Calculus: Mean Value Theorem

Notes, Examples, and Practice Questions (with Solutions)

Topics include MVT definition, Rolle's Theorem, Implicit Differentiation, applications, extrema, and more.

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Derivative Mean Value Theorem

If a function is continuous on the interval [a, b]
and differentiable on the interval (a, b), then

there exists at least one point c where $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$f'(c)$: instanteous rate of change at c

$\dfrac{f(b) - f(a)}{b - a}$: average rate of change between a and b

Example: $h(x) = x^3 - 2$

a) determine the AROC on the interval [-1, 3]

b) find the value "c" to verify the mean value theorem

First, we recognize that this satisfies the necessary parts of the MVT..

It is continuous on [-1, 3] and differentiable on (-1, 3)...

a) Average Rate Of Change (slope) $\dfrac{25 - (-3)}{3 - (-1)} = \boxed{7}$

b) Instantaneous Rate Of Change $h'(x) = 3x^2 - 0$

at point "c" $h'(c) = 7$

$3c^2 = 7$

c = ~~-1.53~~ or $\boxed{1.53}$

not in the interval [-1, 3]

Application: A runner goes 5km is 20 minutes. Show that he ran exactly 12 km/hour at least twice.

Mean Value Theorem

The velocity of the runner is continuous...
Initial rate is 0 ----> (0, 0)

AROC is 5 km/20 minutes = 15 km/hour

As the runner accelerates from 0 to 15 (or more), he must pass a rate of 12 km/hour.
And, when the runner stops, he must slow down from at least 15 km/hour to 0...

If function is continuous and differentiable...
there exists at least one point c where

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \text{interval } [a, b]$$

What does it *mean*?

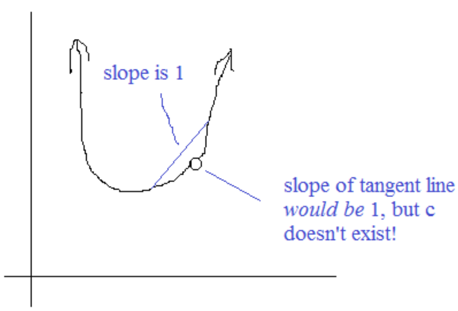
Assume the right side is the formula for slope between two points (AROC) ⟹ secant line

the left side is the expression for the slope at a point (IROC) ⟹ tangent line

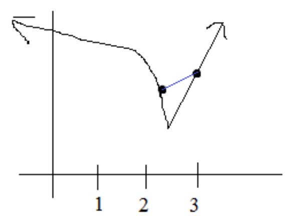
So, whatever the slope between 2 points, there is some point that has the same slope....

Exceptions:

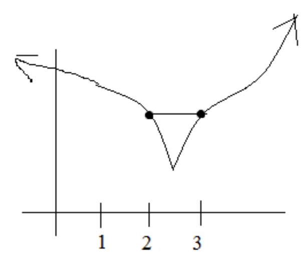
it's not continuous



it's not differentiable



nowhere with the same slope because of the cusp



slope is 0 (horizontal line),
but because of the cusp, there is no slope of 0

function is not differentiable on interval (2, 3)

Example: Let g be a function $g(x) = x^3 - 2x^2$

Find all values c on the interval [-1, 3] that satisfy the conclusion of the mean value theorem

$g(-1) = -3 \qquad g(3) = 9$ slope between (-1, -3) and (3, 9) is 3

MVT ⟹ $g'(c) = \frac{g(3) - g(-1)}{3 - (-1)} = \frac{9 - (-3)}{4} = 3$

$$g'(x) = 3x^2 - 4x$$

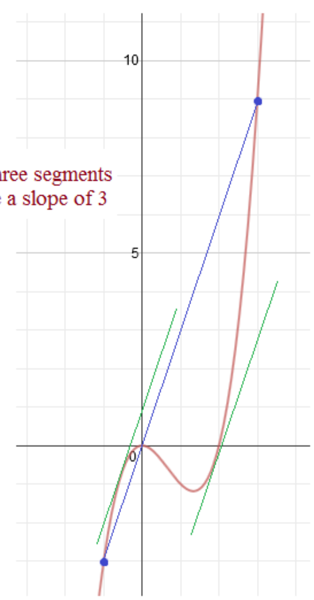
$$3 = 3c^2 - 4c$$

$$3c^2 - 4c - 3 = 0$$

$$c = \frac{4 \pm \sqrt{16 - (-36)}}{2(3)} = \frac{2 \pm \sqrt{13}}{3}$$

approx. -.53 and 1.87

all three segments have a slope of 3



Who is Michel Rolle? A French mathematician from the 1600s and 1700s.

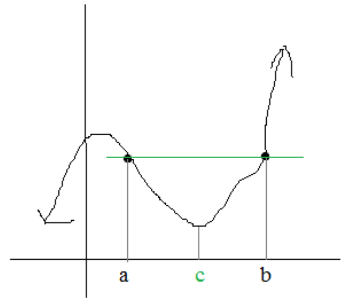
What is Rolle's Theorem? It's a specific "version of the Mean Value Theorem (MVT)", when the slope is zero.

Definition: If function is continuous and differentiable... and $f(a) = f(b)$, then there exists at least one point c where

$$f'(c) = \frac{f(b) - f(a)}{b - a} = 0 \qquad \text{interval } [a, b]$$

Note: if $f(a) = f(b)$, then the line $\overline{ab}$ is horizontal ----> slope is 0

This implies there is (at least) one critical value in [a, b] ---> a maximum or minimum



Example: For the function $f(x) = x^3 + 5x^2 - 17x - 21$,

find an interval such that Rolle's Theorem would apply...
Then, determine the "c" value, such that $f(x)$ is a relative max or relative min.

We'll seek an interval between zeros....

$f(x) = (x - 3)(x + 1)(x + 7)$ so, the zeros are (3, 0), (-1, 0), and (-7, 0)...

So, let's choose the interval [-7, 3]....

Slope between (-7, 0) and (3, 0) is 0
(horizontal tangent)

$f'(x) = 3x^2 + 10x - 17$

where is $f'(x) = 0$??

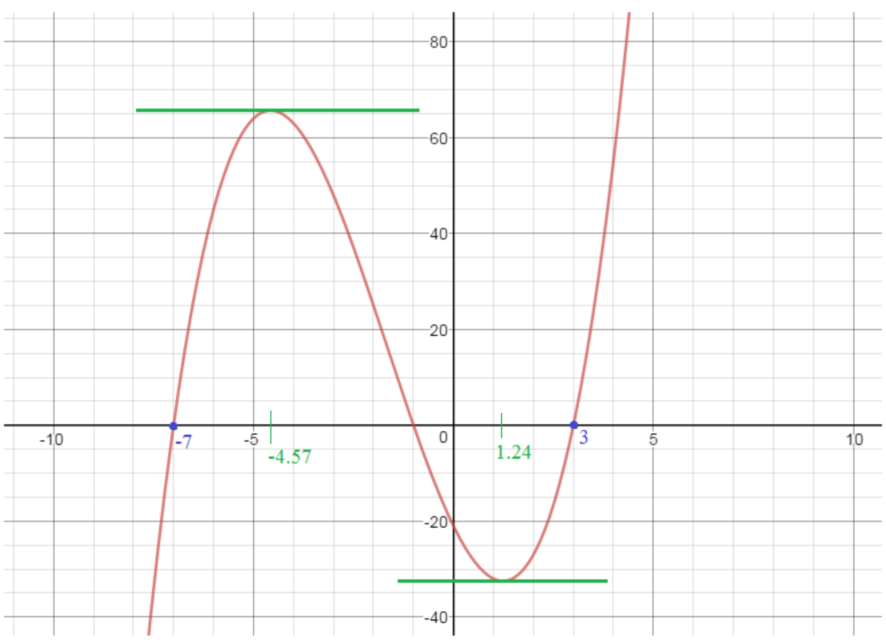
$3x^2 + 10x - 17 = 0$ $\quad$ $x = -4.57$ and $x = 1.24$

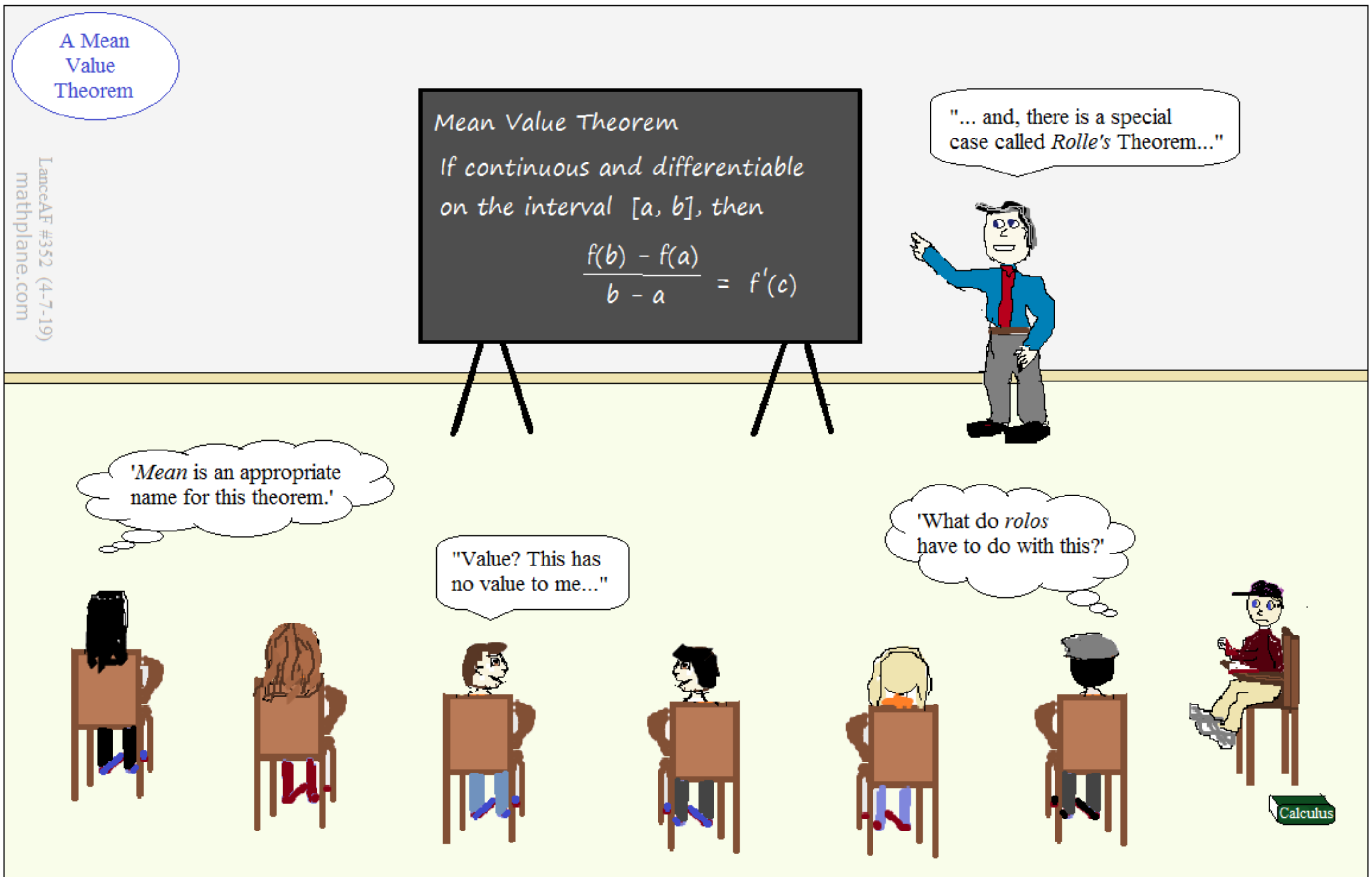
conditions of Rolle's theorem

$f(x)$ is continuous on the closed interval [-7, 3]

$f(x)$ is differentiable on the open interval (-7, 3)

$f(-7) = f(3)$





Alternative Definition: If this class curve is continuous and differentiable, and the maximum grade is a C and the minimum grade is an F, then there exists many unsatisfactory grade points in between.....

Practice Questions-→

Mean Value Theorem Questions

1) On the interval [0, 3], find the value of "c" that satisfies the Mean Value Theorem.

$$f(x) = x^2 - 2x + 5$$

2) For the function $f(x) = x^3 + 2x^2 - 9x - 18$

Apply Rolle's Theorem and explain why there is a (local) minimum between $x = -2$ and $x = 3$...

3) What is the tangent line that is parallel to the secant line with points (-3, 8) and (4, 1) that passes through

$$x^2 + (y - 4)^2 = 25$$

4) The function $f(x) = x^2 - \frac{1}{e^x}$ exists over the interval [0, 3].

Where does the Average Rate of Change equal the Instanteous Rate of Change?

a) -1.99

b) 1.55

c) 2.57

d) 3.32

e) 9.96

5) Explain and show why the MVT applies to [0, 8], but fails in the interval [-1, 8]...

$f(x) = x^{\frac{2}{3}}$

6) $f(x) = \frac{1}{x}$ On the interval [-2, 2], find c that satisfies the mean value theorem.

Why doesn't it work?!?!

Mean Value Theorem Questions

SOLUTIONS

1) On the interval [0, 3], find the value of "c" that satisfies the Mean Value Theorem.

$$f(x) = x^2 - 2x + 5$$

Step 1: Determine if the function satisfies the MVT

-----> it is continuous on [0, 3] and differentiable on (0, 3)
so, it qualifies..

Step 2: Find the AROC (i.e. slope between endpoints)

$f(0) = 5$ and $f(3) = 8$

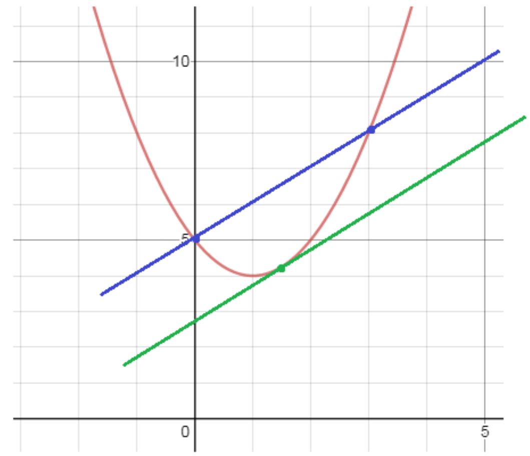
-----> the slope between (0, 5) and (3, 8) is 1

Step 3: Find the IROC

-----> $f'(x) = 2x - 2$ → $f'(c) = 1$

$2(c) - 2 = 1$

$c = 3/2$



2) For the function $f(x) = x^3 + 2x^2 - 9x - 18$

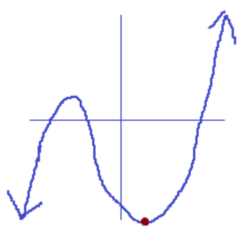
Apply Rolle's Theorem and explain why there is a (local) minimum between x = -2 and x = 3...

factor $f(x)$ --- $x^2(x + 2) - 9(x + 2)$

$(x^2 - 9)(x + 2)$

$(x + 3)(x - 3)(x + 2)$

$f(-2) = f(3) = 0$



$f'(c) = 0$ at a point in the interval (-2, 3)

$f'(x) = 3x^2 + 4x - 9$

$0 = 3x^2 + 4x - 9$ -2.52 and 1.19

since the derivative equals zero, it is a maximum or a minimum.
And, we find that $f(c) < 0$ ----> minimum

$f(1.19) = 1.685 + 2.832 - 10.71 - 18 = -24.2$ ✓

3) What is the tangent line that is parallel to the secant line with points (-3, 8) and (4, 1)
that passes through

$$x^2 + (y - 4)^2 = 25$$

secant line: slope is $\frac{8 - 1}{-3 - 4} = -1$

$y = -x + 5$

tangent line that is parallel will have a slope of -1

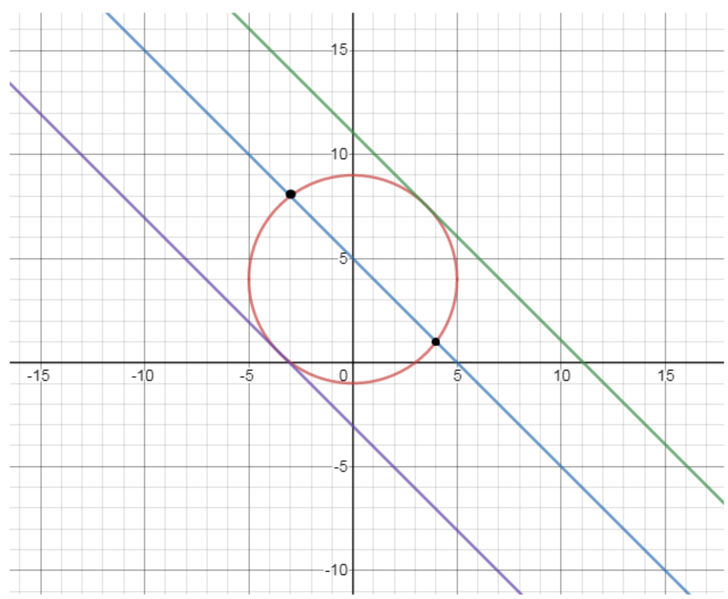
$x^2 + y^2 - 8y + 16 = 25$

$2x + 2y\frac{dy}{dx} - 8\frac{dy}{dx} + 0 = 0$

$\frac{dy}{dx} = \frac{2x}{8 - 2y}$

plug in the slope -1: $-1 = \frac{x}{4 - y}$

$y = x + 4$



Find the intersection: $y = x + 4$ and $x^2 + (y - 4)^2 = 25$ → $2x^2 = 25$

(3.53, 7.53)

(-3.53, .47)

$y - 7.53 = -1(x - 3.53)$

$y - .47 = -1(x + 3.53)$

Mean Value Theorem Questions

SOLUTIONS

4) The function $f(x) = x^2 - \frac{1}{e^x}$ exists over the interval [0, 3].

Where does the Average Rate of Change equal the Instanteous Rate of Change?

a) -1.99

b) 1.55

c) 2.57

d) 3.32

e) 9.96

AROC: $x = 0 \quad f(0) = -1 \quad (0, -1)$

$x = 3 \quad f(3) = 8.95 \quad (3, 8.95)$

Slope between points: $\frac{9.95}{3} = 3.32$

IROC: $f'(x) = 2x + e^{-x}$ so, where is $f'(x) = 3.32$?

$2x + e^{-x} = 3.32$

$x = -1.99$ or 1.55

We cancel -1.99 because it's not in the interval...

5) Explain and show why the MVT applies to [0, 8], but fails in the interval [-1, 8]...

$f(x) = x^{\frac{2}{3}}$

$f'(x) = \frac{2}{3}x^{-1/3}$

Since the function is not differentiable at x = 0,
the MVT does not apply...
(i.e it may or may not work)

However, it can apply to the interval [0, 8],
because a = 0, b = 8 --- c must be in between!

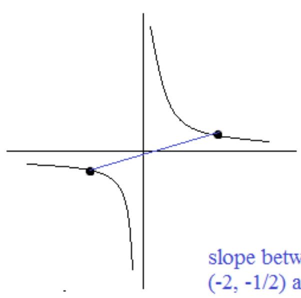
on the interval [0, 8], the function is continuous...

and, differentiable on (0, 8)...

6) $f(x) = \frac{1}{x}$ On the interval [-2, 2], find c that satisfies the mean value theorem.

Why doesn't it work?!?!

Because $f(x) = \frac{1}{x}$ is not continuous (and not differentiable) at $x = 0$



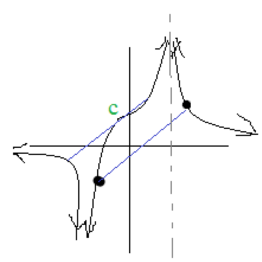
slope between
(-2, -1/2) and (2, 1/2)
is 1/4

and, clearly there is no
spot between -2 and 2 where
the IROC is 1/4

MVT is guaranteed when the interval is differentiable...

Note: if it's not differentiable, it still may work....But, it's not guaranteed..

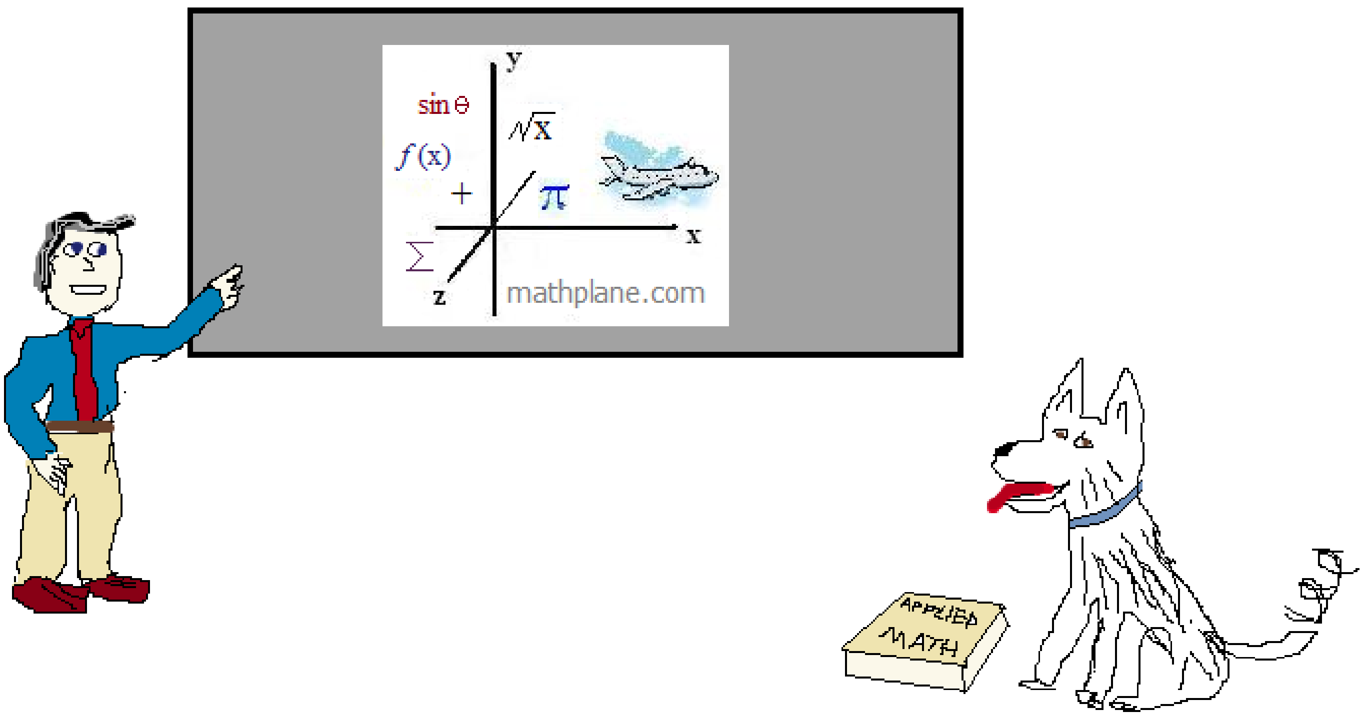
Example:



Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers.



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