = \frac{-1}{3}x + \frac{8}{3} \qquad x + 3y = 8$$

Example: Given the curve $x^2 - xy + y^2 = 16$

Find the coordinate(s) where the tangents are *vertical*:

Implicit Differentiation and Vertical Tangent Lines

SOLUTION:

(If a tangent line is horizontal, then the slope is 0)
If a tangent line is *vertical*, then the slope is undefined!

To find the instantaneous rate of change, find the derivative:

(implicit differentiation)

$$2x - [(1)y + x(1)\frac{dy}{dx}] + 2y\frac{dy}{dx} = 0$$

$$2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

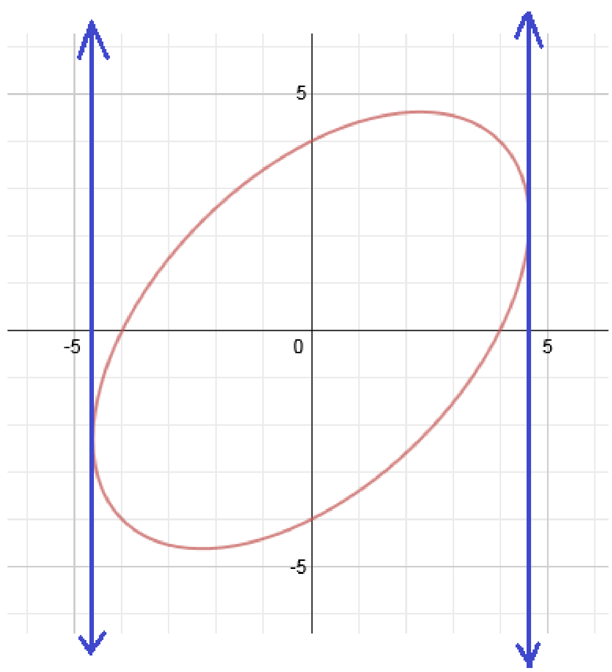
$$\frac{dy}{dx}(2y - x) = y - 2x$$

$$\frac{dy}{dx} = \frac{y-2x}{2y-x}$$

$\frac{y-2x}{2y-x}$ is undefined when the denominator = 0

$$2y - x = 0$$

$$x = 2y$$



Now, find x and y:

$x = 2y$

$x^2 - xy + y^2 = 16$

(substitution)

$$(2y)^2 - (2y)y + y^2 = 16$$

$$4y^2 - 2y^2 + y^2 = 16$$

$$3y^2 = 16$$

$$y = \pm\sqrt{\frac{16}{3}} = \text{approx. } \pm 2.31$$

then, $x = \pm 4.62$ (approx.)

(-4.62, -2.31) and (4.62, 2.31)

Example: $x^2 - \frac{4}{y^2} = x$

Since it is a bit of effort to change equation (from implicit to explicit form), we'll use implicit differentiation.

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

$x^2 - 4y^{-2} = x$

$2x - -8y^{-3}\frac{dy}{dx} = 1$

$\frac{8}{y^3}\frac{dy}{dx} = 1 - 2x$

$\frac{dy}{dx} = \frac{y^3(1-2x)}{8}$

1st derivative

$\frac{dy}{dx} = \frac{1}{8}y^3 - \frac{1}{4}xy^3$

product rule

$\frac{d^2y}{dx^2} = \frac{3}{8}y^2\frac{dy}{dx} - \frac{1}{4}\left(y^3 + x \cdot 3y^2\frac{dy}{dx}\right)$

then, substitute $\frac{dy}{dx}$

$\frac{d^2y}{dx^2} = \frac{3}{8}y^2\left(\frac{y^3(1-2x)}{8}\right) - \frac{1}{4}\left(y^3 + x \cdot 3y^2 \cdot \frac{y^3(1-2x)}{8}\right)$

$= \frac{3y^5(1-2x)}{64} - \frac{y^3}{4} - \frac{3xy^5(1-2x)}{32}$

2nd derivative

$= \frac{3y^5 - 6xy^5 - 16y^3 - 6xy^5 + 12x^2y^5}{64} = \frac{3y^5 - 12xy^5 - 16y^3 + 12x^2y^5}{64}$

Example: $y = x^{3/5}$ Find the 1st and 2nd derivatives.

Then, use implicit differentiation to verify these 1st and 2nd derivatives: $y^5 = x^3$

First derivative (using the power rule)

$\frac{dy}{dx} = (3/5)x^{-2/5}$

Second derivative

$\frac{d^2y}{dx^2} = (-6/25)x^{-7/5}$

$5y^4\frac{dy}{dx} = 3x^2$

$\frac{dy}{dx} = \frac{3}{5}\frac{x^2}{y^4}$

since $y = x^{\frac{3}{5}}$

then $y^4 = x^{\frac{12}{5}}$

$\frac{dy}{dx} = \frac{3}{5}\frac{x^2}{x^{\frac{12}{5}}}$ ✓

First derivative

We continue to find the 2nd derivative.....

$\frac{dy}{dx} = \frac{3}{5}\frac{x^2}{y^4} = \frac{3}{5}x^2y^{-4}$

product rule

$\frac{d^2y}{dx^2} = \frac{3}{5}\left(2xy^{-4} + -4y^{-5}\frac{dy}{dx}x^2\right)$

substitute $\frac{dy}{dx}$

$\frac{d^2y}{dx^2} = \frac{3}{5}\left(\frac{2x}{y^4} - \frac{4x^2}{y^5}\ \frac{3}{5}x^2y^{-4}\right)$

$= \frac{3}{5}\left(\frac{2xy - 4x^2\frac{3x^2}{5y^4}}{y^5}\right)$

$= \frac{3}{5}\left(\frac{2xy - \frac{12x^4}{5y^4}}{y^5}\right)$

$\frac{d^2y}{dx^2} = \frac{3}{5}\left(\frac{\frac{10xy^5}{5y^4} - \frac{12x^4}{5y^4}}{y^5}\right)$

$= \frac{3}{5}\left(\frac{10xy^5 - 12x^4}{5y^9}\right)$

$= \frac{6}{25}x\frac{5y^5 - 6x^3}{y^9}$

Note: $x^3 = y^5$

$y = x^{3/5}$

$= \frac{6}{25}x\frac{5y^5 - 6y^5}{y^9} = -\frac{6}{25}x\frac{y^5}{y^9}$

Second derivative

$= -\frac{6}{25}x \cdot x^{-12/5}$ $= -\frac{6}{25}x^{-7/5}$ ✓

Inverse

Let's compare y' vs x'

derivative of y with respect to x $\frac{dy}{dx}$ vs $\frac{dx}{dy}$ derivative of x with respect to y

Given: $x^3 - xy + y^2 = 4$

Find y' or $\frac{dy}{dx}$

$$3x^2 - [\,(1)y + x(1)\,y'\,] + 2y\,y' = 0$$

$$3x^2 - y - x\,y' + 2y\,y' = 0$$

$$-x\,y' + 2y\,y' = -3x^2 + y$$

$$y'(-x + 2y) = -3x^2 + y$$

$$y' = \frac{-3x^2 + y}{(-x + 2y)}$$

Find x' or $\frac{dx}{dy}$

"To find dx/dy, we insert x' whenever taking the derivative of x"

$$3x^2 x' - [\,(1)x'\,y + x(1)\,] + 2y = 0$$

$$3x^2 x' - x'\,y - x + 2y = 0$$

$$3x^2 x' - x'\,y = x - 2y$$

$$x'(3x^2 - y) = x - 2y$$

$$x' = \frac{x - 2y}{(3x^2 - y)} = \frac{-x + 2y}{-3x^2 + y}$$

Reciprocals

Implicit Differentiation: Word Problem Examples

1) A 25-foot ladder is leaning against a wall. If the top of the ladder is slipping down the wall at a rate of 2 feet/second, how fast will the bottom be moving away from the wall when the top is 20 feet above the ground?

Step 1: Draw diagram, list variables and formulas

length to bottom of ladder = x
length to top of ladder = y

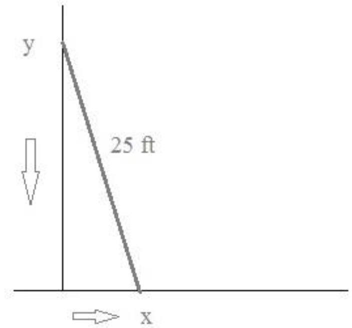
$x^2 + y^2 = 625 \text{ ft}^2$ (pythagorean theorem)

down the wall at a rate of 2 ft/sec $\frac{dy}{dt} = -2$ ft/sec

(change of y *with respect to time*)

moving away from the wall $\frac{dx}{dt} = ?$

(change of x *with respect to time*)



Important note: we're seeking dx/dt, (the change of x *with respect to time*)..
Simply taking the derivative of $y = \sqrt{625 - x^2}$
$1/2\,(625 - x^2)^{(-1/2)} \cdot (-2x) = \frac{-x}{\sqrt{(625 - x^2)}}$
shows us dy/dx, (the change in *y with respect to x*)

Step 2: Set up equation and use implicit differentiation.

$x^2 + y^2 = 625 \text{ ft}^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ derivative with respect to time

Substitute and solve:

$2x\frac{dx}{dt} + 2(20 \text{ ft})(-2 \text{ ft/sec}) = 0$

$x^2 + y^2 = 625 \text{ ft}^2$
$(x)^2 + (20 \text{ ft})^2 = 625 \text{ ft}^2$
When y = 20 ft, x = 15 feet

$2(15 \text{ ft})\frac{dx}{dt} + (-80 \text{ ft}^2/\text{sec}) = 0$

$30 \text{ ft}\frac{dx}{dt} = 80 \text{ ft}^2/\text{sec}$

$\frac{dx}{dt} = \frac{80 \text{ ft}^2/\text{sec}}{30 \text{ ft}} = \boxed{2.67 \text{ ft/sec}}$

Using explicit differentiation & chain rule

$x = \sqrt{625 - y^2}$

$\frac{dx}{dy} = 1/2\,(625 - y^2)^{(-1/2)} \cdot (-2y) = \frac{-y}{\sqrt{(625 - y^2)}}$

$\frac{dx}{dt} = \frac{dy}{dt} \cdot \frac{dx}{dy}$

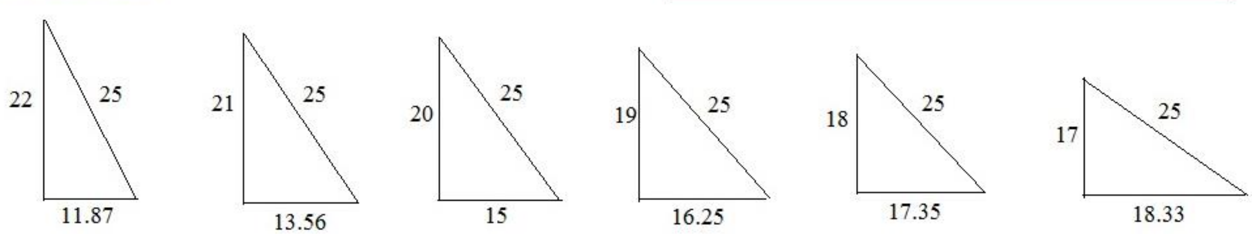
$-2\frac{\text{feet}}{\text{sec}} \cdot \frac{-y}{\sqrt{625 \text{ ft}^2 - y^2}}$

If y = 20 feet, then

$\frac{dx}{dt} = -2\frac{\text{feet}}{\text{sec}} \cdot \frac{-(20 \text{ feet})}{\sqrt{625 \text{ ft}^2 - 400 \text{ ft}^2}}$

$= -2\frac{\text{feet}}{\text{sec}} \cdot \frac{-20 \text{ feet}}{15 \text{ feet}} = \boxed{2.67 \text{ ft/sec}}$

Step 3: Check answer



From 22 to 20 feet (one second), the ladder moved out 3.13 feet

From 21 to 19 feet (one second), the ladder moved out 2.69 feet...

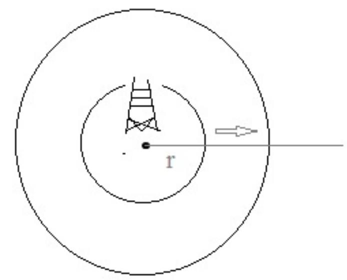
From 20 to 18 feet (one second) the ladder moved 2.35 feet...

2.67 feet per second is a reasonable answer! ✓

Implicit Differentiation: Word Problem Examples (continued)

2) Oil erupts from a ruptured tanker, spreading in a circle whose area increases at a constant rate of 6 square miles per hour. How fast is the radius of the spill increasing when the area is 9π square miles?

Step 1: Draw a diagram, list variables, and consider formulas



spill area $A = \pi r^2$

"area is increasing at a rate of 6 square miles per hour" $\quad \dfrac{dA}{dt} = 6\,\dfrac{\text{miles}^2}{\text{hour}}$

"how fast is the radius of the spill increasing?" $\quad \dfrac{dr}{dt} = ?$

When area is 9π sq. miles, the radius is 3 miles.

Step 2: Implicit differentiation

Take derivative *with respect to t*

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Plug in values and solve

$$6\,\frac{\text{miles}^2}{\text{hour}} = 2\pi(3\text{ miles})\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{6\text{ miles}^2}{\text{hour}\cdot 2\pi(3\text{ miles})} = \boxed{\frac{1}{\pi}\text{ miles/hour}}$$

(or, .318 miles/hour)

When area of spill is 9π square miles, the radius is increasing at .318 miles per hour.

Step 3: Verify Answer

$A = (3.14)r^2$

12 sq miles: $12 = (3.14)r^2$, $r = 1.95$ miles

(one hour later)

18 sq miles: $18 = (3.14)r^2$, $r = 2.39$ miles

(one hour later)

24 sq miles: $24 = (3.14)r^2$, $r = 2.76$ miles

radius changed .33 miles in one hour

(one hour later)

30 sq miles: $30 = (3.14)r^2$, $r = 3.09$ miles

9π sq. miles (28.27 sq miles), $r = 3$ miles → (one hour) → 9π sq. miles + 6 sq. miles = 34.27 sq. miles, $r = 3.30$

radius changed .30 miles in one hour

Find dy/dx if $y = x^x$

Answer: $\ln y = x\ln(x)$ logarithm power rule

dy/dx: $\frac{1}{y}\frac{dy}{dx} = (1)\ln x + x\frac{1}{x}$

$$\frac{1}{y}\frac{dy}{dx} = \ln x + 1$$

$$\frac{dy}{dx} = y(\ln x + 1)$$

$$= x^x(\ln x + 1)$$

Implicit Differentiation & Inverse Trig Derivatives

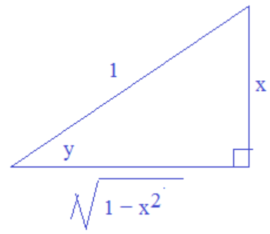
Example: $y = \sin^{-1} x$ What is $\frac{dy}{dx}$?

Step 1: Change the inverse trig term

$$\sin y = \sin(\sin^{-1} x)$$

$$\sin(y) = x$$

Step 2: "Draw the triangle"



$$\text{Sine } y = \frac{\text{opposite}}{\text{hypotenuse}}$$

Pythagorean Theorem

$$a^2 + b^2 = c^2$$

Step 3: Use implicit diffentiation to find *dy*/*dx*

$$\cos(y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)} \Longrightarrow \text{using the triangle, } \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

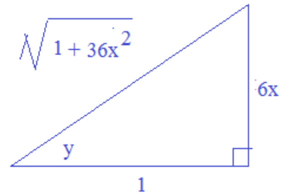
Example: $y = \tan^{-1}(6x)$ Find the derivative.

Step 1: Change the inverse trig term

$$\tan y = \tan(\tan^{-1}(6x))$$

$$\tan(y) = 6x$$

Step 2: "Draw the triangle"



$$\text{Tan } y = \frac{\text{opposite}}{\text{adjacent}}$$

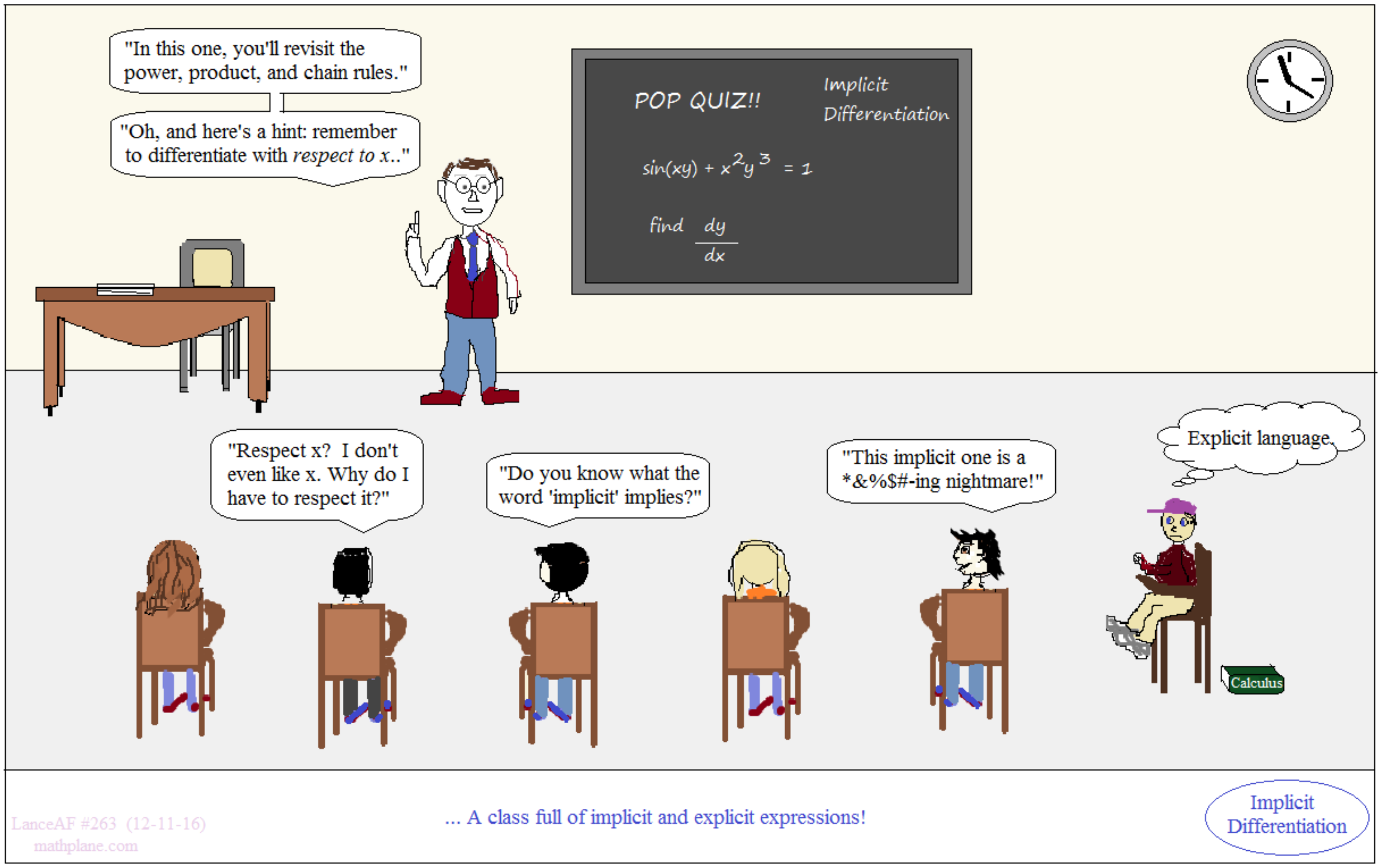
Pythagorean Theorem

$$a^2 + b^2 = c^2$$

Step 3: Use implicit diffentiation to find *dy*/*dx*

$$\sec^2(y)\frac{dy}{dx} = 6$$

$$\frac{dy}{dx} = \frac{6}{\sec^2(y)} \Longrightarrow \text{using the triangle, } \frac{dy}{dx} = \frac{6}{\sqrt{1 + 36x^2}^{\,2}} = \frac{6}{1 + 36x^2}$$



Practice Quiz-→

Implicit Differentiation

1) If $y = \sin(x)\cos(y)$, then @ $(\pi, 0)$ $\frac{dy}{dx} =$

a) -1

b) 0

c) 1

d) π

e) 2π

2) If $x^2 + 2y^2 = 22$, what is the behavior of the graph at $(-2, 3)$

a) increasing, concave up

b) increasing, concave down

c) decreasing, concave up

d) decreasing, concave down

e) increasing, point of inflection

3) A plane flies 6 miles high above the ground.
At the moment, it is directly 10 miles from an airport tower, and it is approaching the tower at a rate of 400 miles per hour.

How fast is the plane traveling?

4) Find the equation of the line tangent to $(x + 5)^2 + y^2 = 40$ @ $(1, 2)$

5) Determine the slope of the following curve at the given point:

$2xy^{1/3} + y = 5$ @ $(2, 1)$

6) $x^2 + y^2 = 36$ What is $\frac{d^2y}{dx^2}$ at $(0, 6)$?

7) Determine the slope of the following curve at the given point.

$$3xy + 6x^{3/2}y^{-1/2} = 189 \quad @ (9, 1)$$

8) $4\cos(xy) = 5x + 7y$ find $\frac{dy}{dx}$

9) Given the equation $x^2 + 2y^2 = 1$

Find the lines with slope = 1 that are tangent to the ellipse.

Implicit Differentiation

SOLUTIONS

Use implicit differentiation to find the derivative (instantaneous rate of change)

1) If $y = \sin(x)\cos(y)$, then @ $(\pi, 0)$ $\frac{dy}{dx} =$

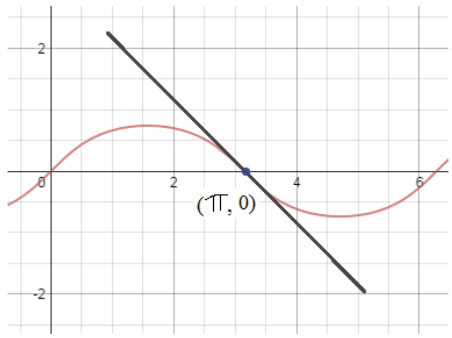
a) -1

b) 0

c) 1

d) π

e) 2π



product rule

$$1 \cdot \frac{dy}{dx} = \cos(x)\cos(y) + \left(-\sin(y)\frac{dy}{dx}\right)\sin(x)$$

to find IROC at point, substitute $(\pi, 0)$

$$\frac{dy}{dx} = \cos(\pi)\cos(0) - \sin(0)\frac{dy}{dx} \cdot \sin(\pi)$$

$$= (-1)(1) + (0)\frac{dy}{dx}(0) = -1$$

2) If $x^2 + 2y^2 = 22$, what is the behavior of the graph at $(-2, 3)$

a) increasing, concave up

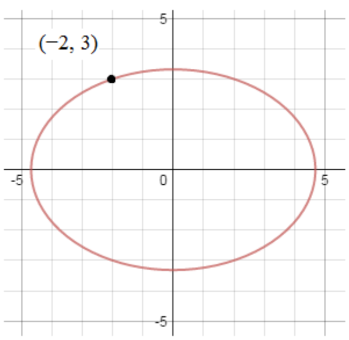
b) increasing, concave down

c) decreasing, concave up

d) decreasing, concave down

e) increasing, point of inflection

Notice, the graph is an ellipse!



To determine increasing or decreasing, find first derivative...

$$2x + 4y\frac{dy}{dx} = 0$$

$$4y\frac{dy}{dx} = -2x \qquad \text{at } (-2, 3) \quad \frac{dy}{dx} = \frac{-(-2)}{2(3)} = \frac{1}{3} > 0$$

increasing...

$$\frac{dy}{dx} = \frac{-x}{2y}$$

To determine concavity, find second derivative...

$$\frac{dy}{dx} = \frac{-x}{2y}$$

Quotient Rule

$$\frac{d}{dx} \cdot \frac{dy}{dx} \longrightarrow \frac{d^2y}{dx^2} = \frac{-1(2y) - 2\frac{dy}{dx}(-x)}{(2y)^2} = \frac{-2y + 2x\frac{dy}{dx}}{4y^2}$$

$$\frac{-y + x\frac{dy}{dx}}{2y^2} = \frac{-y + x\left(\frac{-x}{2y}\right)}{2y^2} \qquad \text{at } (-2, 3) \quad \frac{d^2y}{dx^2} = \frac{-(3) + (-2)\frac{1}{3}}{2(3)^2}$$

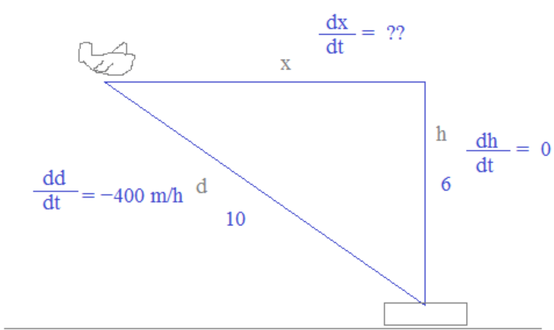
$$= \frac{-11/3}{18} < 0$$

concave down...

3) A plane flies 6 miles high above the ground.
At the moment, it is directly 10 miles from an airport tower, and it is approaching the tower at a rate of 400 miles per hour.

How fast is the plane traveling?

x = horizontal distance
d = direct distance
h = height above ground



$$x^2 + h^2 = d^2$$

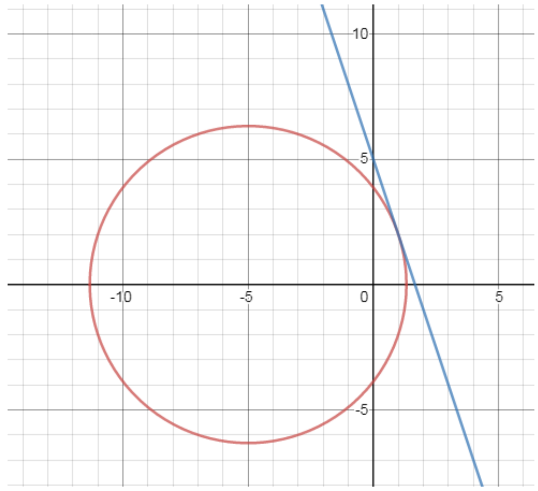
$$2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 2d\frac{dd}{dt}$$

$$16\frac{dx}{dt} + 12(0) = 20(-400)$$

$$\frac{dx}{dt} = 500 \text{ m/h}$$

SOLUTIONS

4) Find the equation of the line tangent to $(x + 5)^2 + y^2 = 40$ @ (1, 2)



$x^2 + 10x + 25 + y^2 = 40$

To find slope, we use implicit differentiation...

$2x + 10 + 0 + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-10 - 2x}{2y}$

@ (1, 2) $\quad \frac{dy}{dx} = \frac{-10 - 2(1)}{2(2)} = -3$

equation of line : $y - 2 = -3(x - 1)$

5) Determine the slope of the following curve at the given point:

$2xy^{1/3} + y = 5 \quad$ @ (2, 1)

The derivative involves implicit differentiation, power rule, and product rule....

$(2x)(y^{1/3}) + y = 5$

$2y^{1/3} + (1/3)y^{-2/3}\frac{dy}{dx}(2x) + 1\frac{dy}{dx} = 0$

We can simplify (i.e. solve for $\frac{dy}{dx}$). However, since we're only seeking the slope, we'll skip that step and substitute now...

@ (2, 1)

$2(1)^{1/3} + (1/3)(1)^{-2/3}\frac{dy}{dx}(2(2)) + 1\frac{dy}{dx} = 0$

$2 + (1/3)\frac{dy}{dx}(4) + 1\frac{dy}{dx} = 0$

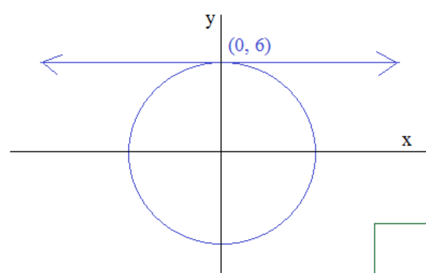
$(7/3)\frac{dy}{dx} = -2$

slope = -6/7

6) $x^2 + y^2 = 36$ What is $\frac{d^2y}{dx^2}$ at (0, 6) ?

Notice: the first derivative equals 0 at (0, 6) -- maximum

the second derivative is negative at (0, 6) -- concave down



1st derivative: $2x + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-x}{y}$

2nd derivative: $\frac{d^2y}{dx^2} = \frac{-1(y) - \frac{dy}{dx}(-x)}{y^2}$

(then, substitute for *dy/dx*)

$\frac{d^2y}{dx^2} = \frac{-1(y) - \frac{-x}{y}(-x)}{y^2} = \frac{-y - \frac{x^2}{y}}{y^2}$

Then, @ (0, 6) ---> $\frac{-6 - 0}{6^2} = \frac{-1}{6}$

Using explicit differentiation:

$y = \pm(36 - x^2)^{\frac{1}{2}}$

$y' = \frac{1}{2}(36 - x^2)^{-\frac{1}{2}}(-2x)$

$y' = (-x)(36 - x^2)^{-\frac{1}{2}}$

$y'' = (-1)(36 - x^2)^{-\frac{1}{2}} + -\frac{1}{2}(36 - x^2)^{\frac{-3}{2}}(-2x)(-x)$

$= -(36 - x^2)^{-\frac{1}{2}} - x^2(36 - x^2)^{\frac{-3}{2}}$

plug in (0, 6) ----> $-1/6 - 0$ ✓

SOLUTIONS

7) Determine the slope of the following curve at the given point.

$$3xy + 6x^{3/2}y^{-1/2} = 189 \quad @\ (9, 1)$$

The derivative will use the power rule, implicit differentiation, and product rule....

$$(3x)(y) + (6x^{3/2})(y^{-1/2}) = 189$$

derivative:

$$3y + 3x\frac{dy}{dx} + 9x^{1/2}y^{-1/2} + 6x^{3/2}(-1/2)y^{-3/2}\frac{dy}{dx} = 0$$

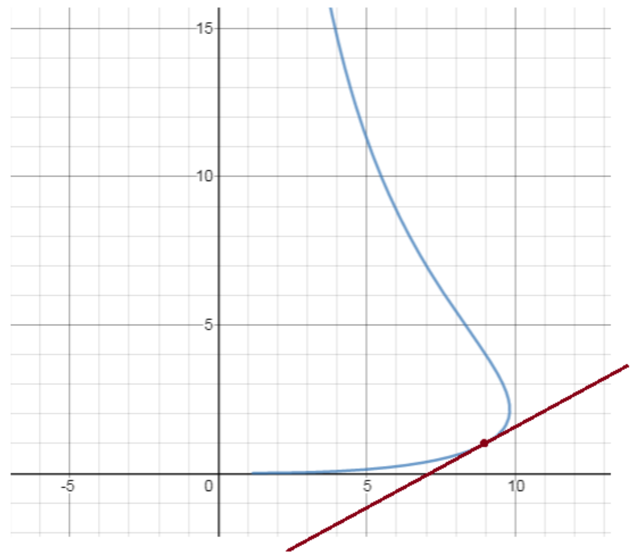
To find the slope, we'll substitute @ (9, 1)

$$3(1) + 3(9)\frac{dy}{dx} + 9(9)^{1/2}(1)^{-1/2} + 6(9)^{3/2}(-1/2)(1)^{-3/2}\frac{dy}{dx} = 0$$

$$3 + 27\frac{dy}{dx} + 27 + 6(27)(-1/2)\frac{dy}{dx} = 0$$

$$30 + 27\frac{dy}{dx} - 81\frac{dy}{dx} = 0$$

$$30 = 54\frac{dy}{dx} \qquad \text{slope} = \frac{5}{9}$$



8) $4\cos(xy) = 5x + 7y$ find $\frac{dy}{dx}$

(Note: it would be extremely difficult to put this in explicit form -- i.e. isolate the y's together... So, implicit differentiation must be used...)

$$4\cos(xy) = 5x + 7y$$

the right side is easy...

$$= 5 + 7\frac{dy}{dx}$$

then, the left side involves constant, trig function, product rule, chain rule..

$$4 \cdot (-\sin(xy)) \cdot \left((1)(y) + (x)(1)\frac{dy}{dx}\right)$$

now, simplify and collect the dy/dx

$$-4\sin(xy)\left((1)(y) + (x)(1)\frac{dy}{dx}\right) = 5 + 7\frac{dy}{dx}$$

distribute the left side...

$$-4\sin(xy)(y) - 4\sin(xy)(x)\frac{dy}{dx} = 5 + 7\frac{dy}{dx}$$

$$-4\sin(xy)(y) - 5 = 4\sin(xy)(x)\frac{dy}{dx} + 7\frac{dy}{dx}$$

factor out the dy/dx

$$-4\sin(xy)(y) - 5 = \frac{dy}{dx}\left(4\sin(xy)(x) + 7\right)$$

factor out the negative

$$-\left(4\sin(xy)(y) + 5\right) = \frac{dy}{dx}\left(4\sin(xy)(x) + 7\right)$$

$$\boxed{\frac{-\left(4\sin(xy)(y) + 5\right)}{\left(4\sin(xy)(x) + 7\right)} = \frac{dy}{dx}}$$

9) Given the equation $x^2 + 2y^2 = 1$

SOLUTIONS

Find the lines with slope = 1 that are tangent to the ellipse.

First, find the slope of any point on the curve...

$$2x + 4y\frac{dy}{dx} = 0$$

$$4y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-x}{2y}$$

Then, we want specifically, when the slope = 1

$$1 = \frac{-x}{2y} \implies x = -2y$$

So, the related slope occurs along the line $x = -2y$....
But, we want it specifically for points in the curve....
So, we'll find where that occurs....

$$x^2 + 2y^2 = 1$$

$$x = -2y$$

Using substitution, we'll solve the system.

$$(-2y)^2 + 2y^2 = 1$$

$$6y^2 = 1$$

$$y = \pm\frac{1}{\sqrt{6}}$$

so, the points are

$$x = -2 \cdot \left(\pm\frac{1}{\sqrt{6}}\right)$$

$$x = \pm\frac{2}{\sqrt{6}}$$

Tangent lines: $y + \frac{1}{\sqrt{6}} = x - \frac{2}{\sqrt{6}}$ $\quad y - \frac{1}{\sqrt{6}} = x + \frac{2}{\sqrt{6}}$

Option 2: Use explicit form to find slope...

$$x^2 + 2y^2 = 1$$

$$2y^2 = -1 - x^2$$

$$y^2 = \frac{1}{2}(-1 - x^2)$$

$$y = \pm\sqrt{\frac{1}{2}(-1 - x^2)}$$

$$y' = \pm\frac{1}{2}\left(\frac{1}{2}(-1 - x^2)\right)^{-1/2}(-x)$$

$$y' = \frac{-x}{\pm 2\sqrt{\frac{1}{2}(-1 - x^2)}}$$

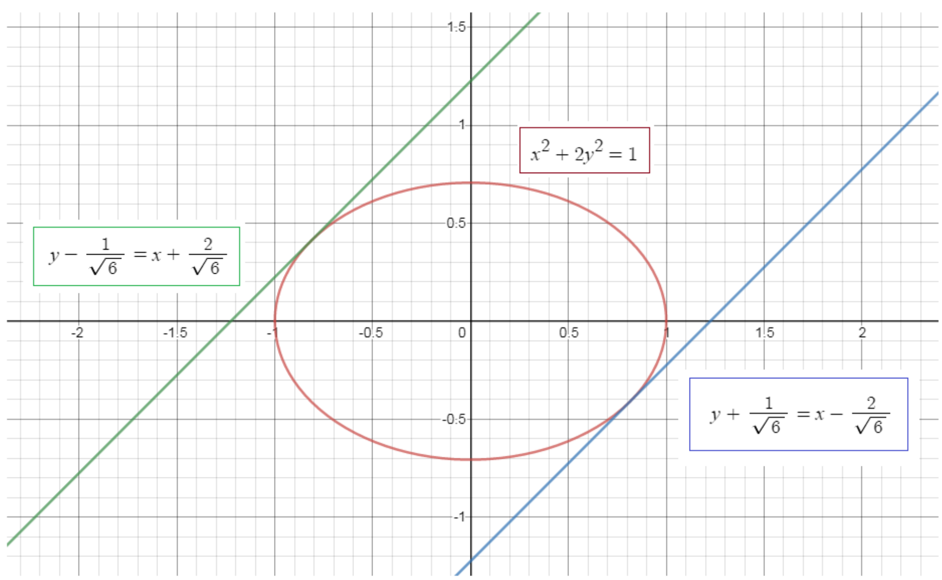
Then, let slope y' = 1

$$1 = \frac{-x}{\pm 2\sqrt{\frac{1}{2}(-1 - x^2)}}$$

$$x^2 = \pm 2(-1 - x^2) \implies 3x^2 = 2$$

$$x = \pm\sqrt{\frac{2}{3}}$$

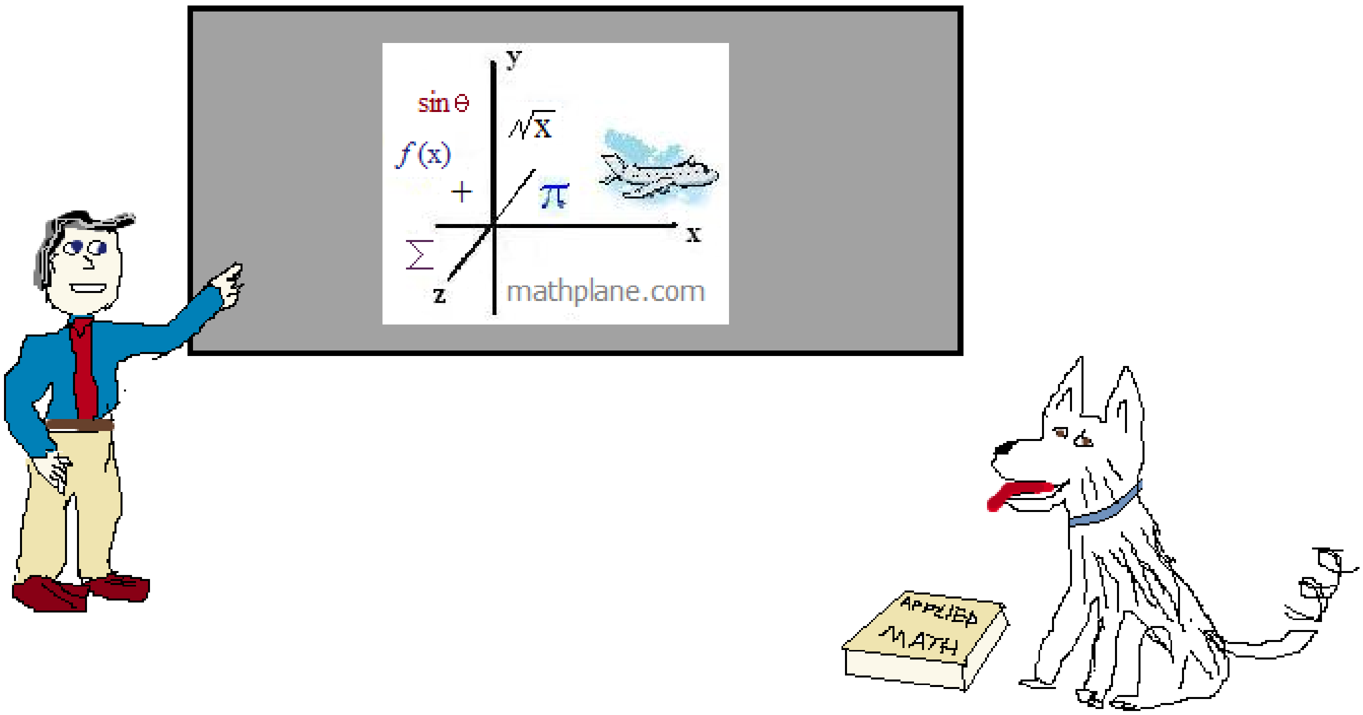
$$\left(+\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right) \quad \left(-\frac{2}{\sqrt{6}}, +\frac{1}{\sqrt{6}}\right)$$



Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers



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