Exponents and Order of Operations

Notes, Examples, and Exercises (with Solutions)

$$12 + 3(4 + 7) \div 3(5) =$$

Topics include PEMDAS or GEMDAS, exponent laws, square roots, and more.

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Exponent Rules: Notes and Examples

Exponent *definition*:

$$X^A = X_1 \cdot X_2 \cdot X_3 \cdot \ \ldots \ \cdot X_{A-2} \cdot X_{A-1} \cdot X_A$$

Examples: $4^5 = 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 1024$

$$\left(\frac{2}{3}\right)^3 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{8}{27}$$

$$(-2)^7 = (-2) \cdot (-2) \cdot (-2) \cdot (-2) \cdot (-2) \cdot (-2) \cdot (-2) = -128$$

$$(-2)^6 = (-2) \cdot (-2) \cdot (-2) \cdot (-2) \cdot (-2) \cdot (-2) = 64$$

Rule #1 ('Addition Rule')

$$X^A \cdot X^B = X^{A+B}$$

Examples:

$$X^3 \cdot X^5 = X^8$$

$$5^3 \cdot 5^2 = 125 \cdot 25 = 3125 = 5^5$$

Note:

$$Y^2 \cdot Y^4 = Y^6$$

$$\underbrace{(Y \times Y)}_{2} \cdot \underbrace{(Y \times Y \times Y \times Y)}_{4} = \underbrace{Y \times Y \times Y \times Y \times Y \times Y}_{\text{6 total Y's}}$$

Rule #2: ('Multiplication Rule')

$$(X^A)^B = X^{AB}$$

Examples: $(X^4)^3 = X^{12}$

$$(4^2)^4 = 4^8 = 16^4 = 65536$$

Note:

$$(Y^5)^3 = \underbrace{(Y \times Y \times Y \times Y \times Y)}_{5} \cdot \underbrace{(Y \times Y \times Y \times Y \times Y)}_{5} \cdot \underbrace{(Y \times Y \times Y \times Y \times Y)}_{5} = Y^{15}$$

3 groups of 5 Y's
total: 3 x 5 = 15 Y's

Rule #3: ('zero exponent')

$$X^0 = 1$$

Examples: $Y^0 = 1$

$$8^0 = 1$$

$$(3cd)^0 = 1$$

What is 0^0 ? $0^A = 0$ because $0 \cdot 0 \cdot 0 \cdot 0 \ldots = 0$

(if $A \neq 0$)

$X^0 = 1$ (zero exponent rule)

$\boxed{0^0 = 1}$

Note: $Z^5 \cdot Z^{-5} = Z^0 = 1$

addition rule --- then, zero exponent rule

$$\frac{Z \times Z \times Z \times Z \times Z}{Z \times Z \times Z \times Z \times Z} = 1$$

Z^5

Z^{-5}

Rule #4: ('negatives' or 'reciprocal rule')

$$X^{(-A)} = \frac{1}{X^A}$$

Examples:

$$X^{-3} = \frac{1}{X^3}$$

$$5^{-2} = \frac{1}{25}$$ It is not equal to -25!!!

$$\left(\frac{1}{3}\right)^{-4} = 81$$

Note: $\boxed{Y^{(-A)} =}\; Y^{(-A)} \cdot \frac{Y^A}{Y^A} = \frac{Y^{(-A)} \cdot Y^A}{Y^A} = \frac{Y^{(-A+A)}}{Y^A} = \frac{Y^0}{Y^A} = \boxed{\frac{1}{Y^A}}$

multiply by one; exponent addition rule; zero exponent

Rule #5: ('base rule')

$$X^A \cdot Y^A = (XY)^A$$

Examples:

$$5^3 \cdot 7^3 = 125 \text{ x } 343 = 42875 = 35^3$$

$$= (5 \text{ x } 5 \text{ x } 5) \text{ x } (7 \text{ x } 7 \text{ x } 7) = (5 \text{ x } 7) \text{ x } (5 \text{ x } 7) \text{ x } (5 \text{ x } 7)$$

$$4^{\frac{1}{2}} \cdot 16^{\frac{1}{2}} = 64^{(1/2)} = 8$$

$$\sqrt{4} \text{ x } \sqrt{16} = \sqrt{4 \text{ x } 16} = \sqrt{64}$$

Rule #6: ('rational exponents')

$$X^{(1/2)} = \sqrt{X} \qquad X^{\left(\frac{A}{B}\right)} = \sqrt[B]{X^A}$$

Examples:

$$25^{(1/2)} = \sqrt{25} = 5$$

$$8^{(1/3)} = \sqrt[3]{8} = 2$$ ('cubed root of 8')

$$121^{(.5)} = 11$$

Note: $Y^{(1/2)} \cdot Y^{(1/2)} = Y^1 \qquad \sqrt{Y} \cdot \sqrt{Y} = Y$

(addition exponent rule)

$$8^{(1/3)} \cdot 8^{(1/3)} \cdot 8^{(1/3)} = 8^1 = 8$$

Order of Operations: PEMDAS

History: Order of operations go back to at least the 1500s (when exponents were introduced). They are necessary to remove ambiguity in mathematics and computer programming. Otherwise, how could we always specify exact answers?

$2 + 6 \times 10 = 8 \times 10 = 80?$
or
$2 + 6 \times 10 = 2 + 60 = 62?$

How does a computer calculate this equation?

PEMDAS: The order of operations are as follows:

1. Parentheses (includes brackets)
2. Exponents (includes roots)
3. Multiplication
   Division
4. Addition
   Subtraction

→ "PEMDAS" or <u>P</u>lease <u>E</u>xcuse <u>M</u>y <u>D</u>ear <u>A</u>unt <u>S</u>ally

Examples:

	--order--
$2 + 6 \times 10 = 62$	(multiplication, addition)
$70 - 10 \div 2 = 65$	(division, subtraction)
$(70 - 10) \div 2 = 30$	(inside parentheses, division)
$(4 + 5)^2 - (8 - 6)^2 = 9^2 - 2^2 = 77$	(inside parentheses, exponents, subtraction)
$5 + 10 \bullet 6 + 7^2 = 5 + 60 + 49 = 114$	(exponents, multiplication, addition)

Three More Rules:

1) 'Grouping Symbols' take precedent. $\sqrt{6 + 3} \times 10 = 30$ (since 6 & 3 are under the radical sign, addition took precedent!)

2) When operations are the same, go left to right. $7 + 5 - 3 + 4 - 1 = 12$

3) Fractions: Numerator and Denominator are solved separately (as if they had parentheses) $\dfrac{7 + 14}{4 - 1} = \dfrac{(7 + 14)}{(4 - 1)} = \dfrac{21}{3} = 7$

A more accurate (updated) acronym for the order of operations when simplifying an expression:

GEMDAS

1. Grouping ( ) { } [ ] $\sqrt{}$ numerators & denominators
2. Exponents
3. Multiplication Division x ÷ (left to right)
4. Addition Subtraction + − (left to right)

Note: multiplication and division are related operations
EX: dividing by 4 is the same as multiplying by 1/4

addition and subtraction are related operations
EX: subtracting 7 is the same as adding (-7)

Examples:

-- order --

$\frac{3+7}{2}$	grouping (numerator), division	$\frac{10}{2} = 5$
$6 \text{ x } 5 - 4 \text{ x } 3 + 2$	multiplication, addition/subtraction (left to right)	$30 - 12 + 2 = 20$
$-3^2 + \sqrt{11-2}$	grouping (sq. root), exponents, multiplication, addition	$-3^2 + \sqrt{9}$ $-3^2 + 3$ $-9 + 3$ -6

Order of operations / PEMDAS / GEMDAS

Example: $12 \times 4 \div 3 \times 2$

Is it 32 or 8?

$12 \times 4 = 48$

$48 / 3 = 16$

$16 \times 2 = \boxed{32}$

Note: $12 \times 4 \div 3 \times 2$ is different from $(12 \times 4) \div (3 \times 2)$

$48 / 6 = 8$

Example: $7 - 6 + 10 =$

Is it 11 or -9?

Addition and subtraction are together...
So, we go left to right...

$7 - 6 + 10 \dashrightarrow 7 + (-6) + 10 \Rightarrow 1 + 10 = 11$

NOTE: PEMDAS is Parenthesis
Exponents
Multiplication/Division (together!)
Addition/Subtraction (together!)

Reminder: Division is Multiplication EX: divide by 5 is same as times 1/5
Subtraction is Addition EX: 5 - 3 is same as 5 + (-3)

Example: $\frac{3}{8} + \frac{5}{8} \times 7$

Left to right: $\frac{8}{8} \times 7 = 7$

~~7 is incorrect...~~

order of operations

multiply: $\frac{3}{8} + \frac{35}{8}$

add: $\boxed{\frac{38}{8}}$

Example: $\frac{8 - 2(8 - 2)}{6 - 5(4)}$ $\frac{8 - 12}{6 - 20} = \frac{2}{7}$

Note: Solve the numerator and denominator separately..

Exponents & PEMDAS

Simplify the following:

$-3^2 =$

$(-3)^2 =$

$-(3^2) =$

$-3^{-2} =$

Solutions:

$-3^2 =$

(Rewrite the equation. Then, use order of operations)

$$-1 \cdot 3^2 = -1 \cdot 9 = -9$$

$(-3)^2 =$

$$(-3) \cdot (-3) = 9$$

$-(3^2) =$

(Order of operations: Parentheses (solve elements inside parentheses first)
Exponent (simplify the exponent)
Multiplication (multiply the -1 and result))

$$-(9) = -9$$

$-3^{-2} =$

(Solve the exponent first!)

$$3^{-2} = \frac{1}{9}$$

therefore, $-3^{-2} = \frac{-1}{9}$

A code is as follows:

Enter a number... then, multiply by 10
subtract 20
divide by 5
add 1
$\longrightarrow$ output

a) If the starting number is 5, what is the last number?

$5 \times 10 = 50$
$50 - 20 = 30$
$30/5 = 6$
$6 + 1 = \boxed{7}$

b) If the output is 13, what is the input?

Working backward: $13 - 1 = 12$
$12 \times 5 = 60$
$60 + 20 = 80$
$80/10 = \boxed{8}$

c) Write an algebraic expression that describes the code.

Remember order of operations!

input number x: $\boxed{\text{output} = \dfrac{(10x - 20)}{5} + 1}$

$x \cdot 10 - 20 \div 5 + 1$ is NOT correct

d) If the output is (k), write an algebraic expression to arrive at the original input.

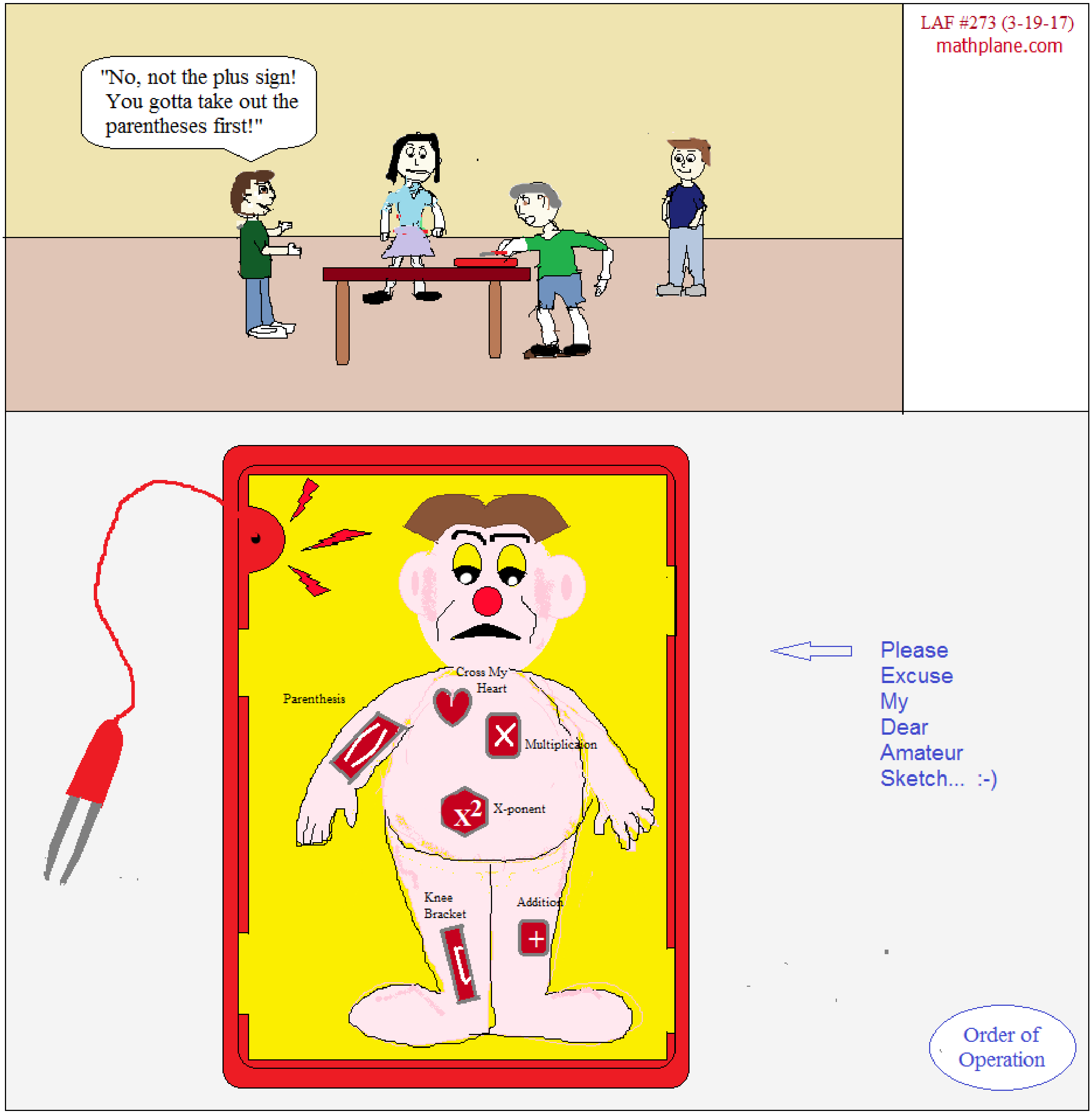
Again, working backward:

$k - 1 \times 5 + 20 \div 10$

then, account for order of operations...

$\dfrac{\left((k - 1) \times 5\right) + 20}{10}$ OR $\left(\left((k - 1) \times 5\right) + 20\right) \div 10$

$\longrightarrow$ original input (x)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"No, not the plus sign! You gotta take out the parentheses first!"
Cross My Heart
Parenthesis
Multiplication
X-ponent
x2
Knee Bracket
Addition
Please
Excuse
My
Dear
Amateur
Sketch... :-)
Order of Operation

Exercises→

Exponents and Roots (Solve the 15 problems below. Then add all the solutions)

1) $(3^3)^2 =$ ________

2) $(2)^{-2} =$ ________

3) $(4)^{3/2} =$ ________

4) $\sqrt{64} - \sqrt[3]{8} =$ ________

5) $9^2 + 9^{1/2} =$ ________

6) $(.3)^3 =$ ________

7) $(32)^{2/5} =$ ________

8) $(1/3)^{-2} =$ ________

9) $(-5)^3 =$ ________

10) $\sqrt{(3)^4} =$ ________

11) $\sqrt{2} \text{ x } \sqrt{50} =$ ________

12) $1^2 - 2^3 + 3^4 =$ ________

13) $(½)^3 =$ ________

14) $8^{1/3} \text{ x } 8^{2/3} =$ ________

15) $\sqrt[3]{(-8)} - \sqrt[3]{27} =$ ________

Now Add them up! The total of all 15 solutions is ________

(Rounded to 3 decimal places)

Hidden Message

Clue: "Name of a Math Mission?"

Letter Key:

0	1	2	3	4	5	6	7	8	9
A	D	E	I	N	O	P	R	S	T

1) $12 - 6 \div 3 - 5 =$

2) $(6 - 4)^2 \cdot 2 - 2 =$

3) $\left(\frac{(2^2 - 2) \text{ x } 2}{2 + 2}\right) \text{ x } 2 =$

4) $6 + 6 \div 3 - 1 =$

5) $0/(5 + 6^2 \div 3^2)^2 =$

6) $\frac{(6 - 4 + 1)(6 + 4 - 1)}{3} =$

7) $[(7 + 2) \div 3] \cdot [9 - 2^3] =$

8) $\sqrt{(7^2 - (3 \text{ x } 8))} =$

9) $1 + 2 - 3 + 4 =$

10) $\frac{4^2 - (6 + 3) + 3}{4 \div 2} =$

11) $(\text{\# of a's in clue box})^2 - \left(\begin{matrix}\text{\# of m's in} \\ \text{clue box}\end{matrix} - \begin{matrix}\text{\# of u's in} \\ \text{clue box}\end{matrix}\right) =$

12) $((6 + 6) \div 2 - 5) =$

13) $\frac{\sqrt{3 + 6} + 5}{6 - 2} =$

14) $\frac{(3 - 5 \text{ x } 2)^2}{9 - 2} =$

Math Electrician

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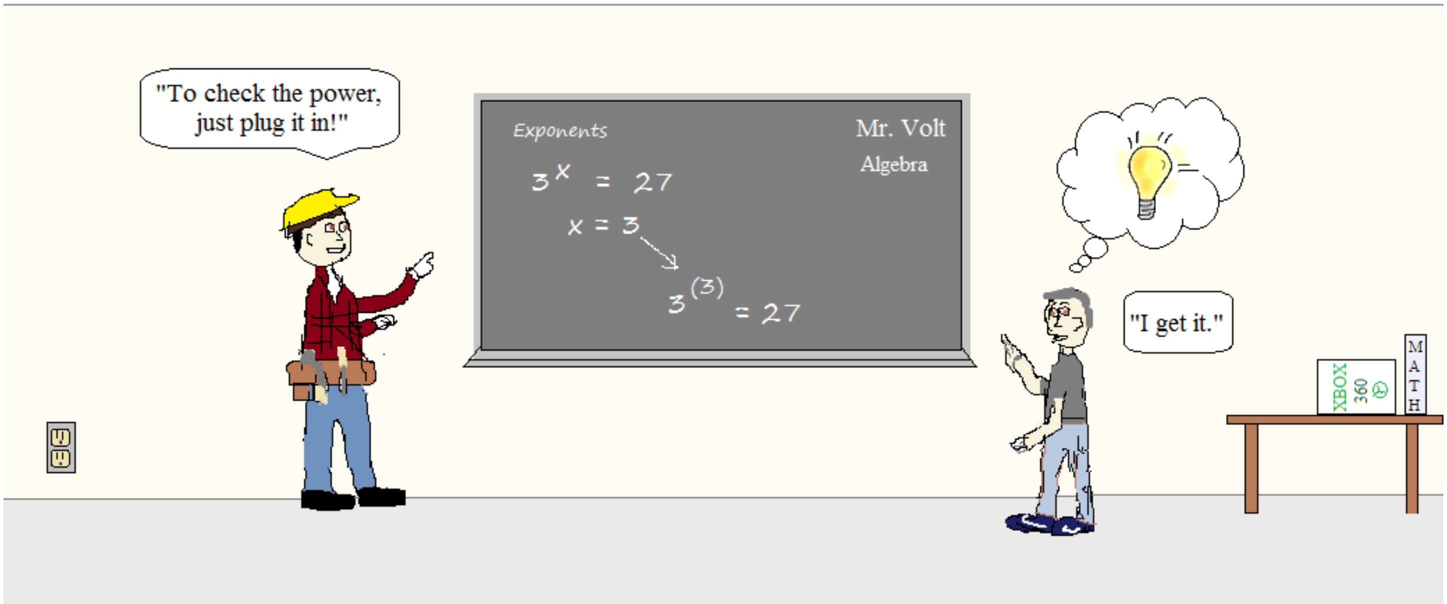

Alex makes the connection!

ANSWERS-→

SOLUTIONS:

1) $(3 \times 3 \times 3) \times (3 \times 3 \times 3) = 27 \times 27 = 729$
2) $2^2 = 4$.. therefore, $2^{-2} = ¼$ or .25
3) 4^3 is 64.. and $64^{1/2}$ is 8.. therefore $4^{3/2} = 8$
4) $8 - 2 = 6$
5) $81 + 3 = 84$
6) $(.3) \times (.3) \times (.3) = (.09) \times (.3)$.. and, $(.09) \times (.3) = .027$
7) $32^{2/5} = 32^{1/5} \times 32^{1/5} = 2 \times 2 = 4$
8) $(1/3)^2$ is 1/9.. therefore $(1/3)^{-2} = 9$ (the inverse of 1/9)
9) $(-5) \times (-5) \times (-5) = -125$
10) $(3)^4 = 81$.. and $\sqrt{81} = 9$
11) $\sqrt{100} = 10$
12) $1 - 8 + 81 = 74$
13) $(½) \times (½) \times (½) = 1/8 = .125$
14) $2 \times 4 = 8$ (or, approach is this way: same base.. therefore add exponents 1/3 and 2/3 to get 1. This leave $8^1 = 8$)
15) $-2 - 3 = -5$

Now add them up!

1-5 -----> 729 + .25 + 8 + 6 + 84 = 827.25

6-10 ---> .027 + 4 + 9 + (-125) + 9 = -102.973

11-15--> 10 + 74 + .125 + 8 + (-5) = 87.125

811.402

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Clue: "Name of a Math Mission?"

Letter Key:									
0	1	2	3	4	5	6	7	8	9
A	D	E	I	N	O	P	R	S	T

SOLUTIONS

1) $12 - 6 \div 3 - 5 = \quad 12 - 2 - 5 = 5$

2) $(6-4)^2 \cdot 2 - 2 = \quad 2^2 \cdot 2 - 2 = 6$

3) $\left(\dfrac{(2^2-2) \times 2}{2+2}\right) \times 2 = \quad \left(\dfrac{(4-2) \times 2}{4}\right) \times 2 = 1 \times 2 = 2$

4) $6 + 6 \div 3 - 1 = \quad 6 + 2 - 1 = 7$

5) $0/(5 + 6^2 \div 3^2)^2 = \quad 0/(5 + 36 \div 9)^2$

$0/(5+4)^2 = 0/81 = 0$

Note: 0 divided by anything is 0

6) $\dfrac{(6-4+1)(6+4-1)}{3} = \quad \dfrac{(2+1)(10-1)}{3} = \dfrac{27}{3} = 9$

7) $[(7+2) \div 3] \cdot [9 - 2^3] = \quad [9 \div 3] \cdot [9 - 8] = 3$

8) $\sqrt{(7^2 - (3 \times 8))} = \quad \sqrt{(49 - (24))} = \sqrt{25} = 5$

9) $1 + 2 - 3 + 4 = \quad 3 - 3 + 4 = 0 + 4 = 4$

10) $\dfrac{4^2 - (6+3) + 3}{4 \div 2} = \dfrac{16 - (9) + 3}{2} = \dfrac{7+3}{2} = 5$

11) $(\text{\# of a's in Clue box})^2 - \left(\begin{matrix}\text{\# of m's in} \\ \text{Clue box}\end{matrix} - \begin{matrix}\text{\# of u's in} \\ \text{Clue box}\end{matrix}\right) = (3)^2 - (3 - 1) =$
$9 - 2 = 7$

12) $((6+6) \div 2 - 5) = \quad (12 \div 2 - 5) = \quad (6 - 5) = 1$

13) $\dfrac{\sqrt{3+6} + 5}{6 - 2} = \quad \dfrac{\sqrt{9} + 5}{4} = \dfrac{8}{4} = 2$

14) $\dfrac{(3 - 5 \times 2)^2}{9 - 2} = \quad \dfrac{(3-10)^2}{7} = \dfrac{(-7)^2}{7} = 7$

Name of a math mission?
"Operation Order"

- 5 → O
- 6 → P
- 2 → E
- 7 → R
- 0 → A
- 9 → T
- 3 → I
- 5 → O
- 4 → N
- 5 → O
- 7 → R
- 1 → D
- 2 → E
- 7 → R

Order of Operations:

$$12 + 3(4 + 7) \div 3(5)$$

$$12 + 33 \div 3 \times 5$$

$$12 + 11 \times 5$$

$$12 + 55 = \boxed{67}$$

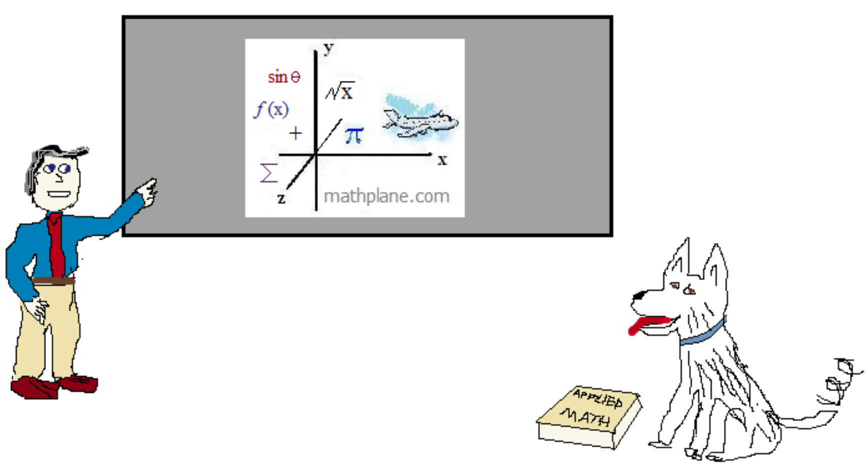
NOT $\cancel{12 + 33 \div 15 = 3}$

NOT $\cancel{12 + 33/15}$

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers



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