# Statistics: Normal Distributions

(Notes, examples, and practice w/solutions)

*Topics include Z-Scores, Standard Deviations, probability intervals, binomial distributions, and more.*

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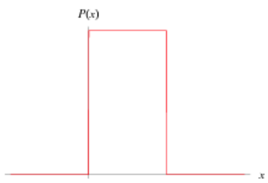
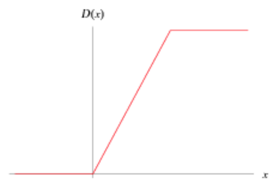
A *Uniform Distribution*, sometimes known as a rectangular distribution, is a distribution that has constant Probability.

Uniform Distribution

Continuous

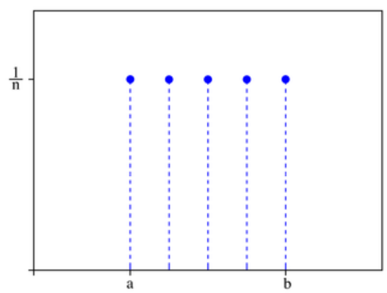



Uniform distribution




Uniform *cumulative* distribution

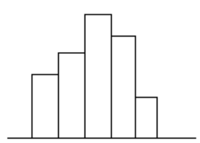
Discrete




n is the number of choices
and, since each outcome has the same chance,
the probability of each is 1/n

Although this distribution is not uniform, it is quite easy to find the area of each rectangle.

Find the probability the player gets an A, B, C..



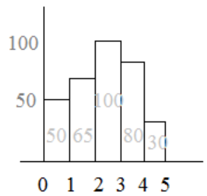

Answer: Find the area of each rectangle.

$$\text{Probability of A, B, or C} = \frac{\text{Area of A} + \text{Area of B} + \text{Area of C}}{\text{Total Area}}$$

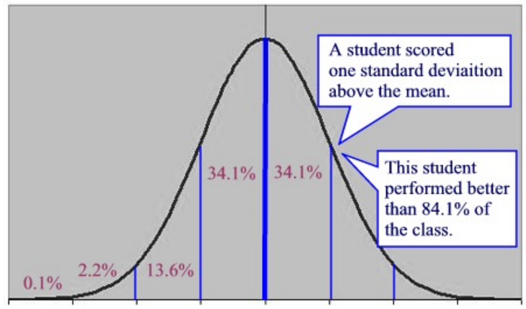
Normal Distribution

So, if we had a random sample of real numbers between 0 and 5, we could determine the amount and probability of each rectangular area....

*Example:* What is the probability of choosing a number less than 3?




$$\text{Probility}(<3) = \frac{50 + 65 + 100}{325} = 66\%$$



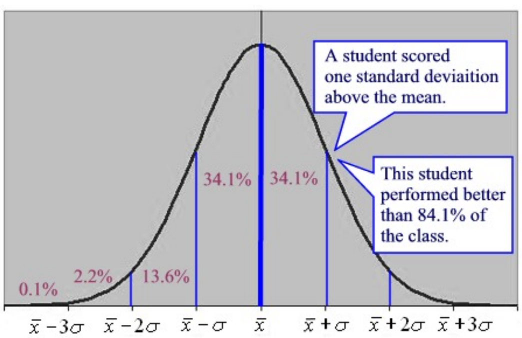

Finding the area under the curve is more difficult. However, since it is a normal distribution, we can convert into a standardized measure (or z-score)

$$z = \frac{(X - \mu)}{\sigma}$$

X is chosen value
M is mean (of population)
$\sigma$ is standard deviation (of population)

$$z = \frac{\text{sample score} - \bar{x}}{\frac{\sigma}{\sqrt{n}}}$$

n is sample size
$\bar{x}$ is the sample mean




$\bar{x}$ is the mean of the sample

The curve is symmetric,
so the area under the curve left of the mean is 50% of the total area...

Note: there are 3 main ways to analyze the data

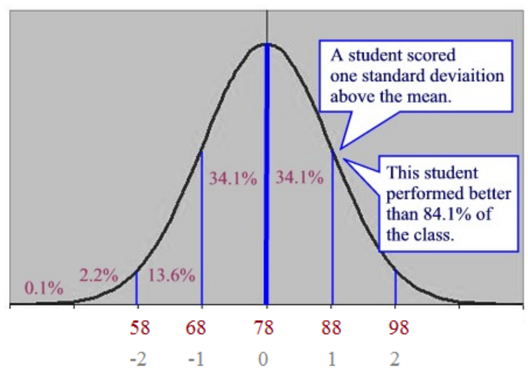
1) measure in standard deviations (from standard error or z-score)

2) measure in percentages (p-score) or percentile

3) measure in quantity

1 standard deviation from the mean is the inflection point of the normal curve!

*Example:* Assume the mean is 78, and the standard deviation is 10...




z-score = 1

value/quantity = 88

percentile = 84.1%

Standard Deviation (binomial samples)

Standard Deviation $= \sqrt{npq}$ where n is the number in the sample

$\sigma_x$

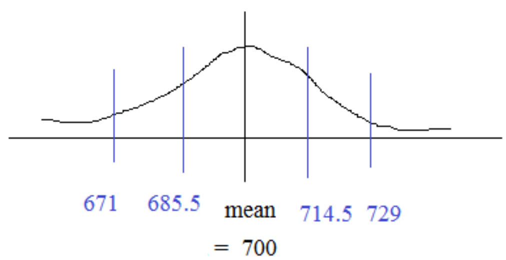
p = probability of 'success'
q = probability of 'failure'

Since it is binomial, p + q = 1

*Example:* Trial of 1000 people.
70% choose up
30% choose down

What is the standard deviation?

$\sqrt{1000(.7)(.3)} = 14.5$ (approx.)




Expected Number of Successes

The graph is labeled by the *number* of people...

Standard Deviation (proportion) $= \sqrt{\frac{pq}{n}}$ where n is the number in the sample

$\sigma_{\hat{p}}$

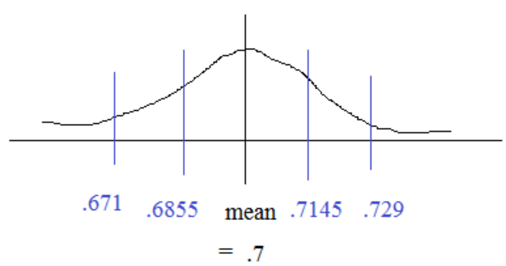
p = probability of 'success'
q = probability of 'failure'

Since it is binomial, p + q = 1

*Example:* Trial of 1000 people.
70% choose up
30% choose down

What is the standard deviation?

$\sqrt{\frac{(.7)(.3)}{1000}} = .0145$ (approx.)




Expected Proportion of Successes

The graph is labeled by the *proportions (percentages)* of people...

There are 3 main ways to describe data in a normal distribution: z-score, value, and percentile (or probability).

*Example:* z-scores

A normal distribution has a population mean of 35 and a standard deviation of 4.
Find the standardized values of 25 and 38

$$Z = \frac{(25-35)}{4} = -2.5$$

25 is 2.5 standard deviations below the mean...

$$Z = \frac{(38-35)}{4} = .75$$

38 is .75 standard deviations above the mean...

Use the z-score formula...
The result describes the value's distance (in standard deviations) from the mean...

*Example:* Actual Value

A normal distribution has a mean of 8.2 and a standard deviation of 2.1.
A sample of size 16 is taken.
What score would 8% of the scores higher than it?

If 8% of the scores are higher than a selected mrak, then 92% of the scores must be below it.
According to the z-score table, a percentage of 92 corresponds to a z-score of 1.4

Since a sample of 16 is taken, here is the z-score:

$$1.4 = \frac{x - 8.2}{\frac{2.1}{\sqrt{16}}} \Rightarrow 1.4 = \frac{x - 8.2}{.525}$$

The actual value of the cut-off score is X = 8.935

*Example:* Percentages

The national ACT test scores have a normal distribution with a mean 21.1 and a standard deviation of 6.4.
Find the proportion of students who score less than 24.

$$Z = \frac{24 - 21.1}{6.4} = .453$$

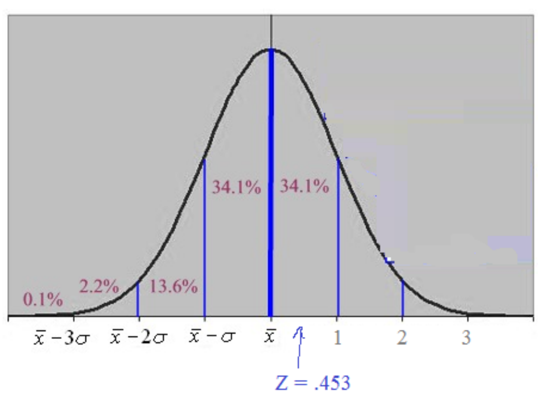
Use the table to translate a z-score to a corresponding percentage.
That percentage represents the amount BELOW -- i.e. the area of the curve to the left...

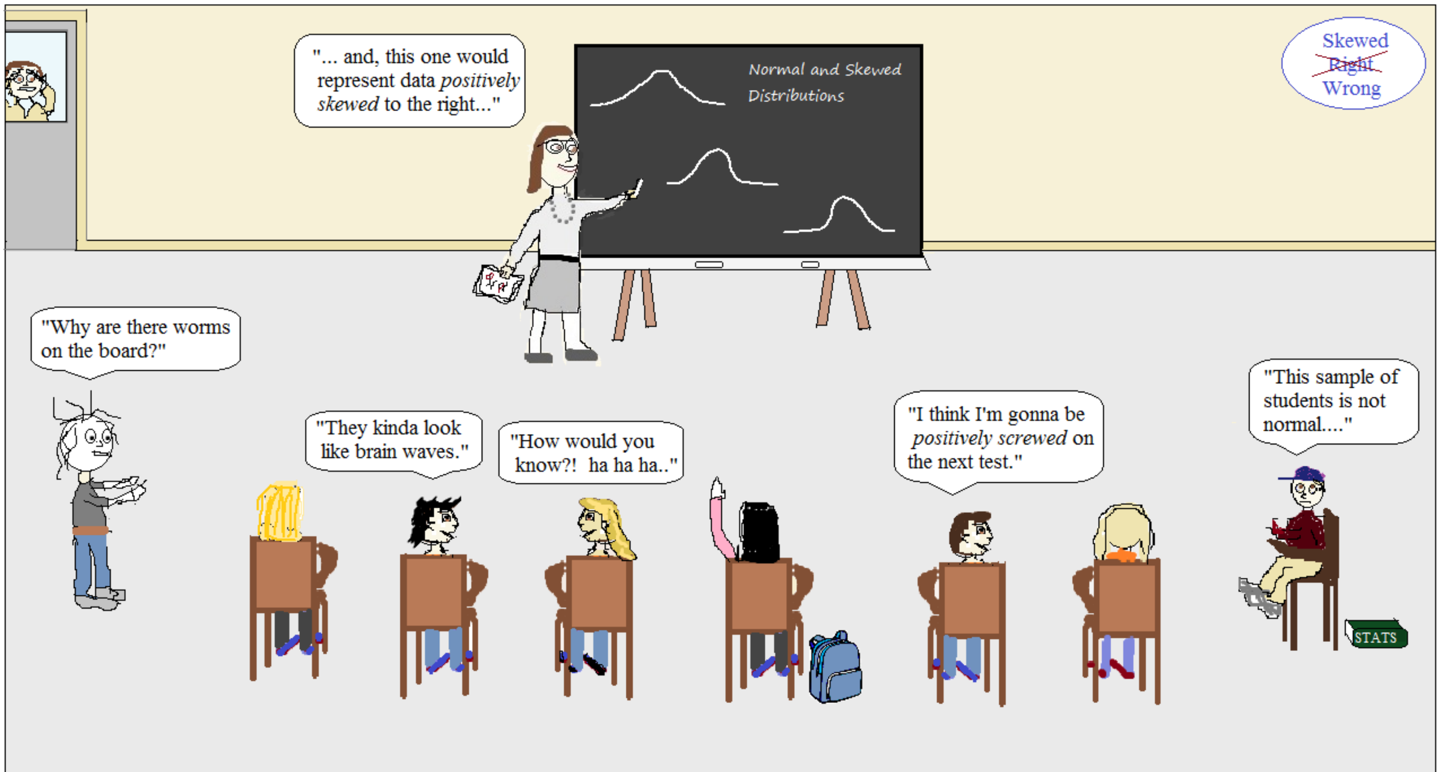
Then, find the corresponding percentage on a table (or calculator)

.453 corresponds to .675...

So, the p(a student scores below 24) is 67.5%

Using percentages, z-scores, probability and standard deviation, you can determine the quantity (portion) of the total sample or population represented in the normal distribution curve...






LanceAF #278 (4-26-17)
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It seems every stats class has some variance, a standard deviant, and outliers...

Practice Exercises-→

Statistics: Normal Distributions Quiz

1) 8 people are randomly chosen from a town. Their ages are 23, 31, 44, 28, 51, 50, 33, 36.
Determine the

a) Determine the mean, median, range, and standard deviation.

b) Using the mean and standard deviation from part a), find the z-score of a 40 year old.

c) Using the result in b), estimate the probability that someone chosen from town is under 40.

2) The average math scores are normally distributed with a mean of 20.9 and standard deviation of 6.5
What range of scores is the middle 50%?

3) In a town, January high has a mean 35 degrees with standard deviation of 9 degrees…

the August average high is 85 degrees with a standard deviation of 11 degrees.

Is it more unusual to see a 60 degree day in January or August?

4) A radar unit is used to measure speeds of cars on a highway. The speeds are normally distributed with a mean of 62 miles per hour and a standard deviation of 8 miles per hour.
What is the probability that a car picked at random is speeding over 70 miles per hour?

5) Albert ran home with his statistics grade. He excitedly claimed, "I scored one and a half standard deviations above the class mean!" If the class has 44 students, where did Albert rank amongst his classmates?

6) The salaries at a large company are normally distributed with a mean of $44,000 and a standard deviation of $8,000.

a) What percent of employees earn less than $36,000?

b) What percent of employees earn more than $48,000?

c) What percent of employees earn between $50,000 and $60,000?

d) What salary level represents the top 20% of employees?

7) The mean basketball score is 78 points with a standard deviation of 6.5 points. The mean baseball score is 6.4 runs with a standard deviation of 1.2 runs.

Are you more likely to see a basketball team score 82 points or a baseball team score 7 runs?

8) To be admitted to the Math Academy, an applicant's score must be in the top 75%. The mean of the MA applicants is 63, and the standard deviation is 8.4. If Tony's score is 69, can he be admitted?

Statistics: Normal Distributions Quiz

SOLUTIONS

1) 8 people are randomly chosen from a town. Their ages are 23, 31, 44, 28, 51, 50, 33, 36.
Determine the

a) Determine the mean, median, range, and standard deviation.

$\overline{x} = 37$ median: 34.5
range: 28

standard deviation of the sample: $= 10.31$

$$\sqrt{\frac{14^2+6^2+7^2+9^2+14^2+13^2+4^2+1^2}{8-1}}$$

b) Using the mean and standard deviation from part a), find the z-score of a 40 year old.

$$z = \frac{x-\overline{x}}{s} = \frac{40-37}{10.31} = .291$$

c) Using the result in b), estimate the probability that someone chosen from town is under 40.

z-score of .291 is .291 standard deviations above the mean...

this corresponds to a probability of 61.4%

2) The average math scores are normally distributed with a mean of 20.9 and standard deviation of 6.5
What range of scores is the middle 50%?

find z-scores for 25% and 75%

$$z = \frac{x-\mu}{\sigma} = \frac{x - 20.9}{6.5}$$

a z-score for 50% is the mean: 0

z-score for <25% : −.674
z-score for <75% : .674

$$-.674 = \frac{x - 20.9}{6.5} \qquad x = 16.52$$

$$.674 = \frac{x - 20.9}{6.5} \qquad x = 25.28$$

The middle range of scores would be between 16.52 and 25.28

3) In a town, January high has a mean 35 degrees with standard deviation of 9 degrees...

the August average high is 85 degrees with a standard deviation of 11 degrees.

Is it more unusual to see a 60 degree day in January or August?

$$z = \frac{x-\mu}{\sigma} = \frac{60-35}{9} = \frac{25}{9} \quad \text{January}$$

$$z = \frac{x-\mu}{\sigma} = \frac{60-85}{11} = \frac{-25}{11} \quad \text{August}$$

It's more unusual to see it in January because it is 25/9 standard deviations..

(in August, a temperature of 60 is 25/11 standard deviations away from the norm...)

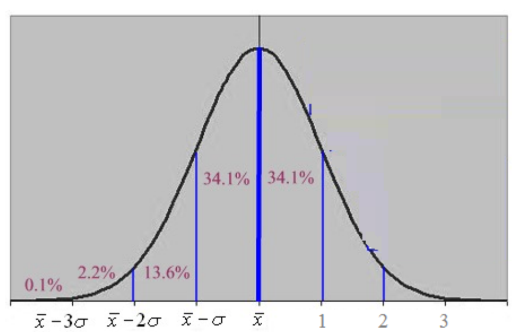
4) A radar unit is used to measure speeds of cars on a highway. The speeds are normally distributed with a mean of 62 miles per hour and a standard deviation of 8 miles per hour.
What is the probability that a car picked at random is speeding over 70 miles per hour?

mean: 62
standard deviation: 8

$$z = \frac{70-62}{8} = 1$$

84.1%

so, 15.9% OVER 70 mph



Statistics: Normal Distributions Quiz

SOLUTIONS

5) Albert ran home with his statistics grade. He excitedly claimed, "I scored one and a half standard deviations above the class mean!" If the class has 44 students, where did Albert rank amongst his classmates?

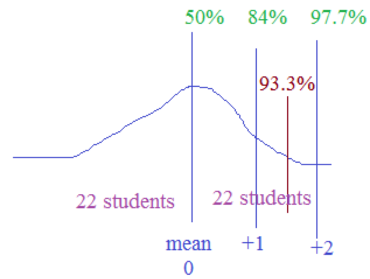
Albert's z-score was 1.5

This corresponds to a percentile of 93.32...

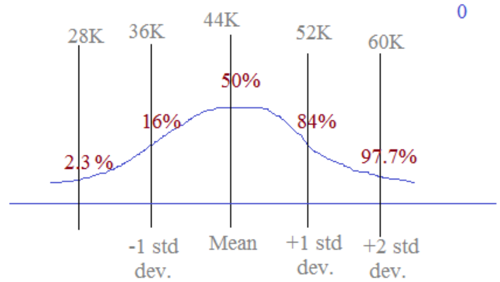
If there are 35 students, 93% of 35 is .93 x 35 = 32.66... below

6.68% of 35 is 2.34 above

Albert ranks 3rd in the class....




6) The salaries at a large company are normally distributed with a mean of $44,000 and a standard deviation of $8,000.




a) What percent of employees earn less than $36,000?

z-score = -1 $\frac{36{,}000 - 44{,}000}{8{,}000}$ ⇨ 16%

b) What percent of employees earn more than $48,000?

z-score = .50 $\frac{48{,}000 - 44{,}000}{8{,}000}$ ⇨ 69% so, 31% earn MORE

c) What percent of employees earn between $50,000 and $60,000?

50,000 z-score = .75 ⇨ 77.3% 60,000 z-score = 2.00 ⇨ 97.7% 20.4%

d) What salary level represents the top 20% of employees?

top 20% is equivalent to bottom 80%

80% corresponds to a z-score of .84

$.84 = \frac{x - 44{,}000}{8{,}000}$

approx. salary level x = 50,720

7) The mean basketball score is 78 points with a standard deviation of 6.5 points. The mean baseball score is 6.4 runs with a standard deviation of 1.2 runs.

Are you more likely to see a basketball team score 82 points or a baseball team score 7 runs?

basketball: mean - 78 std - 6.5 z-score: $z = \frac{82 - 78}{6.5} = .615$

baseball: mean - 6.4 std - 1.2 z-score: $z = \frac{7 - 6.4}{1.2} = .500$

Since the baseball score is closer to the mean, it is more likely to be seen..

8) To be admitted to the Math Academy, an applicant's score must be in the top 75%. The mean of the MA applicants is 63, and the standard deviation is 8.4. If Tony's score is 69, can he be admitted?

In a normal distribution, a percentile of 75% is equivalent to a z-score of .6745

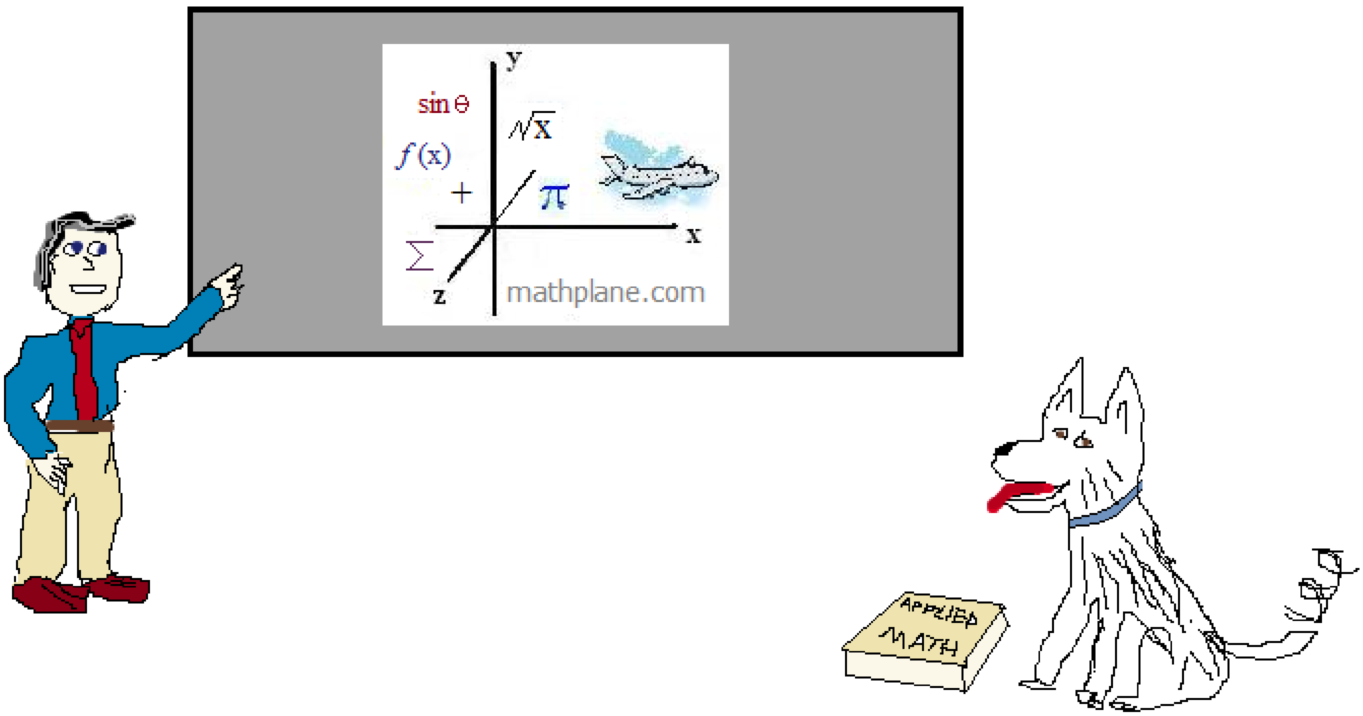
z-score = $\frac{x - \mu}{\sigma}$ $.6745 = \frac{x - 63}{8.4}$ x = 68.67 is the cut-off...

Tony's score of 69 is above it..
Yes, he can be admitted..

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers.

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