Matrix I and Matrix II

Notes, Examples, and Practice Tests

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Includes scalar multiplication, solving linear systems, Determinants, Inverses, applications, Identity matrix, Cramer's Rule, and more.

Mathplane.com

Matrix: Brief Notes and Examples

Entry-wise Addition & Subtraction

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$$

-- matrices have same dimensions
-- add/subtract corresponding entries

Example:

Let $A = \begin{bmatrix} 4 & 0 \\ -3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 9 \\ 4 & -6 \end{bmatrix}$ $C = \begin{bmatrix} 3 & 1 & 7 \\ -5 & 0 & 2 \end{bmatrix}$

$$A + B = \begin{bmatrix} 4+2 & 0+9 \\ -3+4 & 2+-6 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 1 & -4 \end{bmatrix}$$

$$B + A = \begin{bmatrix} 2+4 & 9+0 \\ 4-3 & -6+2 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 1 & -4 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 4-2 & 0-9 \\ -3-4 & 2--6 \end{bmatrix} = \begin{bmatrix} 2 & -9 \\ -7 & 8 \end{bmatrix}$$

$$B - A = \begin{bmatrix} 2-4 & 9-0 \\ 4--3 & -6-2 \end{bmatrix} = \begin{bmatrix} -2 & 9 \\ 7 & -8 \end{bmatrix}$$

$A + C =$ "undefined"
(different dimensions)

(Note: matrix addition is commutative and associative.
Matrix subtraction is not!)

Scalar Multiplication

$$s \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} s\cdot a & s\cdot b \\ s\cdot c & s\cdot d \end{bmatrix}$$

Example:

Let $A = \begin{bmatrix} 4 & 0 \\ -3 & 2/3 \end{bmatrix}$ $3A = \begin{bmatrix} 3(4) & 3(0) \\ 3(-3) & 3(2/3) \end{bmatrix} = \begin{bmatrix} 12 & 0 \\ -9 & 2 \end{bmatrix}$

Other properties of Addition, Subtraction, and Scalar Multiplication

Let A, B, C be matrices (of same dimension)
r, s be scalars

- $A - B = A + (-B)$
- $A + (B + C) = (A + B) + C$
- $r(A + B) = rA + rB$
- $(r + s)A = rA + rA$
- $rsA = r(sA)$

Matrix Multiplication

Multiply ROWS of the first matrix by COLUMNS of the second matrix.

Note: when multiplying a row by a column, multiply corresponding entries and add them up.

columns 1 & 2

row 1, row 2

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 4 & 2 \end{bmatrix}$$

"Row 1 x Column 1" = "1 - 1 slot"

(1 x -3) + (3 x 4) = 9

"Row 1 x Column 2" = "1 - 2 slot"

(1 x 0) + (3 x 2) = 6

$$\begin{bmatrix} 9 & 6 \\ 14 & 10 \end{bmatrix}$$

columns 1 & 2

row 1, row 2

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 4 & 2 \end{bmatrix}$$

And, "row 2 x column 2" = "2 - 2 slot"

(2 x 0) + (5 x 2) = 10

"row 2 x column 1" = "2 - 1 slot"

(2 x -3) + (5 x 4) = 14

Important: # of rows in first matrix must equal the # of columns in the second matrix!

Example:

Let

$$A = \begin{bmatrix} 3 & -6 \\ 0 & 2/3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 4 \\ 1/4 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 1 & 0 \\ 1/5 & -3 & -5 \end{bmatrix}$$

$$AB = \begin{bmatrix} (3 \times 1) + (-6 \times 1/4) & (3 \times 4) + (-6 \times -2) \\ (0 \times 1) + (2/3 \times 1/4) & (0 \times 4) + (2/3 \times -2) \end{bmatrix} = \begin{bmatrix} 3/2 & 24 \\ 1/6 & -4/3 \end{bmatrix}$$

$$BA = \begin{bmatrix} (1 \times 3) + (4 \times 0) & (1 \times -6) + (4 \times 2/3) \\ (1/4 \times 3) + (-2 \times 0) & (1/4 \times -6) + (-2 \times 2/3) \end{bmatrix} = \begin{bmatrix} 3 & -10/3 \\ 3/4 & -17/6 \end{bmatrix}$$

Note: AB ≠ BA
(matrix multiplication is NOT commutative!)

$$AC = \begin{bmatrix} (3 \times 2) + (-6 \times 1/5) & (3 \times 1) + (-6 \times -3) & (3 \times 0) + (-6 \times -5) \\ (0 \times 2) + (2/3 \times 1/5) & (0 \times 1) + (2/3 \times -3) & (0 \times 0) + (2/3 \times -5) \end{bmatrix} = \begin{bmatrix} 24/5 & 21 & 30 \\ 2/15 & -2 & -10/3 \end{bmatrix}$$

$$BC = \begin{bmatrix} (1 \times 2) + (4 \times 1/5) & (1 \times 1) + (4 \times -3) & (1 \times 0) + (4 \times -5) \\ (1/4 \times 2) + (-2 \times 1/5) & (1/4 \times 1) + (-2 \times -3) & (1/4 \times 0) + (-2 \times -5) \end{bmatrix} = \begin{bmatrix} 14/5 & -11 & -20 \\ 1/10 & 25/4 & 10 \end{bmatrix}$$

CA does not exist

CB does not exist

(C has 3 rows and A & B have only 2 columns!)

$$AC = \begin{bmatrix} 3 & -6 \\ 0 & 2/3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1/5 & -3 & -5 \end{bmatrix} = \begin{bmatrix} 24/5 & 21 & 30 \\ 2/15 & -2 & -10/3 \end{bmatrix}$$

$$CA = \begin{bmatrix} 2 & 1 & 0 \\ 1/5 & -3 & -5 \end{bmatrix} \begin{bmatrix} 3 & -6 \\ 0 & 2/3 \end{bmatrix}$$

Rows and columns don't match up!
(CA doesn't exist)

Matrix Example: Linear System Application

Find the solution of the following linear equations:

$$3x + y - z = -1$$
$$x - 3y + 2z = 11$$
$$2y + 4z = 4$$

(Set up a matrix; then, change to Reduced Row Echelon Form.)

Step 1: Set up the matrix

$$\left|\begin{array}{ccc|c} 3 & 1 & -1 & -1 \\ 1 & -3 & 2 & 11 \\ 0 & 2 & 4 & 4 \end{array}\right|$$

Coefficients of x (column 1)
y (column 2)
z (column 3)
and, the solutions (column 4)

The goal is to change the matrix to Reduced Row Echelon Form.

$$\left|\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{array}\right|$$

Step 2: Use strategies to eliminate, switch, and reduce columns & rows and the included elements.

$$\left|\begin{array}{ccc|c} 3 & 1 & -1 & -1 \\ 1 & -3 & 2 & 11 \\ 0 & 2 & 4 & 4 \end{array}\right|$$

"Switch Rows" — Rearrange the 3 rows.

$$\left|\begin{array}{ccc|c} 1 & -3 & 2 & 11 \\ 0 & 2 & 4 & 4 \\ 3 & 1 & -1 & -1 \end{array}\right|$$

"Reduce to 1" (Multiply/Divide to get 0's & 1's) — Divide R2 by 2 $\left(\frac{1}{2}R2\right)$

$$\left|\begin{array}{ccc|c} 1 & -3 & 2 & 11 \\ 0 & 1 & 2 & 2 \\ 3 & 1 & -1 & -1 \end{array}\right|$$

"Add to eliminate" — $-3(R1) + R3$

$$\left|\begin{array}{ccc|c} 1 & -3 & 2 & 11 \\ 0 & 1 & 2 & 2 \\ 0 & 10 & -7 & -34 \end{array}\right|$$

(Column 1 is finished.)

Strategy Note: Row 2 has a '0' and a '1'...

---> We can eliminate the other numbers in column 2 <u>without disrupting Column 1</u>

$$\left|\begin{array}{ccc|c} 1 & -3 & 2 & 11 \\ 0 & 1 & 2 & 2 \\ 0 & 10 & -7 & -34 \end{array}\right| \quad \text{First, } 3R2 + R1 \longrightarrow \quad \left|\begin{array}{ccc|c} 1 & 0 & 8 & 17 \\ 0 & 1 & 2 & 2 \\ 0 & 10 & -7 & -34 \end{array}\right| \quad \text{Then, } -10R2 + R3$$

$$\left|\begin{array}{ccc|c} 1 & 0 & 8 & 17 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -27 & -54 \end{array}\right| \quad \text{Multiply R3 by -1/27} \longrightarrow \quad \left|\begin{array}{ccc|c} 1 & 0 & 8 & 17 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right|$$

Strategy: Again, we have 0s and 1s in a Row (3).. So, we can eliminate the other elements in Column 3!!

-2R3 + R2...
then,
-8R3 + R1...

$$\left|\begin{array}{ccc|c} 1 & 0 & 8 & 17 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right| \longrightarrow \left|\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right|$$

Reduced Row Echelon Form shows
x = 1
y = -2
z = 2

Step 3: "Check your solution"

1) Plug into original linear equations:

3(1) + (-2) - (2) = -1

(1) - 3(-2) + 2(2) = 11

2(-2) + 4(2) = 4

2) Shorthand multiplication inside matrix:

$$\left|\begin{array}{ccc|c} 3 & 1 & -1 & -1 \\ 1 & -3 & 2 & 11 \\ 0 & 2 & 4 & 4 \end{array}\right|$$

Row 1: (3 x 1) + (1 x -2) + (-1 x 2) = -1

Row 2: (1 x 1) + (-3 x -2) + (2 x 2) = 11

Row 3: (0 x 1) + (2 x -2) + (4 x 2) = 4

solution:

$$\left|\begin{array}{c} 1 \\ -2 \\ 2 \end{array}\right|$$

Solving linear system: Comparing an augmented matrix and the linear equations

Example:

$$2x - 2y = 14$$
$$3x + y = 33$$

multiply first equation by 1/2

$$x - y = 7$$
$$3x + y = 33$$

multiply first equation and add to 2nd equation

$$x - y = 7$$
$$-3x + 3y = -21$$
$$3x + y = 33$$
$$4y = 12$$
$$y = 3$$

then, solve for y

Since y = 3, substitute into first equation to get x

$$x - (3) = 7$$
$$x = 10$$

"Gauss-Jordan elimination to reduced row echelon form"

$$\left[\begin{array}{cc|c} 2 & -2 & 14 \\ 3 & 1 & 33 \end{array}\right]$$

coefficients are in the matrix

$\frac{1}{2}$ R1

$$\left[\begin{array}{cc|c} 1 & -1 & 7 \\ 3 & 1 & 33 \end{array}\right]$$

-3R1 + R2 (replace R2)

$$\left[\begin{array}{cc|c} 1 & -1 & 7 \\ 0 & 4 & 12 \end{array}\right]$$

$\frac{1}{4}$ R2

$$\left[\begin{array}{cc|c} 1 & -1 & 7 \\ 0 & 1 & 3 \end{array}\right]$$

R2 + R1 (replace R1)

$$\left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & 3 \end{array}\right]$$

reduced row echelon form, reveals x = 10 and y = 3

Example:

$$3x + 2y = -4$$
$$-2x + y + 5z = 25$$
$$x - 4y + 2z = 2$$

swap equation 1 and 3

$$x - 4y + 2z = 2$$
$$-2x + y + 5z = 25$$
$$3x + 2y = -4$$

multiply 1st equation by 2 and add to 2nd equation

$$x - 4y + 2z = 2$$
$$2x - 8y + 4z = 4$$
$$-2x + y + 5z = 25$$
(A) $$-7y + 9z = 29$$

multiply 1st equation by -3 and add to 3rd equation

$$x - 4y + 2z = 2$$
$$-3x + 12y - 6z = -6$$
$$3x + 2y = -4$$
(B) $$14y - 6z = -10$$

multiply new equation(A) by 2 and add to new equation(B)

$$-7y + 9z = 29$$
$$-14y + 18z = 58$$
$$14y - 6z = -10$$
$$12z = 48$$
$$z = 4$$

divide new equation(A) by -7

$$y - \frac{9z}{7} = \frac{-29}{7}$$

"Gauss elimination to row echelon form"

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & -4 \\ -2 & 1 & 5 & 25 \\ 1 & -4 & 2 & 2 \end{array}\right]$$

coefficients and solution placed into augmented matrix

swap R1 and R3

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & 2 \\ -2 & 1 & 5 & 25 \\ 3 & 2 & 0 & -4 \end{array}\right]$$

2R1 + R2 (replace R2)

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & 2 \\ 0 & -7 & 9 & 29 \\ 3 & 2 & 0 & -4 \end{array}\right]$$

-3R1 + R3 (replace R3)

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & 2 \\ 0 & -7 & 9 & 29 \\ 0 & 14 & -6 & -10 \end{array}\right]$$

2R2 + R3 (replace R3)

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & 2 \\ 0 & -7 & 9 & 29 \\ 0 & 0 & 12 & 48 \end{array}\right]$$

$\frac{1}{4}$ R

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & 2 \\ 0 & -7 & 9 & 29 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$\frac{-1}{7}$ R2

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & 2 \\ 0 & 1 & -9/7 & -29/7 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

(Gauss Elimination leads to row echelon form, enough to easily reveal the solutions)

Matrix Applications

Example: 3 solutions for the quadratic equation $y = ax^2 + bx + c$ are (1, 4), (-3, 10), and (-1, -1).

What is the equation?

If we substitute each solution into the general equation, we end up with 3 equations with 3 unknowns...

$$\begin{aligned} &(1, 4): && a(1)^2 + b(1) + c = 4 && \Longrightarrow && a^2 + b + c = 4 \\ &(-3, 10): && a(-3)^2 + b(-3) + c = 10 && \Longrightarrow && 9a^2 - 3b + c = 10 \\ &(-1, -1): && a(-1)^2 + b(-1) + c = -1 && \Longrightarrow && a^2 - b + c = -1 \end{aligned}$$

Then, to solve the system, establish an augmented matrix...

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 9 & -3 & 1 & 10 \\ 1 & -1 & 1 & -1 \end{array}\right] \xrightarrow{\text{multiply row 1 by (-1) and add to row 3}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 9 & -3 & 1 & 10 \\ 0 & -2 & 0 & -5 \end{array}\right] \xrightarrow{\text{multiply row 3 by (-1/2) and switch with row 2}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 5/2 \\ 9 & -3 & 1 & 10 \end{array}\right]$$

(Utilizing the 2nd row, we can clear the 2nd column)

multiply row 2 by (-1) and add to row 1
multiply row 2 by (3) and add to row 3

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3/2 \\ 0 & 1 & 0 & 5/2 \\ 9 & 0 & 1 & 35/2 \end{array}\right] \xrightarrow{\text{multiply row 1 by (-9) and add to row 3}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3/2 \\ 0 & 1 & 0 & 5/2 \\ 0 & 0 & -8 & 4 \end{array}\right] \xrightarrow{\text{divide row 3 by (-8)}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3/2 \\ 0 & 1 & 0 & 5/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right]$$

Finally, multiply row 3 by (-1) and add to row 1

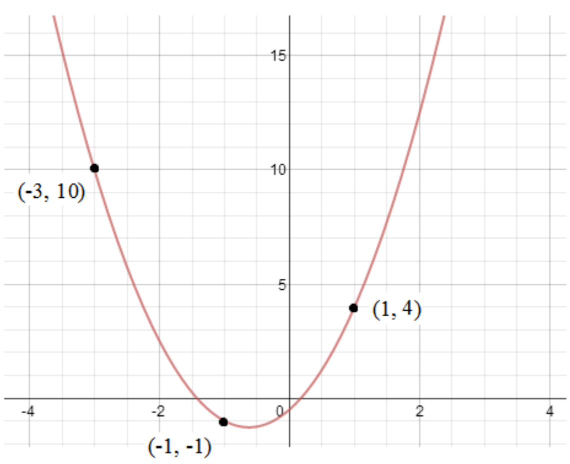
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right]$$

The augmented matrix is in Reduced Row Echelon Form revealing the solution:

$a = 2$
$b = 5/2$
$c = -1/2$

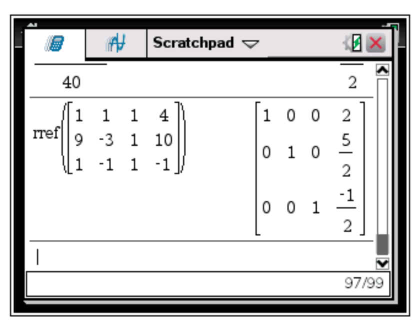
$$y = 2x^2 + \frac{5}{2}x - \frac{1}{2}$$

Note: To check the solution, simply plug in the 3 points above...



Solve using the TI - *n*spire CX Graphing Calculator

Home/ON turn unit on
A Calculate
Menu
7 Matrix & Vector
5 Reduced Row-Echelon Form rref() appears
Template Key (next to the 9) template symbol menu appears
(Select the matrix template)
Number of Rows: 3
Number of Columns: 4
enter matrix appears
(Input the values)
enter



Matrix Application / Models

Example: A business borrows \$45,000 to buy a special math machine.
The money is divided into 3 loans at 6%, 8%, and 10% interest rates.
Annual (simple) interest payments are \$3,740 per year.
If the total amount borrowed at 6% and 8% is twice the amount borrowed at 10%,
what is the amount of each loan?

Step 1: Establish Variables

A = Amount borrowed at 6%
B = Amount borrowed at 8%
C = Amount borrowed at 10%

Step 2: Determine equations that describe the word problem

total amount borrowed: $45{,}000 = A + B + C$

interest payments: $3740 = A(.06) + B(.08) + C(.10)$

borrowed amounts: $A + B = 2C$ (6% and 8% are twice as much as 10%)
(comparison)

or, $A + B - 2C = 0$

Step 3: Set up the system of equations (augmented matrix) and solve

A B C

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 45000 \\ .06 & .08 & .10 & 3740 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

Get rid of decimals multiply row 2 by 100

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 45000 \\ 6 & 8 & 10 & 374{,}000 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

multiply row 1 by (-1) and add to row 3

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 45000 \\ 6 & 8 & 10 & 374{,}000 \\ 0 & 0 & -3 & -45000 \end{array}\right]$$

Divide row 3 by (-3)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 45000 \\ 6 & 8 & 10 & 374{,}000 \\ 0 & 0 & 1 & 15{,}000 \end{array}\right]$$

(since row 3 is in RREF, we'll use it to clear column 3)

R3 x (-1) add to R1
R3 x (-10) add to R2

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 30{,}000 \\ 6 & 8 & 0 & 224{,}000 \\ 0 & 0 & 1 & 15{,}000 \end{array}\right]$$

R1 x (-6) and add to row 2

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 30{,}000 \\ 0 & 2 & 0 & 44{,}000 \\ 0 & 0 & 1 & 15{,}000 \end{array}\right]$$

Multiply row 2 by 1/2

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 30{,}000 \\ 0 & 1 & 0 & 22{,}000 \\ 0 & 0 & 1 & 15{,}000 \end{array}\right]$$

multiply Row 2 by (-1) and add to Row 1

Reduced Row Echelon Form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 8{,}000 \\ 0 & 1 & 0 & 22{,}000 \\ 0 & 0 & 1 & 15{,}000 \end{array}\right]$$

A (6% loan amount) = \$8,000

B (8% loan amount) = \$22,000

C (10% loan amount) = \$15,000

Step 4: Check answer....

6% of \$8,000 = \$480

8% of \$22,000 = \$1760

10% of \$15,000 = \$1,500

Total interest payment: \$3,740 ✓

6% and 8% loan amount: \$30,000
10% loan amount: \$15,000
2x ✓

Matrix application - Modeling data from a table

Example: A restaurant menu offers a variety of items. Or, a customer may order a prepared meal of particular items.

The table at the right displays the item prices in each meal option.

The table at the right displays the item prices (in dollars) in each meal.

Restaurant menu (Items in $)

Meal Option	Salad	Soup	Entree	Beverage	Dessert
1	3	2	7	1.50	2.50
2	4	0	8	2	3
3	6	5	0	2	0
4	0	4	8	1.50	3.50

1) Write the info (in the table) as a 4 x 5 menu matrix M.

2) On Friday, we ordered 15 meal #1's, 22 meal #2's, 3 meal #3's, and 18 meal #4's. Express the Friday orders as a *row matrix* F.

3) Find matrix FM. What does FM represent?

---

1) Menu Table expressed as a 4 x 5 matrix:

Menu Matrix M (4 rows x 5 columns)

$$\begin{bmatrix} 3 & 2 & 7 & 1.5 & 2.5 \\ 4 & 0 & 8 & 2 & 3 \\ 6 & 5 & 0 & 2 & 0 \\ 0 & 4 & 8 & 1.5 & 3.5 \end{bmatrix}$$

Rows: meal options
Columns: menu items

2) Row matrix describing the Friday order:

Friday Order Matrix F (1 row x 4 columns)

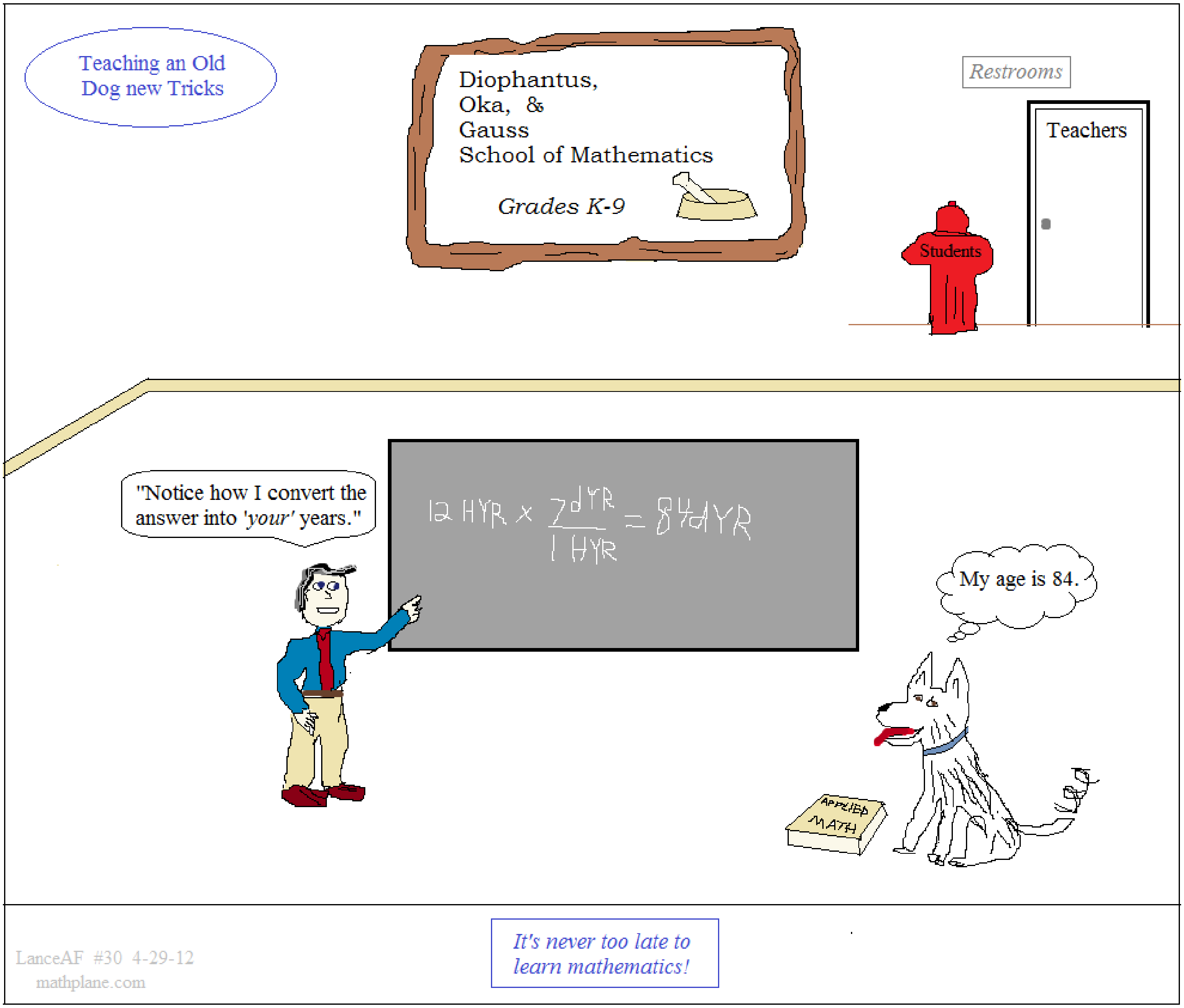
$$\begin{bmatrix} 15 & 22 & 3 & 18 \end{bmatrix}$$

3) FM is the total money spent for each item:

$$\begin{bmatrix} 15 & 22 & 3 & 18 \end{bmatrix} \begin{bmatrix} 3 & 2 & 7 & 1.5 & 2.5 \\ 4 & 0 & 8 & 2 & 3 \\ 6 & 5 & 0 & 2 & 0 \\ 0 & 4 & 8 & 1.5 & 3.5 \end{bmatrix} = \begin{bmatrix} 151 & 117 & 425 & 99.5 & 166.5 \end{bmatrix}$$

note: 1 x 4 4 x 5 ----> 1 x 5 matrix

The restaurant earned
$151 from salads
$117 from soups
$425 from entrees
$99.50 from beverages
$166.50 from desserts...



Practice Quiz- →

Matrix Test

1)

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 0 & 4 \\ -1 & 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -4 & 0 \\ 2 & 3 & -3 \\ 1 & 5 & 2 \end{bmatrix}$$

$A + B =$ $\qquad$ $3A =$ $\qquad$ $BA =$

2) Solve using matrices.

A) $3X + 2Y = 10$
$X - 6Y = 0$

B) $2X - 3Y + Z = 12$
$4Y - Z = -9$
$X + 6Y + 2Z = 6$

Challenge: Find X and Y

C) $2X + Y + 5Z = 10$
$X + 2Y - 3Z = 14$

Matrix Test

3) Find x and y:

$$\begin{bmatrix} x & y \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ -1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 15 \\ 1 & 12 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 2 \\ 3 & 4 & 2 \\ 2 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ x \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 17 \\ y \end{bmatrix}$$

4) For a 2x2 matrix D,

$$D \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{and,} \quad D \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

What matrix should you multiply D by to get

$$\begin{bmatrix} 5 \\ 5 \end{bmatrix} \ ?$$

Matrix Test

SOLUTIONS

1)

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 0 & 4 \\ -1 & 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -4 & 0 \\ 2 & 3 & -3 \\ 1 & 5 & 2 \end{bmatrix}$$

Note: $BA \neq AB$

A + B =

$$\begin{bmatrix} 3+1 & 2+(-4) & -1+0 \\ 2+2 & 0+3 & 4+(-3) \\ -1+1 & 5+5 & 6+2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 & -1 \\ 4 & 3 & 1 \\ 0 & 10 & 8 \end{bmatrix}$$

3A =

$$\begin{bmatrix} 3 \times 3 & 3 \times 2 & 3 \times (-1) \\ 3 \times 2 & 3 \times 0 & 3 \times 4 \\ 3 \times (-1) & 3 \times 5 & 3 \times 6 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 6 & -3 \\ 6 & 0 & 12 \\ -3 & 15 & 18 \end{bmatrix}$$

BA =

$$\begin{bmatrix} 1 & -4 & 0 \\ 2 & 3 & -3 \\ 1 & 5 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & -1 \\ 2 & 0 & 4 \\ -1 & 5 & 6 \end{bmatrix}$$

row 1B/column 1A = (1x3) + (-4x2) + (0x-1) = -5
row 1B/column 2A = (1x2) + (-4x0) + (0x5) = 2
row 3B/column 3A = (1x-1) + (5x4) + (2x6) = 31

$$\begin{bmatrix} \boxed{-5} & \boxed{2} & -17 \\ 15 & -11 & -8 \\ 11 & 12 & \boxed{31} \end{bmatrix}$$

2) Solve using matrices.

A) $3X + 2Y = 10$
$X - 6Y = 0$

$$\left[\begin{array}{cc|c} 3 & 2 & 10 \\ 1 & -6 & 0 \end{array}\right] \quad -3R2 + R1$$

$$\left[\begin{array}{cc|c} 0 & 20 & 10 \\ 1 & -6 & 0 \end{array}\right] \quad \text{Divide R1 by 20}$$

$$\left[\begin{array}{cc|c} 0 & 1 & \frac{1}{2} \\ 1 & -6 & 0 \end{array}\right] \quad 6R1 + R2$$

$$\left[\begin{array}{cc|c} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 3 \end{array}\right]$$

$X = 3$

$Y = \frac{1}{2}$

(Note: to check you work, plug solutions into original equations)

B) $2X - 3Y + Z = 12$
$4Y - Z = -9$
$X + 6Y + 2Z = 6$

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 12 \\ 0 & 4 & -1 & -9 \\ 1 & 6 & 2 & 6 \end{array}\right] \quad -2R3 + R1$$

$$\left[\begin{array}{ccc|c} 0 & -15 & -3 & 0 \\ 0 & 4 & -1 & -9 \\ 1 & 6 & 2 & 6 \end{array}\right] \quad -1/15(R1)$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1/5 & 0 \\ 0 & 4 & -1 & -9 \\ 1 & 6 & 2 & 6 \end{array}\right] \quad \text{rearrange rows}$$

$$\left[\begin{array}{ccc|c} 1 & 6 & 2 & 6 \\ 0 & 1 & 1/5 & 0 \\ 0 & 4 & -1 & -9 \end{array}\right] \quad -4R2 + R3$$

$$\left[\begin{array}{ccc|c} 1 & 6 & 2 & 6 \\ 0 & 1 & 1/5 & 0 \\ 0 & 0 & -9/5 & -9 \end{array}\right] \quad -5/9(R3)$$

$$\left[\begin{array}{ccc|c} 1 & 6 & 2 & 6 \\ 0 & 1 & 1/5 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$Z = 5$
$Y + 1/5Z = 0$
$X + 6Y + 2Z = 6$

$$\boxed{\begin{array}{l} Z = 5 \\ Y = -1 \\ X = 2 \end{array}}$$

Challenge: Find X and Y

C) $2X + Y + 5Z = 10$
$X + 2Y - 3Z = 14$

$$\left[\begin{array}{ccc|c} 2 & 1 & 5 & 10 \\ 1 & 2 & -3 & 14 \end{array}\right] \quad -2R2 + R1$$

$$\left[\begin{array}{ccc|c} 0 & -3 & 11 & -18 \\ 1 & 2 & -3 & 14 \end{array}\right] \quad \text{Switch rows}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 14 \\ 0 & -3 & 11 & -18 \end{array}\right] \quad 2/3(R2) + R1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 13/3 & 2 \\ 0 & -3 & 11 & -18 \end{array}\right] \quad -1/3(R2)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 13/3 & 2 \\ 0 & 1 & -11/3 & 6 \end{array}\right] \quad \text{change back to linear forms}$$

$X + 13/3Z = 2$
$Y - 11/3Z = 6$

$$\boxed{\begin{array}{l} X = 2 - \frac{13}{3}Z \\ Y = 6 + \frac{11}{3}Z \end{array}}$$

Matrix Test

SOLUTIONS

3) Find x and y:

$$\begin{bmatrix} x & y \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ -1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 15 \\ 1 & 12 \end{bmatrix}$$

$1x + (-1)y = 2$ $-3x + 3y = -6$

$3x + 6y = 15$ $3x + 6y = 15$

$9y = 9$

$y = 1$

so, $x = 3$

$$\begin{bmatrix} -2 & 1 & 2 \\ 3 & 4 & 2 \\ 2 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ x \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 17 \\ y \end{bmatrix}$$

use row 1 in A and column 1 in X....

$(-2)(1) + (1)(x) + (2)(3) = 6$

$-2 + x + 6 = 6$ $x = 2$

Since $x = 2$, row 3 in A and column 1 in X....

then $(3)(1) + (4)(2) + (2)(3) = 17$

then, $(2)(1) + (0)(2) + (4)(3) = y$

$2 + 0 + 12 = y$ $y = 14$

4) For a 2x2 matrix D,

$$D \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{and,} \quad D \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

What matrix should you multiply D by to get

$$\begin{bmatrix} 5 \\ 5 \end{bmatrix} \; ?$$

First, let's find D...

$$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$D \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad D \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

$$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}$$

$$D \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$2x + 4y = 5 \Rightarrow 2x + 4y = 5$

$3x - y = 5 \Rightarrow 12x - 4y = 20$

$x = 25/14$ $y = 5/14$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 25/14 \\ 5/14 \end{bmatrix}$$

Identity Matrix

Definition: A square matrix where every element in the *main diagonal* is a 1, and all the other elements are 0

1 x 6 = 6   (1 is the multiplicative identity)

If **I** is an identity matrix and **M** is a matrix (of same dimensions), then **I** x **M** = **M** or **M** x **I** = **M**

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an identity matrix $\begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is not an identity matrix! $\begin{bmatrix} -4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 3 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a 3 x 3 identity matrix

Inverse Matrix

Definition: A square matrix A^{-1} that when multiplied to another matrix A results in the identity matrix.

Non-square matrices do not have inverses; also, some square matrices do not have inverses.

$\frac{1}{10}$ and 10 are multiplicative inverses (reciprocals).

$\frac{1}{10} \times 10 = 1$ $10 \times \frac{1}{10} = 1$

> If a square matrix does not have an inverse, it is non-invertible.
>
> When does that happen? If and only if the determinant of the matrix is 0

$A = \begin{bmatrix} 3 & 6 \\ -2 & 2 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} \frac{1}{9} & \frac{-1}{3} \\ \frac{1}{9} & \frac{1}{6} \end{bmatrix}$

$$AA^{-1} = \begin{bmatrix} (3 \times 1/9) + (6 \times 1/9) & (3 \times -1/3) + (6 \times 1/6) \\ (-2 \times 1/9) + (2 \times 1/9) & (-2 \times -1/3) + (2 \times 1/6) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} (1/9 \times 3) + (-1/3 \times -2) & (1/9 \times 6) + (-1/3 \times 2) \\ (1/9 \times 3) + (1/6 \times -2) & (1/9 \times 6) + (1/6 \times 2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Finding the inverse of a square matrix:

Method 1: Using the formula

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ The inverse of X is $\frac{1}{|X|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ or $\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example: $X = \begin{bmatrix} 1 & 4 \\ -2 & 2 \end{bmatrix}$ Find X^{-1}

"Find the determinant": $|X| = \begin{vmatrix} 1 & 4 \\ -2 & 2 \end{vmatrix} = 2 - (-8) = 10$

"Transform the matrix": $\begin{bmatrix} 1 & 4 \\ -2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -4 \\ 2 & 1 \end{bmatrix}$

"Calculate" : $\frac{1}{10}\begin{bmatrix} 2 & -4 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{-2}{5} \\ \frac{1}{5} & \frac{1}{10} \end{bmatrix}$

<u>Check your answer:</u>

Does $X \cdot X^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$?

$$\begin{bmatrix} 1 & 4 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{5} & \frac{-2}{5} \\ \frac{1}{5} & \frac{1}{10} \end{bmatrix} = \begin{bmatrix} (1/5 + 4/5) & (-2/5 + 4/10) \\ (-2/5 + 2/5) & (4/5 + 2/10) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Yes!

Method 2: Using an augmented matrix

Example: "Transform left side to identity matrix to reveal the inverse"

$$\left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ -1 & 5 & 0 & 1 \end{array}\right] \xrightarrow[\text{(replace R1)}]{3R2 + R1} \left[\begin{array}{cc|cc} 0 & 17 & 1 & 3 \\ -1 & 5 & 0 & 1 \end{array}\right] \xrightarrow[\text{(then swap rows)}]{-1R2} \left[\begin{array}{cc|cc} 1 & -5 & 0 & -1 \\ 0 & 17 & 1 & 3 \end{array}\right] \xrightarrow{\frac{1}{17}R2} \left[\begin{array}{cc|cc} 1 & -5 & 0 & -1 \\ 0 & 1 & \frac{1}{17} & \frac{3}{17} \end{array}\right]$$

A I

5R2 + R1
(replace R1)

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{5}{17} & \frac{-2}{17} \\ 0 & 1 & \frac{1}{17} & \frac{3}{17} \end{array}\right]$$

Left side is the identity matrix; right side is the inverse!

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{5}{17} & \frac{-2}{17} \\ 0 & 1 & \frac{1}{17} & \frac{3}{17} \end{array}\right]$$

I A^{-1}

Note and compare with the above formula:

determinant of A = 17 and, the transformed matrix is $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$

And, the result of A^{-1} is also $\begin{bmatrix} \frac{5}{17} & \frac{-2}{17} \\ \frac{1}{17} & \frac{3}{17} \end{bmatrix}$

Determinants

Definition: A single number obtained from a matrix, revealing some of the matrix properties.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (aei + bfg + dhc) - (ceg + fha + ibd)$$

Even though the symbols are similar, a determinant is <u>not</u> an absolute value.

Notice the diagonal and criss-cross patterns in these examples:

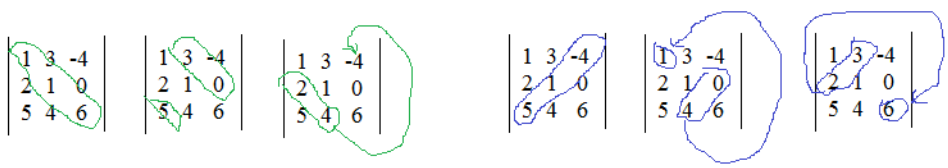
$$\begin{vmatrix} 4 & 5 \\ 2 & 6 \end{vmatrix} = (4 \text{ x } 6) - (5 \text{ x } 2) = 14$$

$$\begin{vmatrix} 3 & -1 \\ 4 & -1 \end{vmatrix} = 3 \text{ x } (-1) - (-1 \text{ x } 4) = -3 - (-4) = 1$$

$$\begin{vmatrix} -2 & 1 \\ 4 & 2 \end{vmatrix} = -4 - (4) = -8 \qquad \begin{vmatrix} 1/2 & 4 \\ 2 & 16 \end{vmatrix} = 8 - 8 = 0$$

$$\begin{vmatrix} 1 & 3 & -4 \\ 2 & 1 & 0 \\ 5 & 4 & 6 \end{vmatrix} = (1 \text{ x } 1 \text{ x } 6) + (3 \text{ x } 0 \text{ x } 5) + (2 \text{ x } 4 \text{ x } (-4)) - [\,(-4 \text{ x } 1 \text{ x } 5) + (0 \text{ x } 4 \text{ x } 1) + (6 \text{ x } 3 \text{ x } 2)\,]$$

$$6 \;+\; 0 \;+\; (-32) \;-\; [\,-20 \;+\; 0 \;+\; 36\,] \;=\; -42$$



The determinant of a larger matrix can be found using method of "expansion by cofactors".

Example: Find the determinant: $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

Method 1: break into separate 2 x 2 matrices...

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \qquad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \qquad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$$1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

1 x (-3) 2 x (-6) 3 x (-3) = -3 - (-12) + (-9) = 0

Method 2: copy and cross

$$\begin{vmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 4 & 5 & 6 & 4 & 5 & 6 \\ 7 & 8 & 9 & 7 & 8 & 9 \end{vmatrix}$$

\+ \+ \+ - - -

$$\begin{vmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 4 & 5 & 6 & 4 & 5 & 6 \\ 7 & 8 & 9 & 7 & 8 & 9 \end{vmatrix} \qquad \begin{vmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 4 & 5 & 6 & 4 & 5 & 6 \\ 7 & 8 & 9 & 7 & 8 & 9 \end{vmatrix}$$

+(1 x 5 x 9) +(2 x 6 x 7) +(3 x 4 x 8)

= 45 + 84 + 96

225

−(1 x 6 x 8) −(2 x 4 x 9) −(3 x 5 x 7) =

= -48 -72 -105

-225

0

Cramer's Rule

What is it? A method of solving linear systems using determinants.

The system must be a "square" (e.g. 3 equations, 3 variables; 2 equations, 2 variables)
The determinant of the coefficients must not equal zero (i.e. the matrix of the coefficients would be invertible)

2 x 2 example:

$$\begin{aligned} 3x + 4y &= 14 \\ 2x - 6y &= 5 \end{aligned}$$

$3x + 4y = 14$, $2x - 6y = 5$ (find determinant of the coefficients)

$3x + 4y = 14$, $2x - 6y = 5$ ("replace the x column with the solution column")

$3x + 4y = 14$, $2x - 6y = 5$ ("replace the y column with the solution column")

$$D = \begin{vmatrix} 3 & 4 \\ 2 & -6 \end{vmatrix} = -18 - 8 = -26 \qquad D_x = \begin{vmatrix} 14 & 4 \\ 5 & -6 \end{vmatrix} = -84 - 20 = -104 \qquad D_y = \begin{vmatrix} 3 & 14 \\ 2 & 5 \end{vmatrix} = -13$$

$$x = \frac{D_x}{D} = \frac{-104}{-26} = 4 \qquad y = \frac{D_y}{D} = \frac{-13}{-26} = \frac{1}{2} \qquad \boxed{(4, \tfrac{1}{2})}$$

3 x 3 example:

$$\begin{aligned} 2x - 3z &= -13 \\ x + 4y + 7z &= 28 \\ 3x - 2y + 4z &= 27 \end{aligned}$$

find "solution" determinant:

$$D = \begin{vmatrix} 2 & 0 & -3 \\ 1 & 4 & 7 \\ 3 & -2 & 4 \end{vmatrix} = (2 \text{ x } 4 \text{ x } 4) + (0 \text{ x } 7 \text{ x } 3) + (-3 \text{ x } 1 \text{ x } -2) - [(-3 \text{ x } 4 \text{ x } 3) + (0 \text{ x } 1 \text{ x } 4) + (2 \text{ x } 7 \text{ x } (-2)]$$

$$= 32 + 0 + 6 - [-36 + 0 + (-28)] = 102$$

find "x determinant" (replace x coefficients with solution column)

$$D_x = \begin{vmatrix} -13 & 0 & -3 \\ 28 & 4 & 7 \\ 27 & -2 & 4 \end{vmatrix} = (-13 \text{ x } 4 \text{ x } 4) + (0 \text{ x } 7 \text{ x } 27) + (-3 \text{ x } 28 \text{ x } (-2)) - [(-3 \text{ x } 4 \text{ x } 27) + (0) + (-13 \text{ x } 7 \text{ x } (-2))]$$

$$= -208 + 0 + 168 - [-324 + 0 + 182] = 102$$

$$x = \frac{D_x}{D} = \frac{102}{102} = 1$$

find "y determinant" (replace y coefficients with solution column)

$$y = \frac{D_y}{D} = \frac{-204}{102} = -2$$

$$D_y = \begin{vmatrix} 2 & -13 & -3 \\ 1 & 28 & 7 \\ 3 & 27 & 4 \end{vmatrix} = (2 \text{ x } 28 \text{ x } 4) + (-13 \text{ x } 7 \text{ x } 3) + (1 \text{ x } 27 \text{ x } (-3)) - [(-3 \text{ x } 28 \text{ x } 3) + (2 \text{ x } 7 \text{ x } 27) + (1 \text{ x } 4 \text{ x } (-13)]$$

$$= 224 + (-273) + (-81) - [-252 + 378 + (-52)] = -204$$

$$z = \frac{D_z}{D} = \frac{510}{102} = 5$$

$\boxed{(1, -2, 5)}$

find "z determinant" (replace z coefficients with solution column)

to check solution, plug answer into linear equations

$$D_z = \begin{vmatrix} 2 & 0 & -13 \\ 1 & 4 & 28 \\ 3 & -2 & 27 \end{vmatrix} = (2 \text{ x } 4 \text{ x } 27) + (0) + (1 \text{ x } (-2) \text{ x } (-13)) - [(-13 \text{ x } 4 \text{ x } 3) + (0) + (2 \text{ x } 28 \text{ x } (-2)] =$$

$$216 + 0 + 26 - [-156 + 0 - 112] = 510$$

Solving Linear Systems using the inverse matrix

Suppose you need to solve $20x = 40$

If you multiply both sides by the multipicative inverse (reciprocal) of 20, you reveal the solution:

$$\frac{1}{20} \cdot 20x = \frac{1}{20} \cdot 40$$

$$x = 2$$

The same approach works with matrices and linear systems.

Suppose A is a <u>square</u> matrix that represents the coefficients
X is a matrix that represents the variables
B is a matrix that represents the solutions

Then, $AX = B$

Now, suppose A^{-1} is the inverse of A.

Since $AX = B$,

then, $A^{-1}AX = A^{-1}B$

$A^{-1}A = I$ (identity matrix)
and, $IX = X$

Therefore, $X = A^{-1}B$

**Important $X = A^{-1}B$
The inverse matrix is *left* of the B matrix.

Note: If the determinant of the matrix = 0, then the matrix is *non-invertible*. Therefore, the linear system does not have a *unique* solution!

Example: $3x - 2y = 10$, $-6x + 4y = 4$ $\quad \begin{vmatrix} 3 & -2 \\ -6 & 4 \end{vmatrix} = 0$ (inconsistent system: no solution)

$5x + 4y = 12$, $-10x - 8y = -24$ $\quad \begin{vmatrix} 5 & 4 \\ -10 & -8 \end{vmatrix} = 0$ (infinite number of solutions)

Example: $4x + 3y = 23$
$x - 6y = -28$

$$A = \begin{bmatrix} 4 & 3 \\ 1 & -6 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad B = \begin{bmatrix} 23 \\ -28 \end{bmatrix}$$

Notice: $AX = B$

Find A^{-1}

$$\det A = \begin{vmatrix} 4 & 3 \\ 1 & -6 \end{vmatrix} = -24 - 3 = -27$$

("Transpose the matrix")

$$\begin{bmatrix} 4 & 3 \\ 1 & -6 \end{bmatrix} \xrightarrow{\text{("swap a and d")}} \begin{bmatrix} -6 & 3 \\ 1 & 4 \end{bmatrix} \xrightarrow{\text{("negative b and c")}} \begin{bmatrix} -6 & -3 \\ -1 & 4 \end{bmatrix}$$

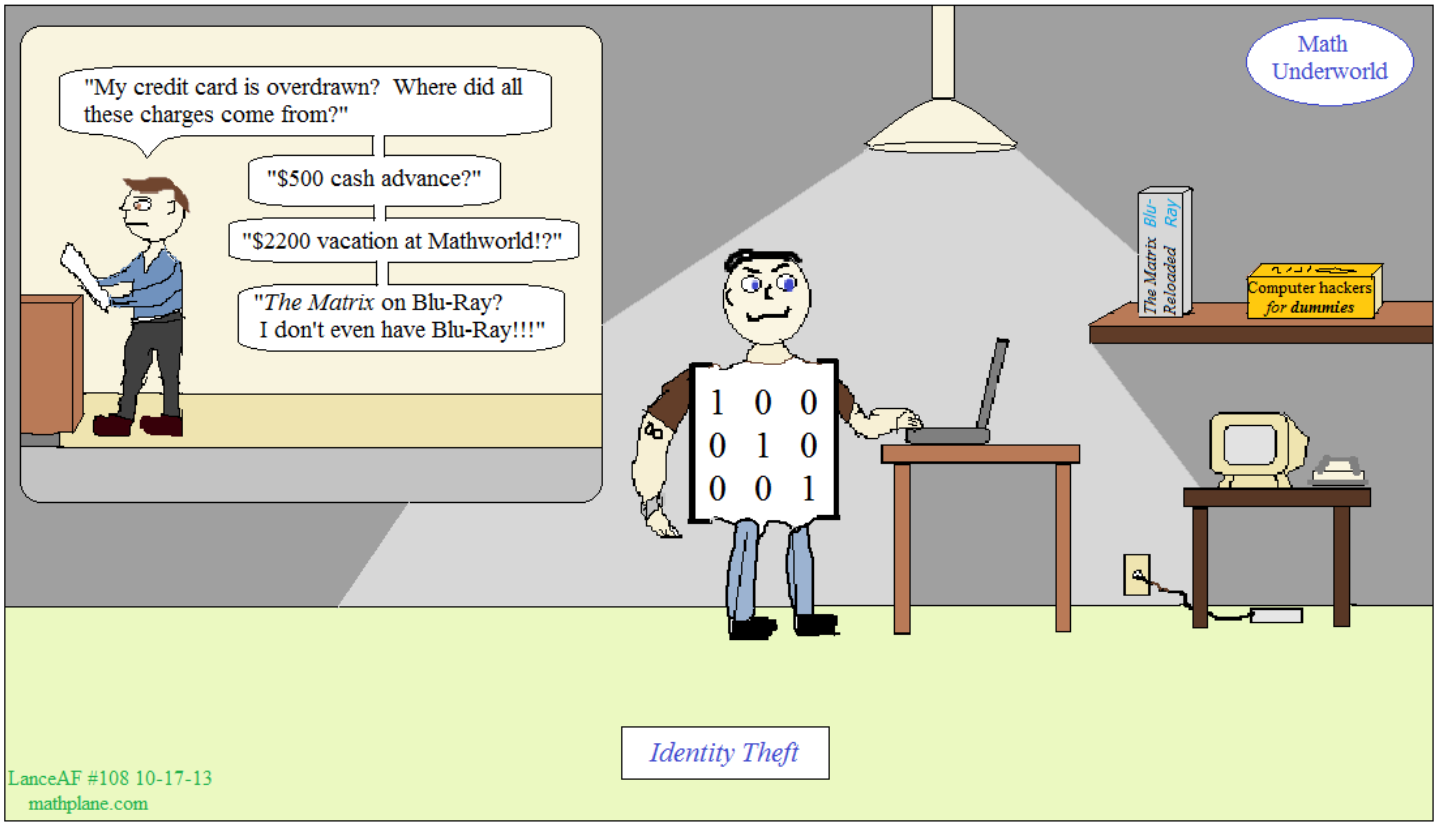
$$A^{-1} = \frac{1}{-27} \begin{bmatrix} -6 & -3 \\ -1 & 4 \end{bmatrix}$$

$AX = B$
then, $A^{-1}AX = A^{-1}B$

$$\underset{A^{-1}}{\begin{bmatrix} 2/9 & 1/9 \\ 1/27 & -4/27 \end{bmatrix}} \underset{A}{\begin{bmatrix} 4 & 3 \\ 1 & -6 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{A^{-1}}{\begin{bmatrix} 2/9 & 1/9 \\ 1/27 & -4/27 \end{bmatrix}} \underset{B}{\begin{bmatrix} 23 \\ -28 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18/9 \\ 135/27 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$



Practice Quiz →

Matrix Inverse and Determinant Quiz

Find the determinant of the following:

a) $\begin{vmatrix} 5 & 8 \\ 6 & 10 \end{vmatrix} =$

b) $\begin{vmatrix} -3 & -1 \\ 9 & 2 \end{vmatrix} =$

c) $\begin{vmatrix} 3 & 4 & -1 \\ 0 & 5 & 6 \\ -2 & -3 & 1/2 \end{vmatrix} =$

d) $\begin{vmatrix} 5 & -1 & 3 \\ 2 & 4 & 1 \end{vmatrix} =$

e) $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} =$

Find the inverse of the following:

a) Use the formula method:

$A = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$ $\quad A^{-1} =$

b) Use the augmented matrix method:

$B = \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}$ $\quad B^{-1} =$

c) Find the determinant of matrix C; What is the inverse C^{-1} ?

$C = \begin{bmatrix} 6 & -2 \\ 3 & -1 \end{bmatrix}$

What is the 4x4 *identity matrix*?

Use Cramer's Rule to solve the following linear system:

$4x + 7y = 43$
$5x - y = 5$

Use Cramer's Rule to find x:

$3x + 5y = 11$
$-2x + y - 3z = 4$
$x + 10z = 17$

Use an augmented matrix to solve the following system:

$7x + 3y + 2z = 6$
$2x - 2y + 10z = 15$
$x + 5y - 12z = -19$

Determine the unknown variable(s) in each determinant.

$\begin{vmatrix} 2 & x \\ 3 & -2 \end{vmatrix} = 11$

$\begin{vmatrix} x & 8 \\ x & x \end{vmatrix} = 20$

$\begin{vmatrix} a & -1 \\ b & 2 \end{vmatrix} = 1$ AND $\begin{vmatrix} b & a \\ 5 & 7 \end{vmatrix} = -107$

Solve using Cramer's Rule

$2x - 3y = 5$

$-8x + 12y = 2$

$x + 5y = 14$

$3x + 15y = 42$

Use Cramer's rule to solve the following system:

$a - 2b - 3c = -1$

$2a + b + c = 6$

$a + 3b - 2c = 13$

Solve the system using $A \cdot X = B$ method...

$$-3x + y = -3$$
$$9x - 5y = 3$$

Given $\begin{vmatrix} a & 3 & -2 \\ 2 & 5 & 4 \\ 3 & -1 & 2a \end{vmatrix} = 184$ Find a.

Matrix Inverse and Determinant Quiz

SOLUTIONS

Find the determinant of the following:

a) $\begin{vmatrix} 5 & 8 \\ 6 & 10 \end{vmatrix} = 50 - 48 = 2$

b) $\begin{vmatrix} -3 & -1 \\ 9 & 2 \end{vmatrix} = (-3 \times 2) - (-1 \times 9) = -6 - (-9) = 3$

c) $\begin{vmatrix} 3 & 4 & -1 \\ 0 & 5 & 6 \\ -2 & -3 & 1/2 \end{vmatrix} = (3 \times 5 \times 1/2) + (4 \times 6 \times -2) + (-1 \times 0 \times -3) - [(-1 \times 5 \times -2) + (4 \times 0 \times 1/2) + (3 \times 6 \times -3)]$

$15/2 + (-48) + 0 - [10 + 0 + (-54)]$

$-40\ 1/2 + 44 = 3\ 1/2$

d) $\begin{vmatrix} 5 & -1 & 3 \\ 2 & 4 & 1 \end{vmatrix} = \emptyset$

It's not a square matrix; determinant cannot be calculated!

e) $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

Find the inverse of the following:

a) Use the formula method:

$A = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ \frac{-3}{10} & \frac{2}{5} \end{bmatrix}$

$\det A = \begin{vmatrix} 4 & -2 \\ 3 & 1 \end{vmatrix} = (4 \times 1) + (-2 \times 3) = 10$

"transform A": $\begin{bmatrix} 1 & \\ & 4 \end{bmatrix}$ (switch) $\begin{bmatrix} & 2 \\ -3 & \end{bmatrix}$ (opposites) $\longrightarrow \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$

inverse is $\frac{1}{10}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ \frac{-3}{10} & \frac{2}{5} \end{bmatrix}$

to check your answer, confirm that $AA^{-1} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

b) Use the augmented matrix method:

$B = \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} -1 & 3 \\ -1 & 2 \end{bmatrix}$

$\left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right]$ (B | I) $\xrightarrow{-2R + R1 \text{ (replace R1)}} \left[\begin{array}{cc|cc} 0 & -1 & 1 & -2 \\ 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{\text{switch rows}} \left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right] \xrightarrow{-1R2} \left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right]$

R1 + R2 (replace R1)

$\left[\begin{array}{cc|cc} 1 & 0 & -1 & 3 \\ 0 & 1 & -1 & 2 \end{array}\right]$ (I | B^{-1})

note: $\begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

c) Find the determinant of matrix C; What is the inverse C^{-1} ?

$C = \begin{bmatrix} 6 & -2 \\ 3 & -1 \end{bmatrix}$

$\det C = \begin{vmatrix} 6 & -2 \\ 3 & -1 \end{vmatrix} = (6 \times -1) - (-2 \times 3) = 0$

Since the determinant = 0, the matrix is *non-invertible.*

There is NO C^{-1}

SOLUTIONS

What is the 4x4 *identity matrix*?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Use Cramer's Rule to solve the following linear system:

$$4x + 7y = 43$$
$$5x - y = 5$$

$$D = \begin{vmatrix} 4 & 7 \\ 5 & -1 \end{vmatrix} = -39 \qquad D_x = \begin{vmatrix} 43 & 7 \\ 5 & -1 \end{vmatrix} = -78 \qquad D_y = \begin{vmatrix} 4 & 43 \\ 5 & 5 \end{vmatrix} = -195$$

$$x = \frac{D_x}{D} = \frac{-78}{-39} = 2 \qquad y = \frac{D_y}{D} = \frac{-195}{-39} = 5$$

(2, 5)

Check: plug x and y into the linear equations

$4(2) + 7(5) = 43$ ✓

$5(2) - (5) = 5$ ✓

Use Cramer's Rule to find x:

$$3x + 5y = 11$$
$$-2x + y - 3z = 4$$
$$x + 10z = 17$$

(To find x, we only need D and D_x)

$$D = \begin{vmatrix} 3 & 5 & 0 \\ -2 & 1 & -3 \\ 1 & 0 & 10 \end{vmatrix} = (3 \times 1 \times 10) + (5 \times -3 \times 1) + 0 - [(0) + (5 \times (-2) \times 10) + (0)] = 15 - (-100) = 115$$

$$D_x = \begin{vmatrix} 11 & 5 & 0 \\ 4 & 1 & -3 \\ 17 & 0 & 10 \end{vmatrix} = (11 \times 1 \times 10) + (5 \times (-3) \times 17) + 0 - [(0) + (5 \times 4 \times 10) + (0)] = 110 - 255 - (200) = -345$$

$$x = \frac{D_x}{D} = \frac{-345}{115} = -3$$

Use an augmented matrix to solve the following system:

$$7x + 3y + 2z = 6$$
$$2x - 2y + 10z = 15$$
$$x + 5y - 12z = -19$$

$$\left[\begin{array}{ccc|c} 7 & 3 & 2 & 6 \\ 2 & -2 & 10 & 15 \\ 1 & 5 & -12 & -19 \end{array}\right] \xrightarrow[\text{(replace R2)}]{-2R3 + R2} \left[\begin{array}{ccc|c} 7 & 3 & 2 & 6 \\ 0 & -12 & 34 & 53 \\ 1 & 5 & -12 & -19 \end{array}\right] \xrightarrow[\text{(replace R1)}]{-3R3 + R1} \left[\begin{array}{ccc|c} 0 & -32 & 86 & 139 \\ 0 & -12 & 34 & 53 \\ 1 & 5 & -12 & -19 \end{array}\right]$$

$$\xrightarrow[\text{then, } -1/12 \text{ (R2)}]{\text{switch R1 \& R3}} \left[\begin{array}{ccc|c} 1 & 5 & -12 & -19 \\ 0 & 1 & -17/6 & -53/12 \\ 0 & -32 & 86 & 139 \end{array}\right] \xrightarrow[\text{(replace R3)}]{32R2 + R3} \left[\begin{array}{ccc|c} 1 & 5 & -12 & -19 \\ 0 & 1 & -17/6 & -53/12 \\ 0 & 0 & -14/3 & -7/3 \end{array}\right]$$

$$\xrightarrow{(-3/14)R3} \left[\begin{array}{ccc|c} 1 & 5 & -12 & -19 \\ 0 & 1 & -17/6 & -53/12 \\ 0 & 0 & 1 & 1/2 \end{array}\right] \xrightarrow[\text{replace R2; } (12)R3 + R1 \text{ replace R1}]{(17/6)R3 + R2} \left[\begin{array}{ccc|c} 1 & 5 & 0 & -13 \\ 0 & 1 & 0 & -36/12 \\ 0 & 0 & 1 & 1/2 \end{array}\right]$$

$$\xrightarrow[\text{(replace R1)}]{-5R2 + R1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1/2 \end{array}\right]$$

x = 2 y = -3 z = 1/2

Determine the unknown variable(s) in each determinant.

SOLUTIONS

$$\begin{vmatrix} 2 & x \\ 3 & -2 \end{vmatrix} = 11$$

$2(-2) - x(3) = 11$

$-4 - 3x = 11$

$x = -5$

$$\begin{vmatrix} x & 8 \\ x & x \end{vmatrix} = 20$$

$x^2 - 8x = 20$

$x^2 - 8x - 20 = 0$

$(x - 10)(x + 2) = 0$

$x = -2, 10$

AND

$$\begin{vmatrix} a & -1 \\ b & 2 \end{vmatrix} = 1 \qquad \begin{vmatrix} b & a \\ 5 & 7 \end{vmatrix} = -107$$

$2a - -1b = 1$

$7b - 5a = -107$

$2a + b = 1$

$-5a + 7b = -107$

$-14a - 7b = -7$

$-19a = -114$

$a = 6 \quad b = -11$

Solve using Cramer's Rule

$2x - 3y = 5$

$-8x + 12y = 2$

$$D_x = \begin{vmatrix} 5 & -3 \\ 2 & 12 \end{vmatrix} = 66 \qquad D_y = \begin{vmatrix} 2 & 5 \\ -8 & 2 \end{vmatrix} = 44$$

$$D = \begin{vmatrix} 2 & -3 \\ -8 & 12 \end{vmatrix} = 0$$

Since the determinant is 0,

there is no solution!!

(inconsistent system)

$\frac{D_x}{D}$ is undefined..

$\frac{D_y}{D}$ is undefined..

$x + 5y = 14$

$3x + 15y = 42$

$$D = \begin{vmatrix} 1 & 5 \\ 3 & 15 \end{vmatrix} = 0 \qquad D_x = \begin{vmatrix} 14 & 5 \\ 42 & 15 \end{vmatrix} = 0 \qquad D_y = \begin{vmatrix} 1 & 14 \\ 3 & 42 \end{vmatrix} = 0$$

Same equations!

(consistent and dependent system)

Use Cramer's rule to solve the following system:

$a - 2b - 3c = -1$

$2a + b + c = 6$

$a + 3b - 2c = 13$

$$D = \begin{vmatrix} 1 & -2 & -3 \\ 2 & 1 & 1 \\ 1 & 3 & -2 \end{vmatrix} = -30$$

$$\begin{vmatrix} 1 & -2 & -3 \\ 2 & 1 & 1 \\ 1 & 3 & -2 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} - -2\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} + -3\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = -5 - 10 - 15 = -30$$

"Replace each respective column with the constant column"

$$D_x = \begin{vmatrix} -1 & -2 & -3 \\ 6 & 1 & 1 \\ 13 & 3 & -2 \end{vmatrix} = -60 \qquad D_y = \begin{vmatrix} 1 & -1 & -3 \\ 2 & 6 & 1 \\ 1 & 13 & -2 \end{vmatrix} = -90 \qquad D_z = \begin{vmatrix} 1 & -2 & -1 \\ 2 & 1 & 6 \\ 1 & 3 & 13 \end{vmatrix} = 30$$

$$\frac{D_x}{D} = 2 \qquad \frac{D_y}{D} = 3 \qquad \frac{D_z}{D} = -1$$

Solve the system using $A \cdot X = B$ method...

SOLUTIONS

$$-3x + y = -3$$
$$9x - 5y = 3$$

Step 1: Identify the matrices

$$A = \begin{bmatrix} -3 & 1 \\ 9 & -5 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad B = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

Step 2: Find the inverse of A

Method 1: Apply the formula

$$\frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} =$$

$$\frac{1}{\begin{vmatrix} -3 & 1 \\ 9 & -5 \end{vmatrix}}\begin{bmatrix} -5 & -1 \\ -9 & -3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -5 & -1 \\ -9 & -3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{-5}{6} & \frac{-1}{6} \\ \frac{-3}{2} & \frac{-1}{2} \end{bmatrix}$$

Method 2: Use Augmented Matrix

$$\left[\begin{array}{cc|cc} -3 & 1 & 1 & 0 \\ 9 & -5 & 0 & 1 \end{array}\right] \quad \text{3R1 added to R2}$$

$$\left[\begin{array}{cc|cc} -3 & 1 & 1 & 0 \\ 0 & -2 & 3 & 1 \end{array}\right] \quad \frac{-1}{2}\text{R2}$$

$$\left[\begin{array}{cc|cc} -3 & 1 & 1 & 0 \\ 0 & 1 & -3/2 & -1/2 \end{array}\right] \quad \text{-1R2 added to R1}$$

$$\left[\begin{array}{cc|cc} -3 & 0 & 5/2 & 1/2 \\ 0 & 1 & -3/2 & -1/2 \end{array}\right] \quad \frac{-1}{3}\text{R1}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -5/6 & -1/6 \\ 0 & 1 & -3/2 & -1/2 \end{array}\right]$$

To check:

$$\begin{bmatrix} \frac{-5}{6} & \frac{-1}{6} \\ \frac{-3}{2} & \frac{-1}{2} \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 9 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A^{-1} A   I

Step 3: Multiply Matrices

$$A \cdot X = B$$

$$A^{-1} A \cdot X = A^{-1} B$$

$A^{-1} A = I$ (the identity matrix)

$$X = A^{-1} B$$

$$X = \begin{bmatrix} \frac{-5}{6} & \frac{-1}{6} \\ \frac{-3}{2} & \frac{-1}{2} \end{bmatrix}\begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{matrix} x \\ y \end{matrix}$$

Step 4: Check Solution $x = 2 \quad y = 3$

$-3x + y = -3$ $-6 + 3 = -3$ ✓

$9x - 5y = 3$ $18 - 15 = 3$ ✓

Given $\begin{vmatrix} a & 3 & -2 \\ 2 & 5 & 4 \\ 3 & -1 & 2a \end{vmatrix} = 184$ Find a.

simply evaluate the determinant....

3 x 3 determinant

$$a\begin{vmatrix} 5 & 4 \\ -1 & 2a \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 3 & 2a \end{vmatrix} + -2\begin{vmatrix} 2 & 5 \\ 3 & -1 \end{vmatrix}$$

then, solve for a....

$$a\begin{vmatrix} 5 & 4 \\ -1 & 2a \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 3 & 2a \end{vmatrix} + -2\begin{vmatrix} 2 & 5 \\ 3 & -1 \end{vmatrix} = 184$$

$$a(10a + 4) - 3(4a - 12) + -2(-2 - 15) = 184$$

$$10a^2 + 4a - 12a + 36 + 34 = 184$$

$$10a^2 - 8a - 114 = 0$$

$$5a^2 - 4a - 57 = 0$$

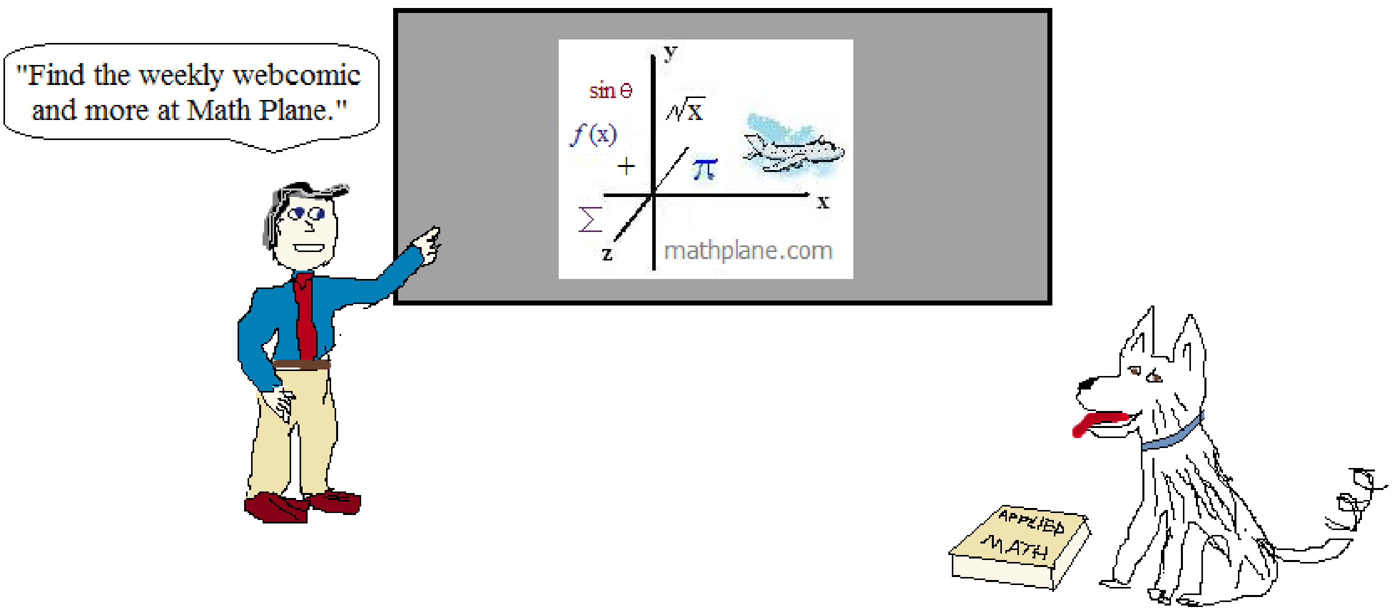
$a = -3$ or $19/5$

then, check your answers!

Thanks for visiting the site. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers



Also at Facebook, Google+, Pinterest, TES and TeachersPayTeachers

Mathplane *Express* for mobile at Mathplane.ORG