Integrals using Trig Substitution

Notes, Examples, and Practice Exercises (w/ solutions)

Topics include U-substitution, trig identities, natural log, and more.

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Integration using trigonometry substitution

Introduction:

$$\int 2x\sqrt{3x^2+7}\;dx$$

Since the derivative of $3x^2$ is $6x$, we can integrate this function rather easily.

$$\frac{1}{3}\int 6x\sqrt{3x^2+7}\;dx$$

$$\frac{1}{3}\,\frac{(3x^2+7)^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2\,(3x^2+7)^{\frac{3}{2}}}{9} + C$$

But, what happens if the integral involves a radical WITHOUT the derivative on the outside?

$$\int \frac{1}{x^2\sqrt{1-x^2}}\,dx$$

the derivative of x^2 is 2x, so direct substitution will not work..
(i.e. there is "no U and U'")

Integration by Trig Substitution is a technique to evaluate integrals involving particular radical forms.

trigonometry identity: $1+\tan^2 x = \sec^2 x$

$\cos^2 x + \sin^2 x = 1$

$\sqrt{a^2-x^2}$	use sine substitution
$\sqrt{a^2+x^2}$	use tangent substitution
$\sqrt{x^2-a^2}$	use secant substitution

In the above integral, we can try sine substitution

let $a = 1$
$x = \sin(u)$

$$\int \frac{1}{x^2\sqrt{1-x^2}}\,dx \longrightarrow \int \frac{1}{(\sin(u))^2\sqrt{1-(\sin(u))^2}}\,dx$$

then, let $\frac{dx}{du} = \cos(u)$

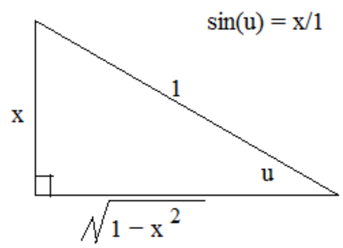
$dx = \cos(u)\,du$

$\sin^2(u) = (\sin(u))^2$

$$\int \frac{1}{(\sin(u))^2\sqrt{1-(\sin(u))^2}}\cos(u)\,du$$

$$\int \frac{1}{(\sin(u))^2\sqrt{1-(\sin(u))^2}}\cos(u)\,du$$

sin(u) = x/1



$$\int \frac{\cos(u)}{\sin^2(u)\,\cos(u)}\,du$$

Trig substitution has eliminated the radical and created an equation that can be integrated!

$$\int \csc^2(u)\,du$$

$$-\cot(u)+C \xrightarrow{\text{(using the triangle)}} \frac{-\sqrt{1-x^2}}{x} + C$$

Using Trig Substitution: Secant

$\sqrt{x^2 - a^2}$

let $x = a \cdot \sec(u)$

Pythagorean Trig Identity:

$1 + \tan^2 x = \sec^2 x$

then, $\tan^2 x = \sec^2 x - 1$

Example:

$$\int \frac{1}{\sqrt{x^2 - 9}}\, dx$$

Step 1: Identify "a" and "x" and other parts

Assuming $a = 3$,

let $x = 3\sec(u)$

then, $\dfrac{dx}{du} = 3\sec(u)\tan(u)$

$dx = 3\sec(u)\tan(u)\, du$

Step 2: Substitute and solve

$$\int \frac{1}{\sqrt{x^2 - 9}}\, dx$$

$$\int \frac{1}{\sqrt{(3\sec(u))^2 - 9}}\, 3\sec(u)\tan(u)\, du$$

Don't forget to substitute for *dx*

$$\int \frac{3\sec(u)\tan(u)}{\sqrt{9\,((\sec^2 u) - 1)}}\, du$$

$$\int \frac{3\sec(u)\tan(u)}{\sqrt{9\tan^2(u)}}\, du$$

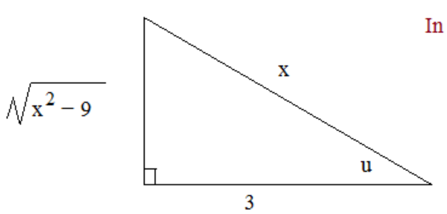
$$\int \frac{3\sec(u)\tan(u)}{3\tan(u)}\, du$$

Note: We take advantage of trig identity to get rid of the square root!

$$\int \frac{\sec(u)}{1}\, du \longrightarrow \ln|\sec(u) + \tan(u)| + C$$

Step 3: Return to "original x's"

To find sec(u) and tan(u) in terms of x, we construct a triangle:



In the beginning, we let $x = 3\sec(u)$

Therefore, $\sec(u) = \dfrac{x}{3}$ $\dfrac{\text{(hypotenuse)}}{\text{(adjacent)}}$

And, using Pythagorean Theorem, we obtain the opposite side...

$\tan(u) = \dfrac{\sqrt{x^2 - 9}}{3}$ $\dfrac{\text{(opposite)}}{\text{(adjacent)}}$

$$\ln\left|\frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3}\right| + C$$

which is equivalent to

$$\ln\left|x + \sqrt{x^2 - 9}\right| + C$$

Using Trig Substitution: Sine

$$\sqrt{a^2 - x^2}$$

let $x = a \cdot \sin(u)$

Pythagorean Trig Identity:

$$\sin^2 x + \cos^2 x = 1$$

then, $1 - \sin^2 x = \cos^2 x$

Example:

$$\int \frac{1}{(25 - x^2)^{3/2}}\, dx$$

There is no way to directly integrate this function with the power rule (because the "U" term x^2 has no "U'" term x)

But, recognizing the term $(25 - x^2)$ fits $a^2 - u^2$

we can try using sine substitution:

Step 1: Identify the "x's, a's, and variables"

let $x = 5\sin(u)$ then, $\frac{dx}{du} = 5\cos(u)$

$dx = 5\cos(u)\, du$

Don't forget to find *dx* !!

Step 2: Substitute and Solve

$$\int \frac{1}{(25 - (5\sin(u))^2)^{3/2}}\, 5\cos(u)\, du$$

$$\int \frac{5\cos(u)}{(25 - 25\sin^2 u)^{3/2}}\, du$$

$$\int \frac{5\cos(u)}{(25[1 - \sin^2 u])^{3/2}}\, du$$

$$\int \frac{5\cos(u)}{(25\cos^2 u)^{3/2}}\, du$$

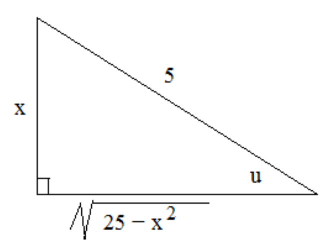
$$\int \frac{5\cos(u)}{125\cos^3 (u)}\, du$$

The radical is gone, and we have an equation that can be integrated!

$$\int \frac{1}{25} \sec^2 (u)\, du \longrightarrow \frac{1}{25} \tan(u) + C$$

$x = 5\sin(u)$

so, $\sin(u) = \frac{x}{5}$ $\frac{\text{(opposite)}}{\text{(hypotenuse)}}$



(Pythagorean Theorem)

Therefore, $\tan(u) = \frac{x}{\sqrt{25 - x^2}}$

Step 3: Return to "original x's"

$$\frac{1}{25} \frac{x}{\sqrt{25 - x^2}} + C$$

Use Trig functions to return to "x terms"

Using Trig Substitution: Tangent

$\sqrt{a^2 + x^2}$

let $x = a \cdot \tan(u)$

Pythagorean Trig Identity

$1 + \tan^2 x = \sec^2 x$

Example: $\displaystyle\int \frac{x}{\sqrt{x^2+9}}\, dx$

Trig Substitution:

Since there is an "a squared plus x squared" under a radical, let's try trig substitution with tangent:

Find terms: let $x = 3 \cdot \tan(u)$

then, $\dfrac{dx}{du} = 3\sec^2(u)$ $\qquad x^2 = 9\tan^2(u)$

$dx = 3\sec^2(u)\, du$

Substitute:

$$\int \frac{x}{\sqrt{x^2+9}}\, dx$$

$$\int \frac{3\tan(u)}{\sqrt{9\tan^2(u) + 9}}\, 3\sec^2(u)\, du$$

$$\int \frac{9\tan(u)\sec^2(u)}{\sqrt{9(1+\tan^2(u))}}\, du$$

Trig identity: $1 + \tan^2 x = \sec^2 x$

$$\int \frac{9\tan(u)\sec^2(u)}{3\sec(u)}\, du$$

$$\int 3\sec(u)\tan(u)\, du$$

$$3\sec(u) + C$$

"Return to 'x' terms":

$$3 \cdot \frac{\sqrt{x^2+9}}{3} + C$$

$$\sqrt{x^2+9} + C$$

Basic Method:

(Traditional Substitution)

$$\int \frac{x}{\sqrt{x^2+9}}\, dx$$

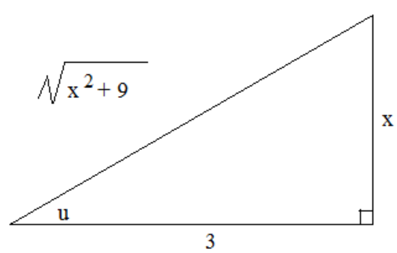
$$\int x\,(x^2+9)^{\frac{-1}{2}}\, dx$$

$$\frac{1}{2}\int 2x\,(x^2+9)^{\frac{-1}{2}}\, dx$$

$$\frac{1}{2}\,\frac{(x^2+9)^{\frac{1}{2}}}{\frac{1}{2}}$$

$$(x^2+9)^{\frac{1}{2}} + C$$

Pythagorean Theorem: $a^2 + b^2 = c^2$



$x = 3\tan(u)$

therefore, $\tan(u) = \dfrac{x}{3}$ $\quad\dfrac{\text{(opposite)}}{\text{(adjacent)}}$

$\sec(u) = \dfrac{\sqrt{x^2+9}}{3}$ $\quad\dfrac{\text{(hypotenuse)}}{\text{(adjacent)}}$

Example: $\displaystyle\int \frac{dx}{\sqrt{9 + x^2}}$

Step 1: Select the trig substitution...

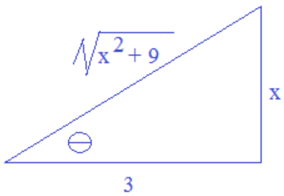
use identity $1 + \tan^2 = \sec^2$

Step 2: Substitute for all terms

let $x = 3\tan\theta$

$\dfrac{dx}{d\theta} = 3\sec^2\theta$

$dx = 3\sec^2\theta \, d\theta$



Step 3: Solve

$$\int \frac{3\sec^2\theta \, d\theta}{\sqrt{9 + 9\tan^2\theta}}$$

$$\int \frac{3\sec^2\theta \, d\theta}{3\sqrt{1 + \tan^2\theta}}$$

$$\int \sec\theta \, d\theta$$

$$\ln(\sec\theta + \tan\theta) + C$$

Step 4: Substitute to original variables and simplify

$$\ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + C$$

$$\ln\frac{1}{3}\left|\sqrt{x^2+9} + x\right| + C$$

$$\ln\frac{1}{3} + \ln\left|\sqrt{x^2+9} + x\right| + C$$

$$\boxed{\ln\left|\sqrt{x^2+9} + x\right| + C}$$

since $\ln\frac{1}{3}$ is a constant, it can be combined with C...

Example: $\displaystyle\int \frac{\sqrt{16 - n^2}}{n} \, dn$

Step 1: select the trig subsitution..

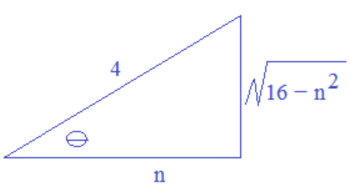
use identity $\sin^2 = 1 - \cos^2$

Step 2: substitute for all the terms

Let $n = 4\cos\theta$

so $\dfrac{dn}{d\theta} = -4\sin\theta$

$dn = -4\sin\theta \, d\theta$



Step 3: Solve

$$\int \frac{\sqrt{16 - (4\cos\theta)^2}}{4\cos\theta} \, -4\sin\theta \, d\theta$$

$$\int \frac{\sqrt{16(1 - \cos^2\theta)}}{4\cos\theta} \, -4\sin\theta \, d\theta$$

$$\int \frac{4\sin\theta \cdot -4\sin\theta \, d\theta}{4\cos\theta}$$

$$\int \frac{-4\sin^2\theta \, d\theta}{\cos\theta}$$

$$-4\int \frac{1 - \cos^2\theta}{\cos\theta} \, d\theta$$

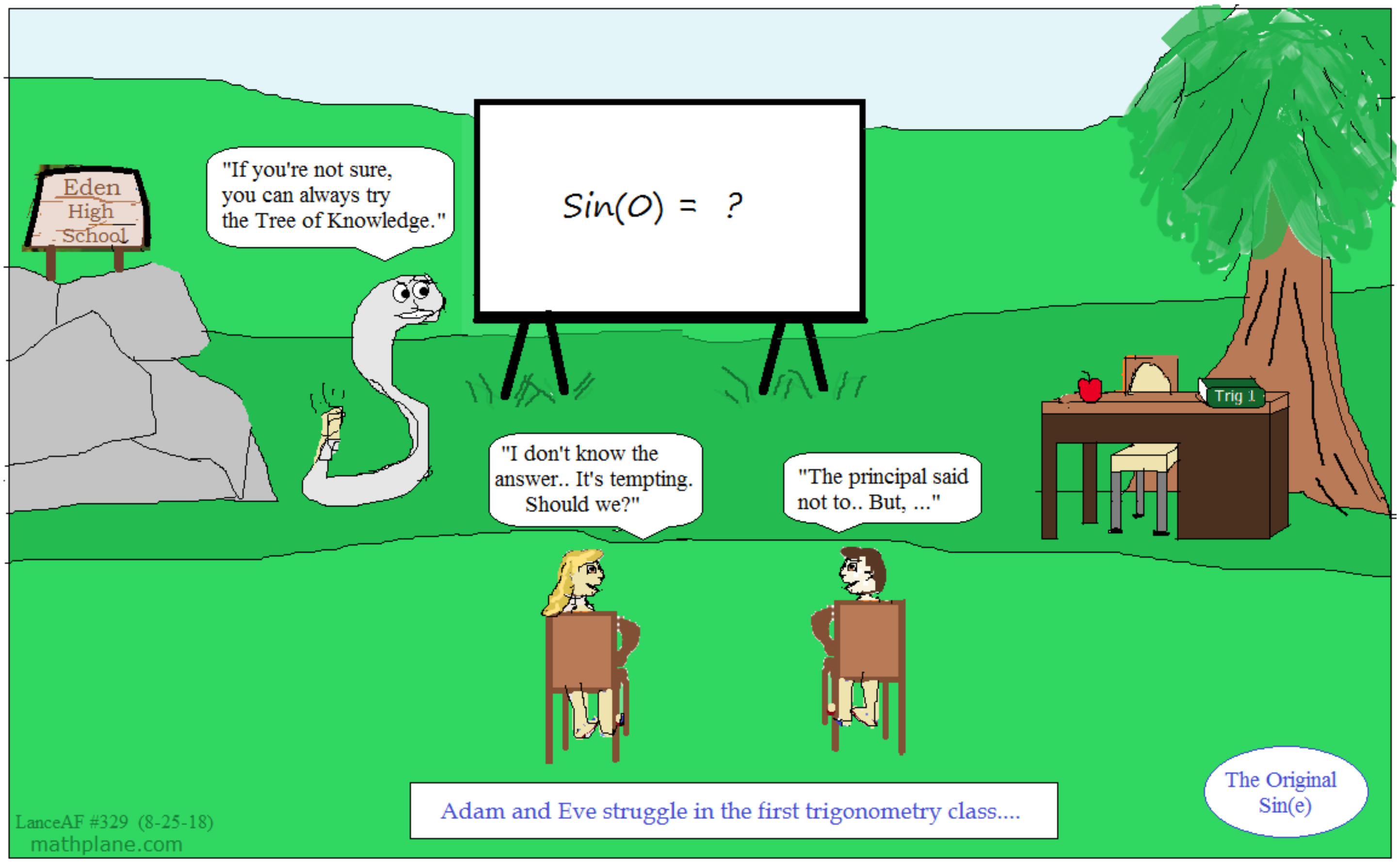
$$-4\int \sec\theta - \cos\theta \, d\theta$$

$$-4\ln(\sec\theta + \tan\theta) + 4\sin\theta + C$$

Step 4: Substitute

$$-4\ln\left|\frac{4}{n} + \frac{\sqrt{16 - n^2}}{n}\right| + \frac{4\sqrt{16 - n^2}}{4} + C$$

$$\boxed{4\ln\left(\frac{\sqrt{16 - n^2} - 4}{|n|}\right) + \sqrt{16 - n^2} + C}$$



Practice Questions-→

Using Trig Substitution

1) $$\int_0^1 x^3 \sqrt{1 - x^2}\; dx$$

2) $$\int \frac{dx}{x(x^4 + 1)}$$

3) $$\int \frac{\sqrt{25x^2 - 4}}{x}\, dx$$

4) $$\int \frac{1}{\sqrt{x^2 - 1}}\, dx$$

SOLUTIONS

Using Trig Substitution

1) $\displaystyle\int_0^1 x^3\sqrt{1-x^2}\;dx$

Recognizing the trig identity

$1 + \sin^2(x) = \cos^2(x)$

$x = \sin\theta$

$dx = \cos\theta\, d\theta$

note: we also change the x boundaries to θ boundaries

$\displaystyle\int_0^1 \Rightarrow \int_0^{\pi/2}$

$$\int_0^{\frac{\pi}{2}} \sin^3\theta\sqrt{1-\sin^2\theta}\;\cos\theta\, d\theta = \int_0^{\frac{\pi}{2}} \sin^3\theta\,\cos^2\theta\, d\theta$$

Split the sine term...

$$\int_0^{\frac{\pi}{2}} \sin\theta\,\sin^2\theta\,\cos^2\theta\, d\theta = \int_0^{\frac{\pi}{2}} \sin\theta\,(1-\cos^2\theta)\cos^2\theta\, d\theta$$

$$\int_0^{\frac{\pi}{2}} \sin\theta\cos^2\theta - \sin\theta\cos^4\theta\; d\theta = -\frac{\cos^3\theta}{3} + \frac{\cos^5\theta}{5}\Bigg|_0^{\frac{\pi}{2}}$$

$-0/3 + 0/5 - (-1/3 + 1/5) =$ **2/15**

2) $\displaystyle\int \frac{dx}{x(x^4+1)}$

The sum of two squares $(x^4 + 1)$
can be a signal to try using tangent or arctangent.

$1 + \tan^2 x = \sec^2 x$

Let $x^2 = \tan U$

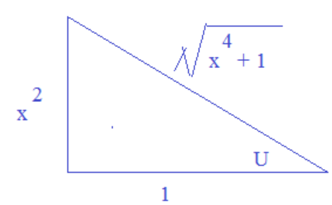
$2x\, dx = \sec^2 U\; dU$

$dx = \dfrac{\sec^2 U}{2x}\; dU$

$$\int \frac{\frac{\sec^2 U}{2x}}{x(\tan^2 U + 1)}\, dU = \int \frac{\sec^2 U}{2x^2\,(\sec^2 U)}\, dU = \int \frac{1}{2\tan U} \Rightarrow \frac{1}{2}\,\frac{\cos U}{\sin U}$$

$$= \frac{1}{2}\, ln\,|\sin U| + C$$

Then, substitute the U for x....



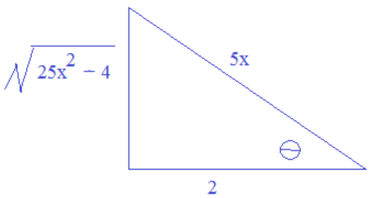
$$\frac{1}{2}\, ln\left|\frac{x^2}{\sqrt{x^4+1}}\right| + C \Rightarrow ln\left|\frac{x}{(x^4+1)^{1/4}}\right| \Rightarrow ln\,x - \frac{1}{4}\, ln\,(x^4+1) + C$$

SOLUTIONS

3) $\displaystyle\int \frac{\sqrt{25x^2-4}}{x}\,dx \Rightarrow \sqrt{4\left(\frac{25}{4}x^2-1\right)}$

$2\sqrt{\left(\frac{25}{4}x^2-1\right)}$

$\boxed{\sec^2 x - 1 = \tan^2 x}$



Let $\frac{25}{4}x^2 = \sec^2\theta$

$x^2 = \frac{4}{25}\sec^2\theta$

$x = \frac{2}{5}\sec\theta$

$dx = \frac{2}{5}\sec\theta\tan\theta\,d\theta$

$\displaystyle\int \frac{\sqrt{25\left(\frac{2}{5}\sec\theta\right)^2-4}}{\frac{2}{5}\sec\theta}$

$\displaystyle\int \frac{\sqrt{4\sec^2\theta-4}}{\frac{2}{5}\sec\theta}$

$\displaystyle\int \frac{\sqrt{4\tan^2\theta}}{\frac{2}{5}\sec\theta}\;\frac{2}{5}\sec\theta\tan\theta\,d\theta$

$\displaystyle\int 2\tan^2\theta\,d\theta$

$\displaystyle 2\int \sec^2\theta - 1\;d\theta$

$2\tan\theta - 2\theta + C$

$\boxed{\sqrt{25x^2-4} - 2\tan^{-1}\left(\frac{\sqrt{25x^2-4}}{2}\right) + C}$

4) $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,dx$

Integration using trigonometry substitution	$\sqrt{a^2-x^2}$	use sine substitution
trigonometry identity: $1+\tan^2 x = \sec^2 x$	$\sqrt{a^2+x^2}$	use tangent substitution
$\cos^2 x + \sin^2 x = 1$	$\sqrt{x^2-a^2}$	use secant substitution

Step 1: Identify the trig substitution

secant trig substitution...

Step 2: Determine the values

let $x = \sec\theta$

then, $dx = \sec\theta\tan\theta\;d\theta$

Step 3: Substitute and Solve

$\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,dx \longrightarrow \int \frac{1}{\sqrt{(\sec\theta)^2-1}}\;\sec\theta\tan\theta\;d\theta$

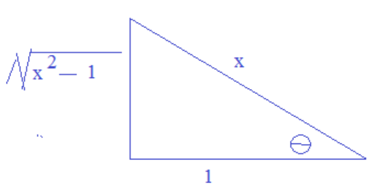
$\displaystyle\int \frac{\sec\theta\tan\theta}{\sqrt{\tan^2\theta}}\,d\theta$

$\displaystyle\int \frac{\sec\theta\tan\theta}{\tan\theta}\,d\theta$

By choosing this trig identity, we've eliminated the radical!

Step 4: Substitute

$\displaystyle\int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C$ ✓



$\ln|\sec\theta+\tan\theta| + C = \boxed{\ln\left|x+\sqrt{x^2-1}\right| + C}$

Finding integral of secant...

$\displaystyle\int \sec(x)\;dx$

$\displaystyle\int \sec(x)\;\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx$

$\displaystyle\int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx$

Let $U = \sec(x)+\tan(x)$

$\frac{dU}{dx} = \sec(x)\tan(x) + \sec^2(x)$

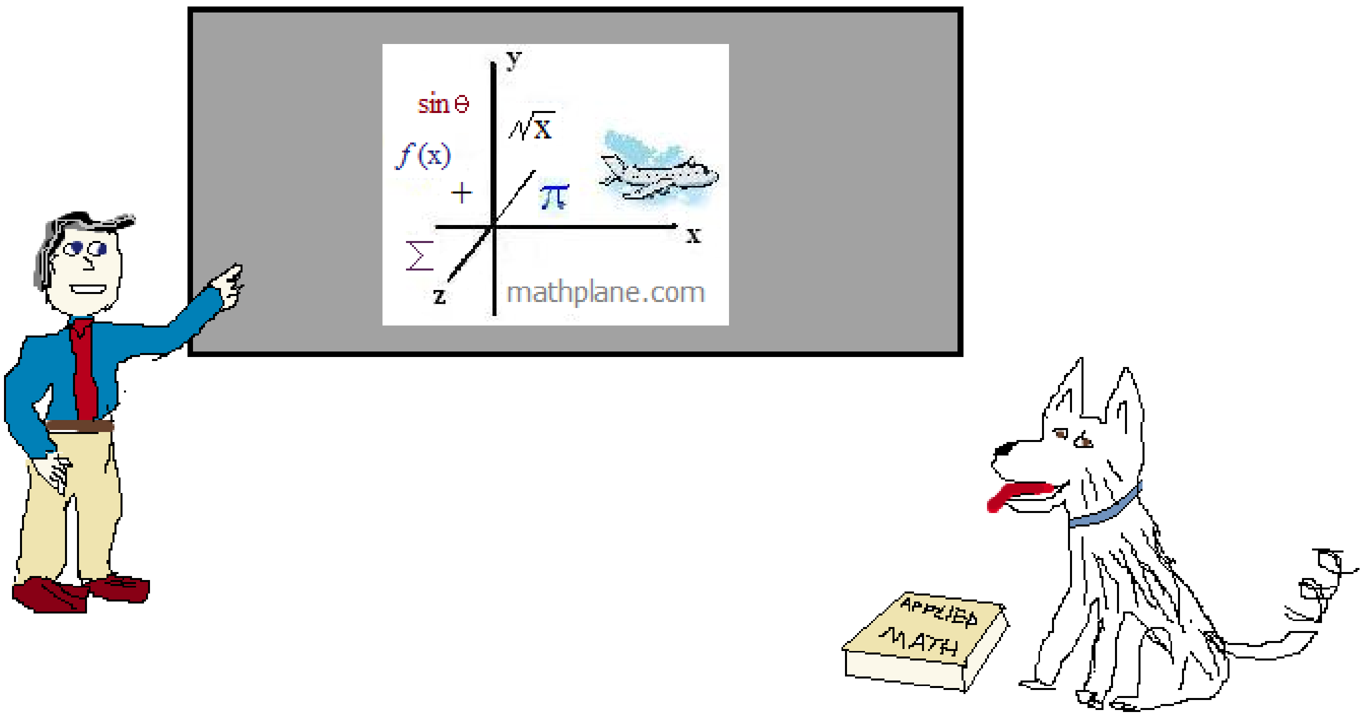
$\displaystyle\int \frac{dU}{U} = \ln|U|$

$\ln|\sec(x)+\tan(x)|$ ✓

Thanks for Visiting!

If you have questions, suggestions, or requests, let us know.

Cheers.



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